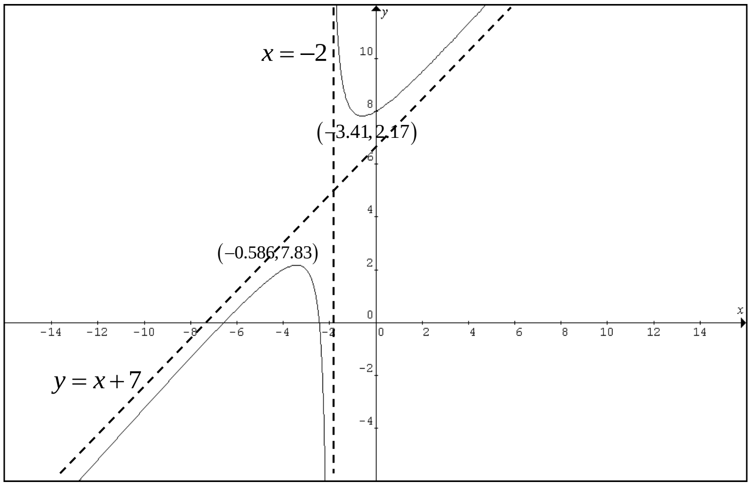
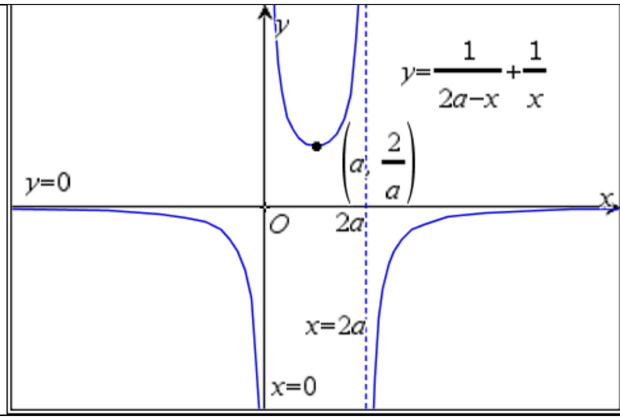
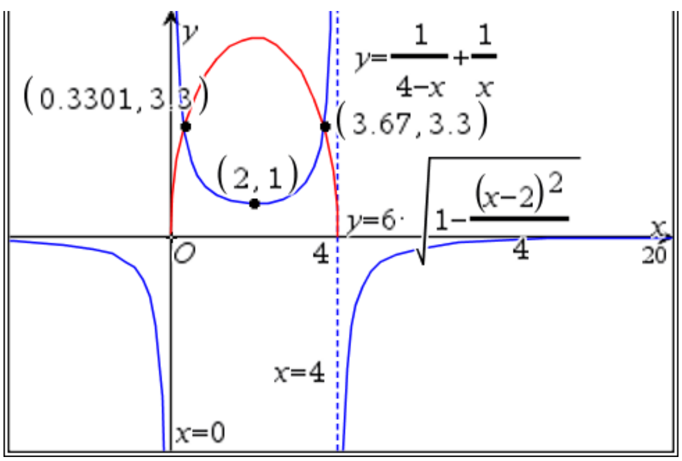
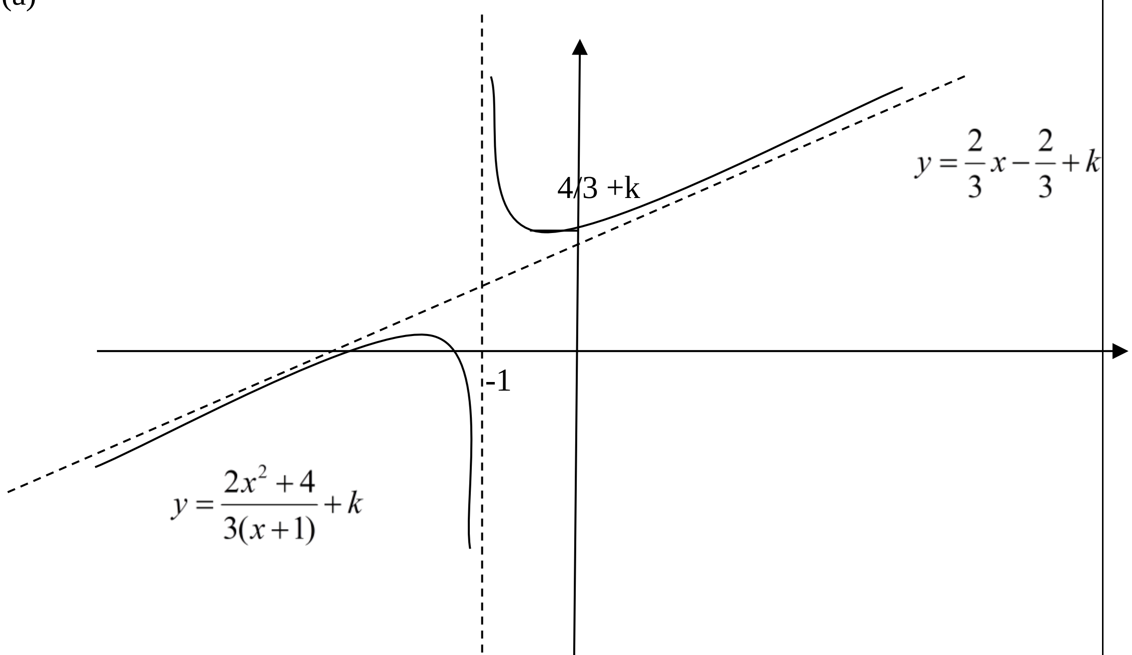
Solutions

Curve Sketching

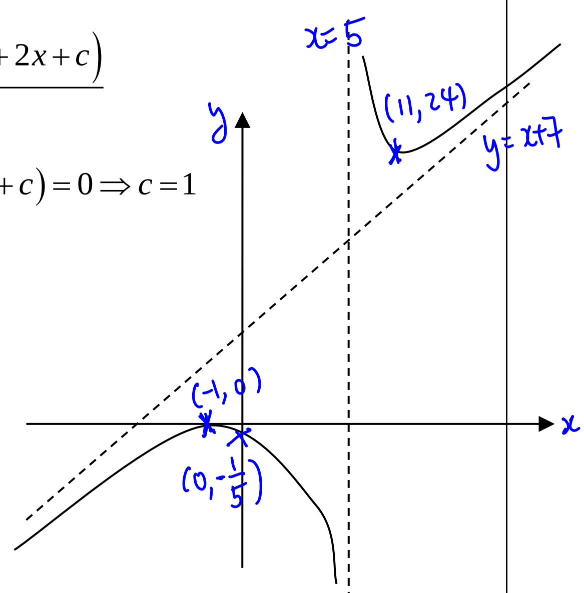
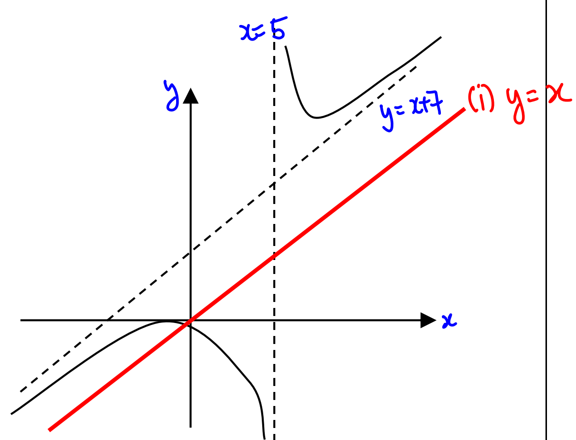
1(i)	Asymptotes: $y = x+7$ and $x = -2$
(ii)	$5-2\sqrt{2} < k < 5+2\sqrt{2}$
(iii)	

