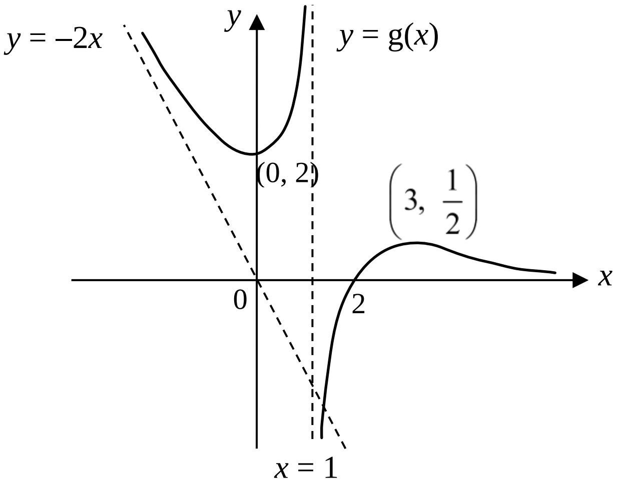
2015 Preliminary Examination H2 Mathematics 9740 Paper 1 (Solutions)

1 The equation of a circle M is given by $x^2+y^2+Ax+By+C=0$ where A, B and C are real constants. The line $y = -2(x + 1)$ passes through the centre of M and the graph of $y = |x|$ intersects M at the points where $x = -2$ and $x = -8$. Find the equation of M. [4]

1	M: $\quad x^2+y^2+Ax+By+C=0$ $\Rightarrow \left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2-\frac{A^2}{4}-\frac{B^2}{4}+C=0$ Centre of M : $\left(-\frac{A}{2},-\frac{B}{2}\right)$ $y=-2(x+1)$ passes through the centre: $-\frac{B}{2}=-2\left(-\frac{A}{2}+1\right)$ $2A+B=4$ ----- (1) At intersection between $y=\|x\|$ and M, we have $x^2+\|x\|^2+Ax+B\|x\|+C=0$ At $x=-2$, $(-2)^2+(\|-2\|)^2+A(-2)+B(\|-2\|)+C=0$ $-2A+2B+C=-8$ ----- (2) At $x=-8$, $(-8)^2+(\|-8\|)^2+A(-8)+B(\|-8\|)+C=0$ $-8A+8B+C=-128$ ----- (3) Solving (2), (3) and (4) using GC: $A = 8$, $B = -12$, $C = 32$ M: $\quad x^2+y^2+8x-12y+32=0$

2 The diagram below shows the graph of $y = \mathrm{g}(x)$. The graph has a minimum point at $(0,\ 2)$ and a maximum point at $\left(3,\ \frac{1}{2}\right)$. The equations of the asymptotes are $x = 1$, $y = 0$ and $y = -2x$.