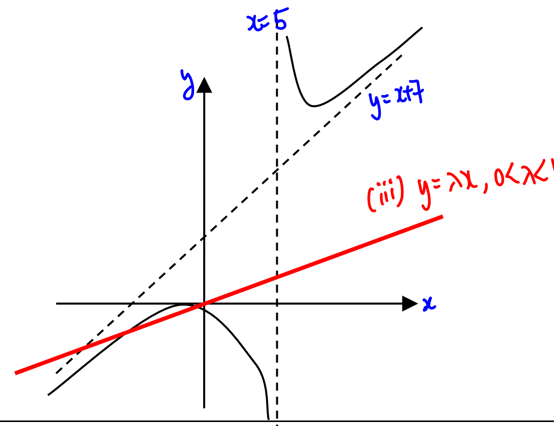
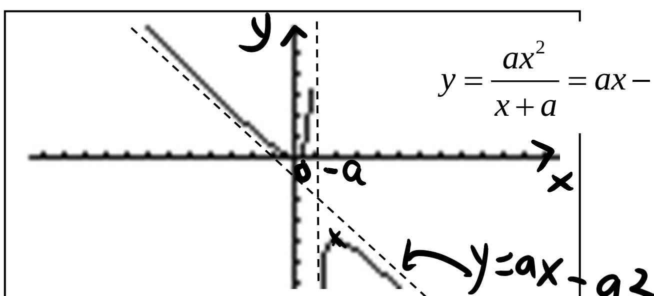
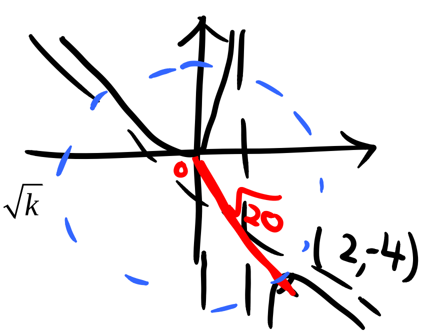
2(i)	**Method 1: Using Differentiation** $\frac{dy}{dx} = \frac{1}{(2a-x)^2} - \frac{1}{x^2}$ Putting $\frac{dy}{dx} = 0$, $\frac{1}{(2a-x)^2} = \frac{1}{x^2}$ $(2a-x)^2 = x^2 \quad \Rightarrow \quad x = \pm(2a-x)$ $x = a$ $y = \frac{1}{2a-a} + \frac{1}{a} = \frac{2}{a}$ $\frac{d^2y}{dx^2} = \frac{2}{(2a-x)^3} + \frac{2}{x^3}$ When $x = a$, $\frac{d^2y}{dx^2} = \frac{2}{a^3} + \frac{2}{a^3} > 0$ $\left(a\ ,\ \frac{2}{a}\right)$ is a min point. **Method 2: Non-differentiation method** $y = \frac{1}{2a-x} + \frac{1}{x} = \frac{2a}{x(2a-x)}$ is the reciprocal graph of quadratic eqn $y = x(2a-x)$ and multiplied by scale factor $2a$ in the y-direction.

	Since the graph of $y = x(2a - x)$ is a parabola with a max point at $(a\,, a^2)$, the curve C_1 will have a min point at $x = a$, and $y = 2a\left(\frac{1}{a^2}\right) = \frac{2}{a}$. Thus $\left(a\ ,\ \frac{2}{a}\right)$ is a min point.
(ii)	
(iii)	$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1$. The graph of C_2 is an ellipse with centre $(a, 0)$, with 2 end-pts at (0,0) and (2a,0). For the ellipse to intersect C_1 at more than one pt, $b > \frac{2}{a}$.
(iv)	$a = 2$ and $b = 6$ $\frac{1}{2a - x} + \frac{1}{x} > b\left[\sqrt{\left(1 - \frac{(x-a)^2}{a^2}\right)}\right]$  From the graph, $0 < x < 0.330$ or $3.67 < x < 4$ (3 s.f.)

3	(i) Since C has a vertical asymptote at $x=0$, thus $\lambda+1=0, \Rightarrow \lambda=-1$.
	(ii) $y=\dfrac{2x^2+4}{3x}=\dfrac{2}{3}x+\dfrac{4}{3x}$; Oblique asymptote is $y=\dfrac{2}{3}x$
	(iii) $y=\dfrac{2x^2+4}{3x}$ $3xy=2x^2+4$ $2x^2-3xy+4=0$ The equation above has no real roots when discriminant < 0. $(-3y)^2-4(2)(4)<0$ $9y^2-32<0$ M1, for getting $b^2-4ac<0$ correctly $(y+\dfrac{\sqrt{32}}{3})(y-\dfrac{\sqrt{32}}{3})<0$ $-\dfrac{4\sqrt{2}}{3}<y<\dfrac{4\sqrt{2}}{3}$ $\therefore a=2$.
	(iv) $y=\dfrac{2x^2+4}{3(x+1)}+k=\dfrac{2}{3}x-\dfrac{2}{3}+k+\dfrac{6}{3(x+1)}$ (a)  x – intercepts are $-\dfrac{3k}{4}+\sqrt{\dfrac{9k^2}{16}-\dfrac{3k}{2}-2}$ and $-\dfrac{3k}{4}-\sqrt{\dfrac{9k^2}{16}-\dfrac{3k}{2}-2}$

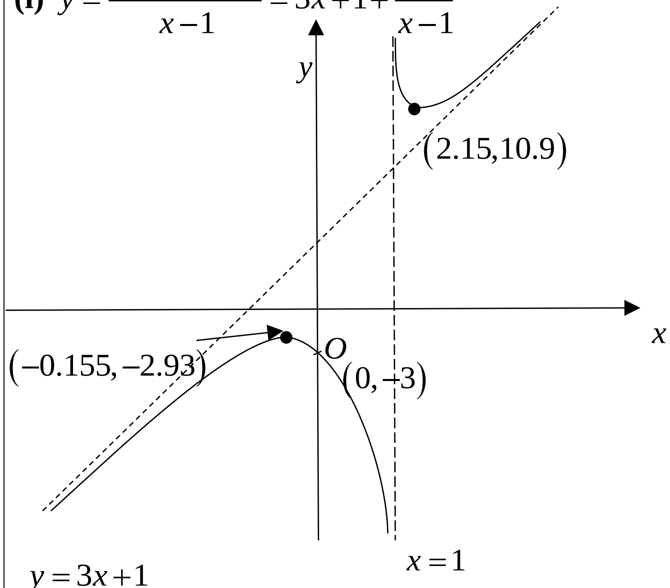
	(b) For the graph of $y=\frac{2x^2+4}{3(x+1)}+k$ to cut the x-axis at 2 distinct points, $\sqrt{\frac{9k^2}{16}-\frac{3k}{2}-2}>0$, =>least integer value of k is 4

4	$x=5$ is an asymptote $\Rightarrow d=-5$ $y=\frac{ax^2+bx+c}{x-5}=ax+(b+5a)+\frac{R}{x-5}$ $y=x+7$ is an asymptote $\Rightarrow \begin{cases} a=1, \\ b+5a=7 \Rightarrow b=2 \end{cases}$ $y=\frac{x^2+2x+c}{x-5} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x-5)(2x+2)-(x^2+2x+c)}{(x-5)^2}$ At $x=-1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow (-1-5)(-2+2)-(1-2+c)=0 \Rightarrow c=1$ Sketch $y=\frac{x^2+2x+1}{x-5}$  $(1-\lambda)x^2+(2+5\lambda)x+1=0$ $\Rightarrow x^2-\lambda x^2+2x+5\lambda x+1=0$ $\Rightarrow x^2+2x+1=\lambda x^2-5\lambda x$ $\Rightarrow \frac{x^2+2x+1}{x-5}=\lambda x$ Therefore, sketch the graph of $y=\lambda x$ on the same diagram. **(i)** $y=\lambda x$, $\lambda=1$ From graph, only one pt of intersection $\Rightarrow$ one real root 

	(ii) $y = \lambda x,\ 0 < \lambda < 1$ From graph, two pts of intersection $\Rightarrow$ two real roots 
5	$y = \dfrac{ax^2}{x+a}$ (i) $\dfrac{dy}{dx} = \dfrac{(x+a)2ax - ax^2}{(x+a)^2} = \dfrac{ax^2 + 2a^2x}{(x+a)^2}$ Set $\dfrac{dy}{dx} = 0 \Rightarrow ax^2 + 2a^2x = 0 \Rightarrow ax(x+2a) = 0 \Rightarrow x = 0 \text{ or } x = -2a$ For all negative values of *a*, there will be two distinct values of *x* thus two stationary pts. (shown) Or $B^2 - 4AC = 4a^2 > 0$ for all negative values of a.
	(ii)  $y = \dfrac{ax^2}{x+a} = ax - a^2 + \dfrac{a^3}{x+a}$ Max pt is $(-2a, -4a^2)$ Min pt is (0,0)
	(iii) $x^4 = (k - x^2)(x-1)^2 \Rightarrow a = -1$ $\Rightarrow \left(\dfrac{-x^2}{x-1}\right)^2 = k - x^2 \Rightarrow y^2 = k - x^2$ $\Rightarrow$ insert a circle, centre (0,0), radius $\sqrt{k}$ to cut curve twice. $\Rightarrow 0 < k < 20$ 

6. Suggested solution

(i) $y=\dfrac{3x^2-2x+3}{x-1}=3x+1+\dfrac{4}{x-1}$



(ii)

C_2 is a horizontal hyperbola with center $(1,4)$.

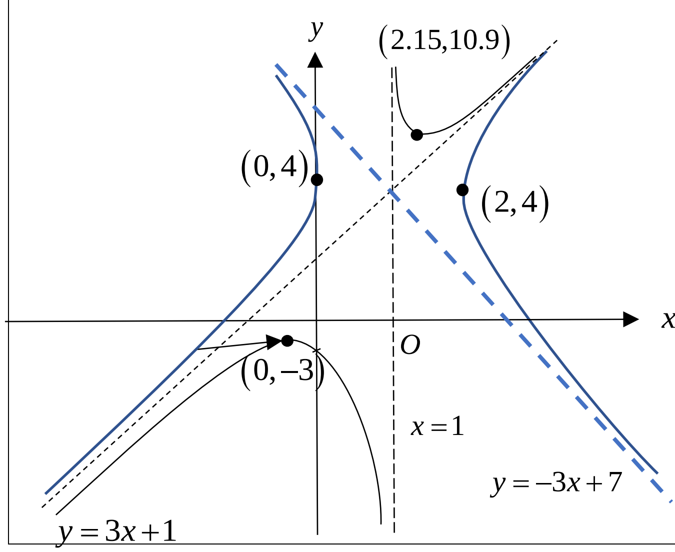
Asymptotes: $y=\pm b(x-1)+4$

For no point of intersection between C_1 and C_2, the asymptote of hyperbola must be as steep or less steep than the oblique asymptote of C_1. Note that the gradient of oblique asymptote of C_1 is 3.

$\therefore 0<b\le 3$

For $b=3$,

Asymptotes: $y=\pm 3(x-1)+4=3x+1,-3x+7$



7(i)	$y=\dfrac{2x^2-a^2x+4a^2}{x^2-a^2}$ By long division, $y=2+\dfrac{6a^2-a^2x}{x^2-a^2}$ Vertical asymptotes: $x=-a$ or $x=a$ Horizontal asymptote: $y=2$
(ii)	From $y=2+\dfrac{6a^2-a^2x}{x^2-a^2}$, $\dfrac{dy}{dx}=0+\dfrac{\left(-a^2\right)\left(x^2-a^2\right)-\left(6a^2-a^2x\right)\left(2x\right)}{\left(x^2-a^2\right)^2}$ $=\dfrac{\left(a^2\right)\left(-x^2+a^2-12x+2x^2\right)}{\left(x^2-a^2\right)^2}$ $=\dfrac{\left(a^2\right)\left(2x^2-12x+a^2\right)}{\left(x^2-a^2\right)^2}$ $\dfrac{dy}{dx}=0\Rightarrow\dfrac{\left(a^2\right)\left(x^2-12x+a^2\right)}{\left(x^2-a^2\right)^2}=0$ $\Rightarrow x^2-12x+a^2=0$ Given C has two turning points, $\therefore b^2-4ac>0\Rightarrow\left(-12\right)^2-4\left(1\right)\left(a^2\right)>0$ $144-4a^2>0$ $a^2-36<0$ $\left(a-6\right)\left(a+6\right)<0$ $-6<a<6$ Since a is a positive constant, $0<a<6$. (Shown)

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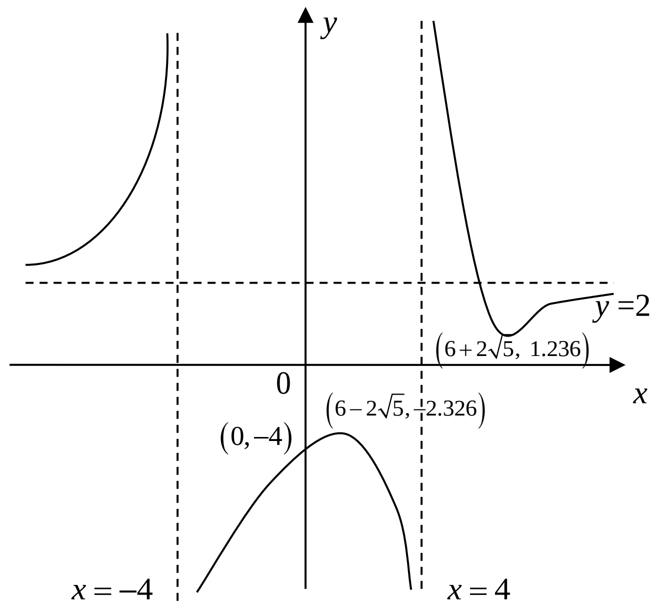

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<td>(iv)</td>
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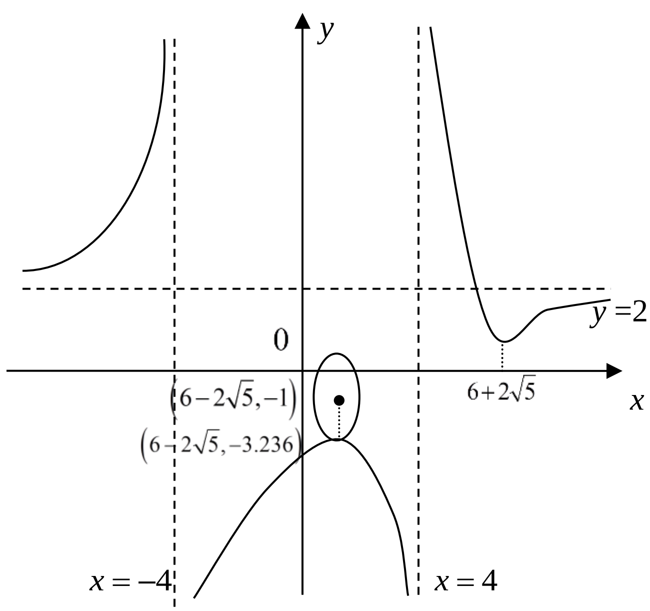
$h^2\left(x-6+2\sqrt{5}\right)^2+\left(y+1\right)^2=h^2$

$\Rightarrow\left(x-6+2\sqrt{5}\right)^2+\dfrac{\left(y+1\right)^2}{h^2}=1$

It is an ellipse centred at $\left(6-2\sqrt{5},-1\right)$.



Maximum turning point of C occurs at $\left(6-2\sqrt{5},-3.236\right)$.

For the ellipse to intersect the curve C more than once, $h>2.236..$
Since h is a positive integer, $h\ge 3$.
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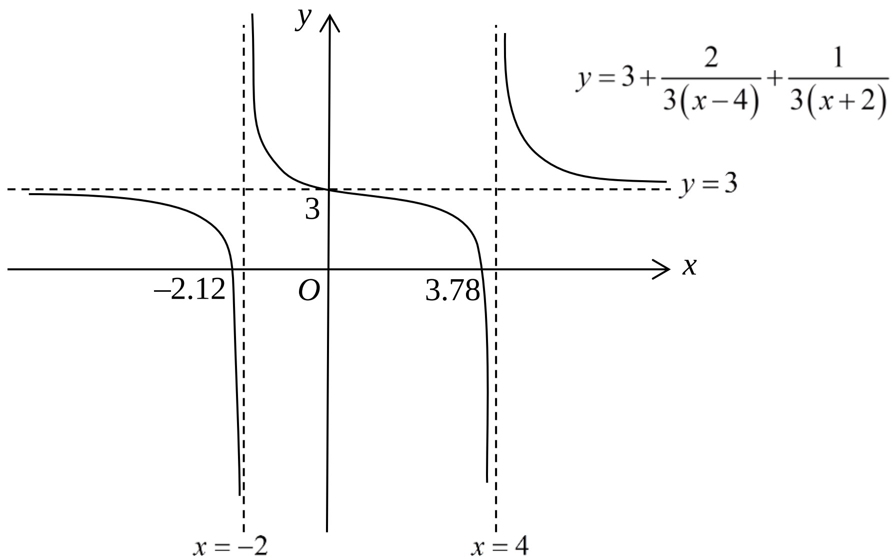
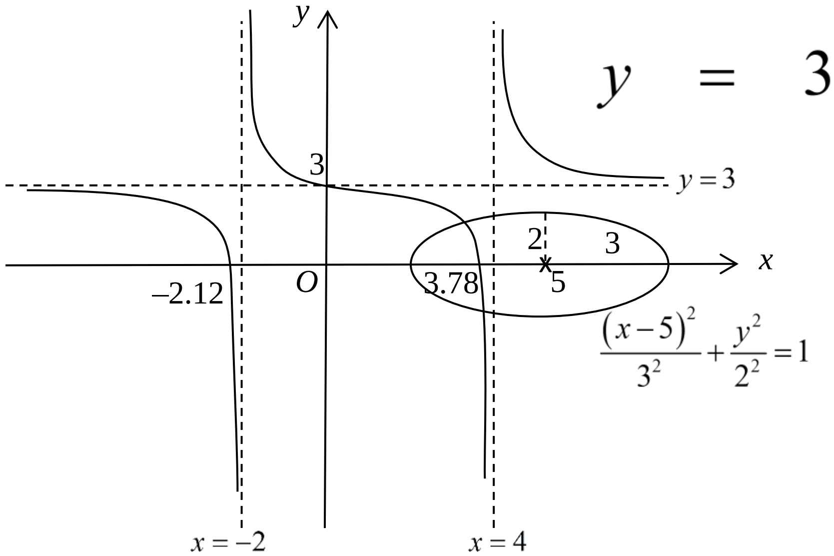
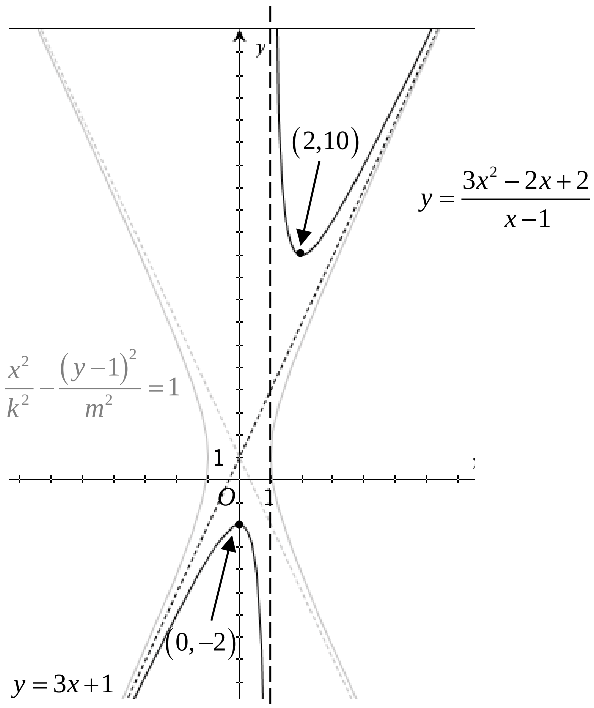
8(i)	$y = 3 + \frac{x}{x^2 - 2x - 8}$ $= 3 + \frac{2}{3(x-4)} + \frac{1}{3(x+2)}$ $\frac{dy}{dx} = -\frac{2}{3(x-4)^2} - \frac{1}{3(x+2)^2}$ < 0 Therefore, C_1 has no stationary points.
(ii)	
(iii)	Required area $= \int_0^2 3 + \frac{2}{3(x-4)} + \frac{1}{3(x+2)}\ dx$ $= \left[3x + \frac{2}{3}\ln\lvert x-4\rvert + \frac{1}{3}\ln\lvert x+2\rvert\right]_0^2$ $= \left(6 + \frac{2}{3}\ln 2 + \frac{1}{3}\ln 4\right) - \left(\frac{2}{3}\ln 4 + \frac{1}{3}\ln 2\right)$ $= 6 + \frac{1}{3}\ln 2 - \frac{1}{3}\ln 4$ units^2 or $6 - \frac{1}{3}\ln 2$ units^2

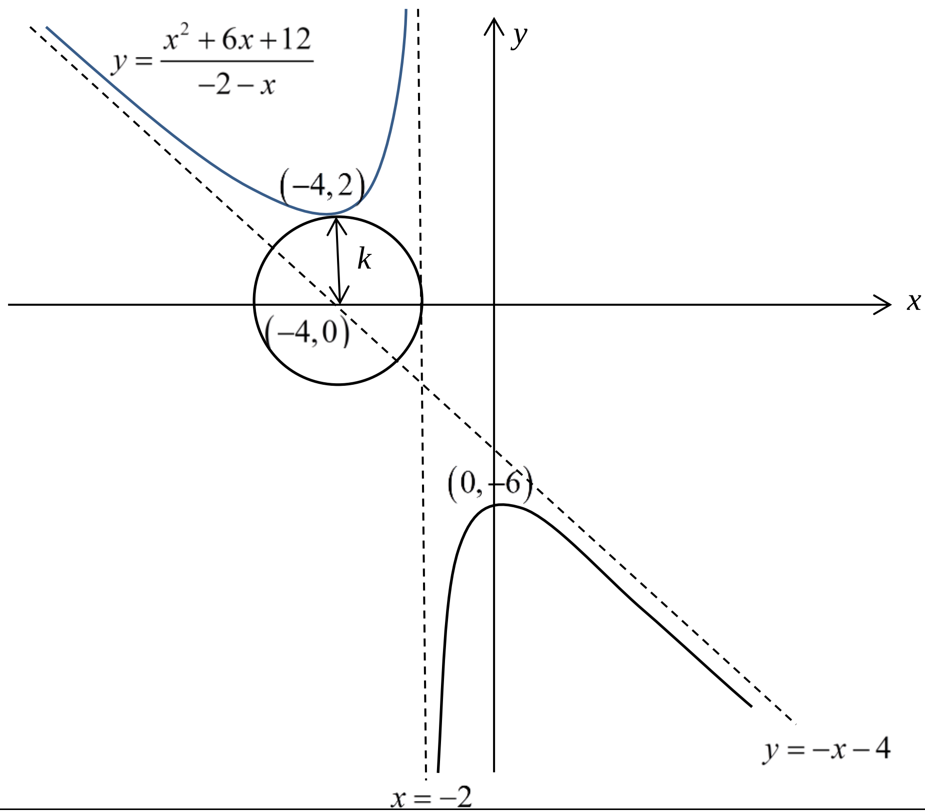
(iv)	 Using G.C., the coordinates of the points of intersection are (3.50,1.73) and (3.86, –1.85).

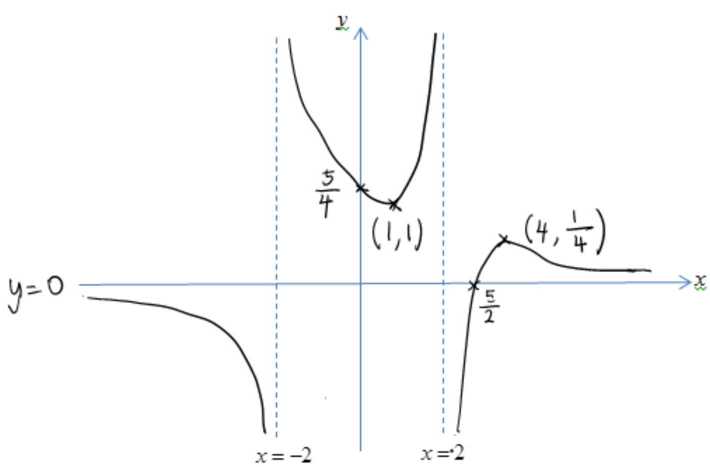
9(i)	$y=\frac{ax^2+bx+c}{x-1}$ Since C passes through the point $\left(3,\frac{23}{2}\right)$, $\frac{23}{2}=\frac{a(3)^2+3b+c}{3-1}\Rightarrow 9a+3b+c=23$ C passes through the point $(2, 10)$ too, so $10=\frac{a(2)^2+2b+c}{2-1}\Rightarrow 4a+2b+c=10$ Since (2, 10) is a minimum point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ when $x=2$. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(2ax+b)(x-1)-(ax^2+bx+c)(1)}{(x-1)^2}$ $=\frac{ax^2-2ax-b-c}{(x-1)^2}$ So $0=4a-4a-b-c\Rightarrow b+c=0.$ Solving the three equations using the GC, $a=3,\ b=-2,\ c=2.$
(ii)	$y=\frac{3x^2-2x+2}{x-1}$ Performing long division, $3x+1$ $x-1\overline{)3x^2-2x+2}$ $\underline{3x^2-3x}$ $x+2$ $\underline{x-1}$ 3 Hence, $y=3x+1+\frac{3}{x-1}$



(iii)	$\frac{x^2}{k^2} - \frac{(y-1)^2}{m^2} = 1$ describes a hyperbola that has asymptotes with equations $y = 1 \pm \frac{m}{k} x$. From the graph, comparing asymptotes, we can observe that the gradient of the asymptote with positive gradient of the hyperbola must be at most 3 if the two graphs are not to intersect. Hence, the set of values of $\frac{m}{k}$ is $\left\{\frac{m}{k} \in \mathbb{R} : 0 < \frac{m}{k} \le 3\right\}$.
(iv)	When *c* = 2, $y = \frac{-x^2 - x + 2}{x-1}$ becomes, $y = -x - 2,\ x \ne 1$ and has no stationary point. From our working in part (i), when $a = b = -1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x^2 + 2x + 1 - c}{(x-1)^2}$ Stationary points of *c* occur when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, i.e. $-x^2 + 2x + 1 - c = 0$. For there to be no stationary points, the equation $-x^2 + 2x + 1 - c = 0$ must have no real solutions for *x*. Hence, the discriminant $2^2 - 4(-1)(1-c) < 0$. $4 + 4(1-c) < 0$ $\Rightarrow c > 2$ So the set of values of *c* is $\{c \in \mathbb{R} : c \ge 2\}$.

10(i)	$x = -2$ is a vertical asymptote, $c = -2$ *C* passes through $(-4, 2)$ and $(0, -6)$ $2 = \frac{(-4)^2 + a(-4) + b}{-2-(-4)}$ $-4a + b = -12$ Eqn 1

	$-6=\dfrac{(0)^2+a(0)+b}{-2-(0)}$ $b=12$ Eqn 2 From Eqn 1, $a=6$
(ii)	
(iii)	$(x+4)^2+\left(\dfrac{x^2+ax+b}{c-x}\right)^2=k^2$ $\left[x-(-4)\right]^2+(y)^2=k^2$ The appropriate graph to draw is a circle with radius k units, centred at $(-4,0)$. For the equation to have no real roots, $0<k<2$
(iv)	Translate C by -3 in the direction of y-axis, hence $p=-3$ Or Translate C by more than 5 units in the direction of y-axis, $p>5$

11(i)	Since there is an asymptote at $x=2$, $2^2+r=0 \Rightarrow r=-4$ Therefore equation of C: $y=\dfrac{px+q}{x^2-4}$ $\left(4,\dfrac{1}{4}\right)$ is a point on C: $\dfrac{1}{4}=\dfrac{4p+q}{12} \Rightarrow q=3-4p \quad ...(1)$ $\dfrac{dy}{dx}=\dfrac{p(x^2-4)-2x(px+q)}{\left(x^2-4\right)^2}=\dfrac{-px^2-4p-2qx}{\left(x^2-4\right)^2}$ $\left(4,\dfrac{1}{4}\right)$ is a turning point on C: $-16p-4p-8q=0$ $5p=-2q...(2)$ Solving (1) and (2), $p=2, q=-5$
(ii)	Therefore equation of C: $y=\dfrac{2x-5}{x^2-4}$ 

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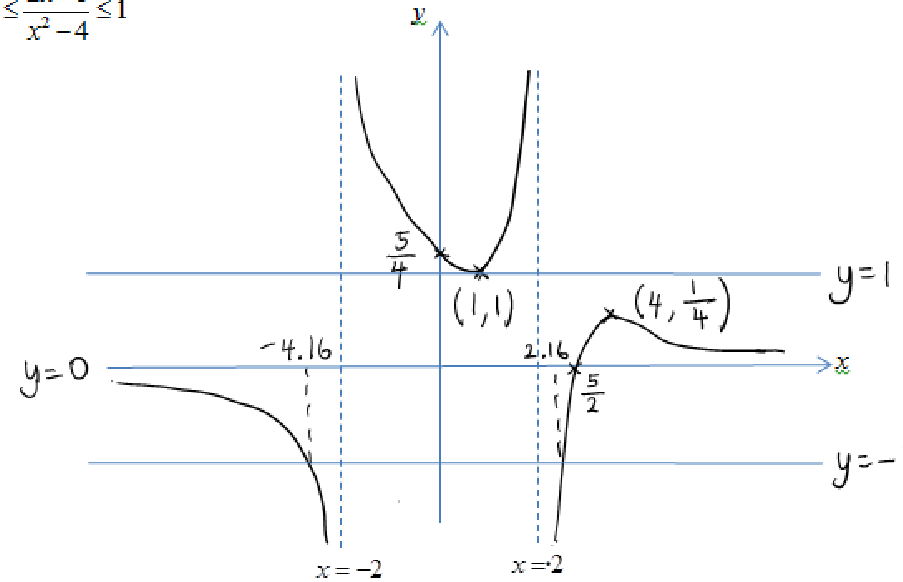
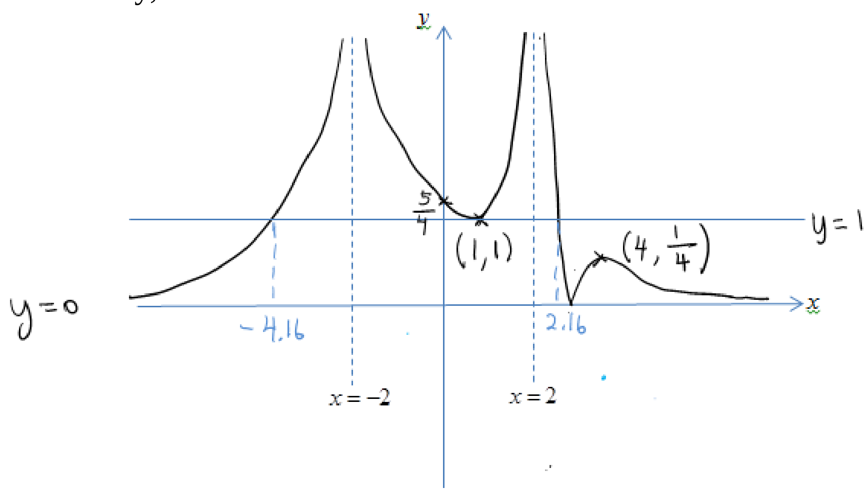
$-1 \leq \frac{2x-5}{x^2-4} \leq 1$



$x \leq -4.16$ or $x = 1$ or $x \geq 2.16$

Alternatively,



$x \leq -4.16$ or $x = 1$ or $x \geq 2.16$

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In the empty region of the graph, the horizontal line $y = p$ does not intersect the curve at any points.

Consider $p = 2x + \frac{8k^2}{x-2}$:

$p(x-2) = (2x)(x-2) + 8k^2$

$xp - 2p = 2x^2 - 4x + 8k^2$

$2x^2 - (4+p)x + 8k^2 + 2p = 0$

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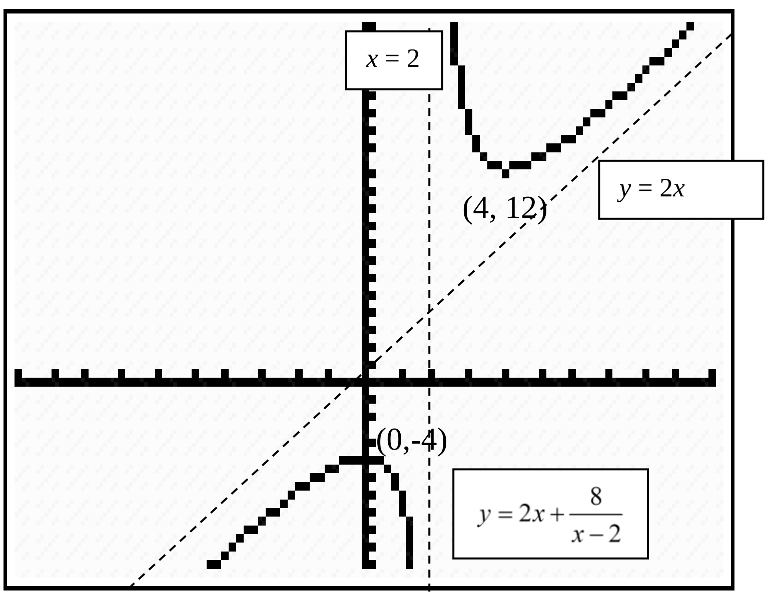
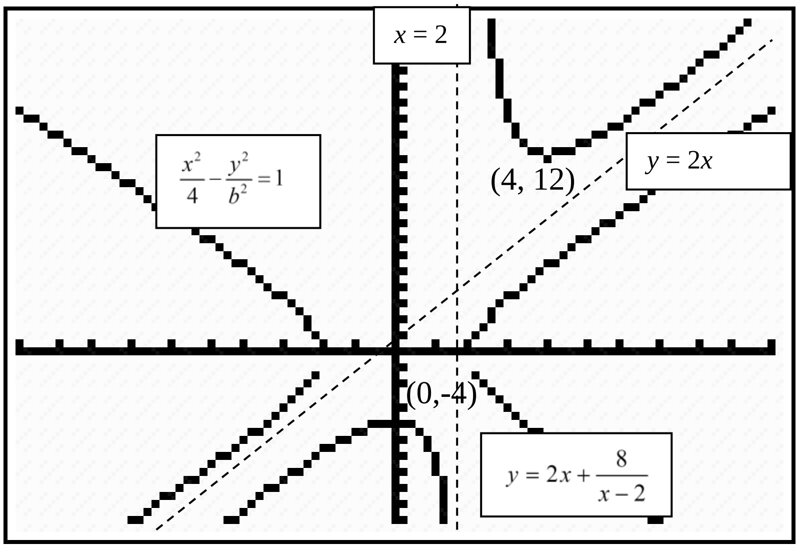
	Discriminant, $b^2-4ac=(4+p)^2-4(2)(8k^2+2p)$ $=p^2+8p+16-64k^2-16p$ $=p^2-8p+16-64k^2$ When $p^2-8p+16-64k^2=0$, $p=\frac{8\pm\sqrt{8^2-4(16-64k^2)}}{2}=4\pm 8k$ In empty region, curve does not intersect the line $y=p$, i.e. the equation $p=2x+\frac{8k^2}{x-2}$ has no real solutions. This occurs when the discriminant is less than 0, i.e. $p^2-8p+16-64k^2<0$ $4-8k<p<4+8k$ Therefore, y cannot lie between 4 – 8k and 4 + 8k.
(ii)	

(iii)	 $b^2x^2-4y^2=4b^2 \quad \Rightarrow \quad \frac{x^2}{4}-\frac{y^2}{b^2}=1$ This is a hyperbola with *x*-intercepts (-2, 0) and (2, 0) and asymptotes $y=\pm\frac{b}{2}x$. For the hyperbola curve not to intersect curve *C*, the absolute value of the gradient of the asymptote of the hyperbola must not be greater than 2. $\frac{b}{2}\le 2$ and $b>0$ $\therefore \quad 0<b\le 4$

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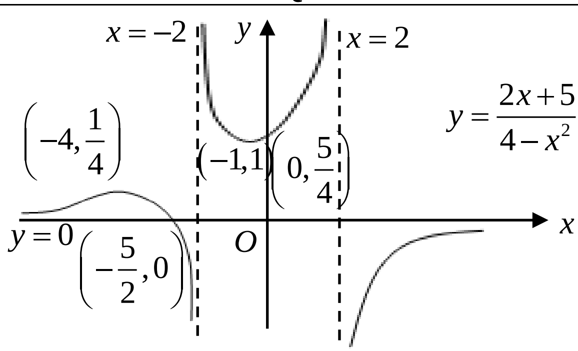
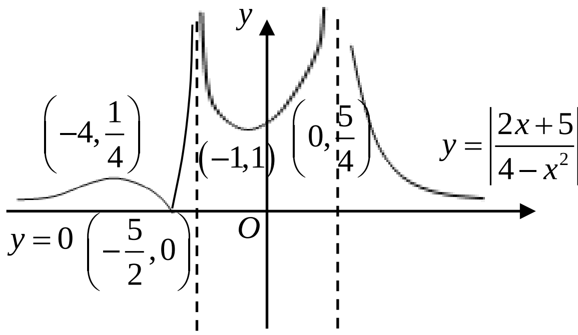
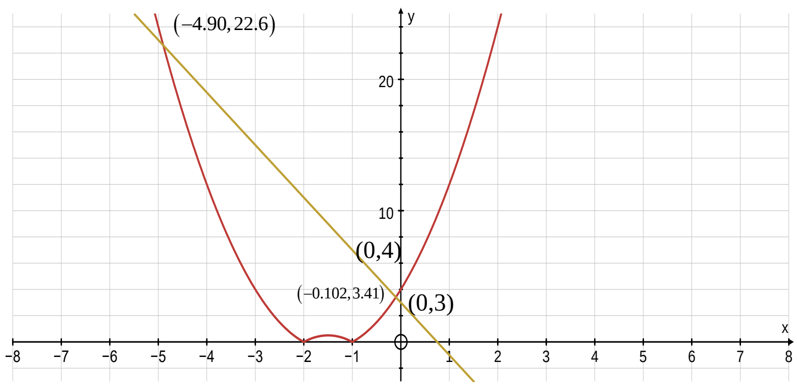
13 i	
ii	When *y* = 2, *x* = 0.82288 or -1.8229 Area formed = $\int_{-1.8229}^{0.82288} 2-\frac{2x+5}{4-x^2}\,dx$ $=1.95 \text{ unit}^2$

iii



From graph, $\left\{k\in\mathbb{R}:0<k<\frac{1}{4}\text{ or }1<k<\frac{5}{4}\right\}$.

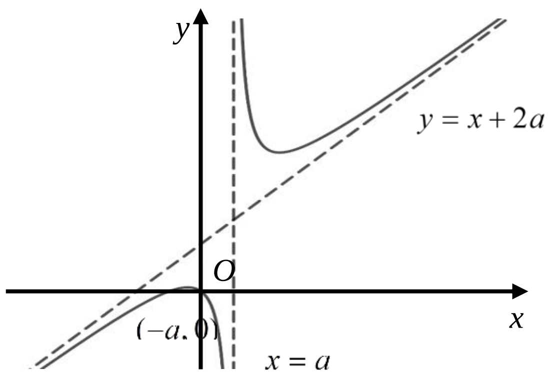
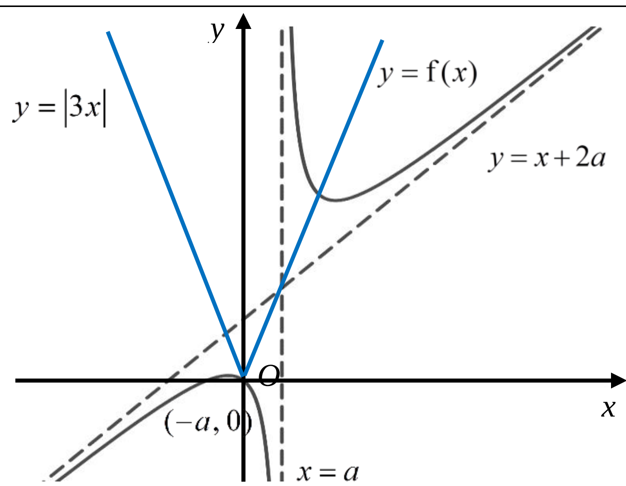
14(i)



Hence solutions to inequality is
$x<-4.90$ or $x>-0.102$ (3sf)

(ii)

$$\int_{-3}^{-1}\left(3-4x-|2x^2+6x+4|\right)dx$$
$$=\int_{-3}^{-2}\left(3-4x-2x^2-6x-4\right)dx+\int_{-2}^{-1}\left(3-4x+2x^2+6x+4\right)dx$$
$$=\int_{-3}^{-2}\left(-2x^2-10x-1\right)dx+\int_{-2}^{-1}\left(2x^2+2x+7\right)dx$$
$$=\left[-\frac{2}{3}x^3-5x^2-x\right]_{-3}^{-2}+\left[\frac{2}{3}x^3+x^2+7x\right]_{-2}^{-1}$$
$$=(\frac{16}{3}-20+2)-(18-45+3)+(-\frac{2}{3}+1-7)-(-\frac{16}{3}+4-14)$$
$$=20.$$

15(i)	Consider $y=k$, k is a constant $\frac{x(x+a)}{x-a}=k$ $x^2+ax=xk-ak$ $x^2+(a-k)x+ak=0$ For the range of y can take, the line $y=k$ and the curve C should have point(s) of intersection. $b^2-4ac\geq 0$ $(a-k)^2-4ak\geq 0$ $a^2-2ak+k^2-4ak\geq 0$ $a^2-6ak+k^2\geq 0$ Consider $k^2-6ak+a^2=0$ $k=\frac{6a\pm\sqrt{36a^2-4a^2}}{2}=\left(3\pm 2\sqrt{2}\right)a$ $\therefore k\geq\left(3+2\sqrt{2}\right)a$ or $k\leq\left(3-2\sqrt{2}\right)a$ Hence, $y\geq\left(3+2\sqrt{2}\right)a$ or $y\leq\left(3-2\sqrt{2}\right)a$
(ii)	
(iii)	

	Consider $\frac{x(x+a)}{x-a}=3x$ $x^2+ax=3x^2-3ax$ $2x^2-4ax=0$ $x(x-2a)=0$ $x=0 \quad or \quad x=2a$ Hence, $y=\lvert 3x\rvert$ and $(-a,0)$ intersect at $x=0$ and $x=2a$. From the graph, $x<0$ or $0<x<a$ or $x>2a$. Checking: Consider $\frac{x(x+a)}{x-a}=-3x$ $x^2+ax=-3x^2+3ax$ $4x^2-2ax=0$ $x(2x-a)=0$ $x=0 \quad or \quad x=\frac{a}{2}$ (N.A. since $a>0$)

16. ECJC/2022/I/Q5

The curve C has equation

$$y=\frac{2x-6}{x^2+2x-3}.$$

(a) State the equations of the asymptotes of C. [2]

(b) Without using a calculator, find the range of values that y can take. [4]

(c) Sketch the graph of C, stating the equations of any asymptotes, the coordinates of the points where the curve crosses the axes and the stationary point(s). [4]

(d) Describe one transformation that will transform the curve C onto the curve $y=\frac{2x-8}{x^2-4}$. [1]

5(a) **Horizontal Asymptote:** $y=0$ **Vertical Asymptote:** $x^2+2x-3=(x+3)(x-1)$ **Vertical asymptotes are** $x=-3,\ x=1$**.**

(b) $yx^2 + 2yx - 3y = 2x - 6$

$yx^2 + (2y-2)x + (6-3y) = 0$

Since $x \in \mathbb{R}$ **,**

Discriminant $(2y-2)^2 - 4y(6-3y) \geq 0$

$16y^2 - 32y + 4 \geq 0$

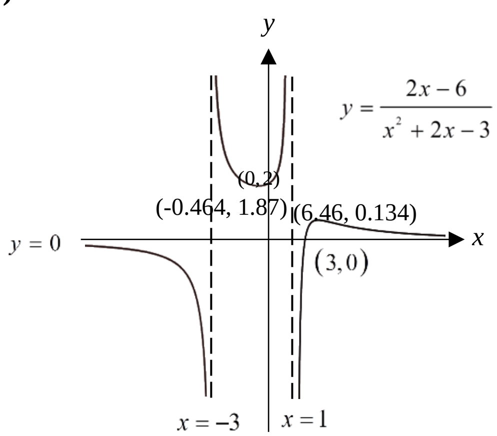
$4y^2 - 8y + 1 \geq 0$

As roots of $4y^2 - 8y + 1 = 0$ **are:**

$$y = \frac{8 \pm \sqrt{8^2 - 4(4)(1)}}{2(4)} = 1 \pm \frac{\sqrt{3}}{2} \text{ ,}$$

$y \leq 1 - \frac{\sqrt{3}}{2}$ **OR** $y \geq 1 + \frac{\sqrt{3}}{2}$ **.**

(c)



(d) **Translation of *C* by 1 unit in the positive *x*-direction.**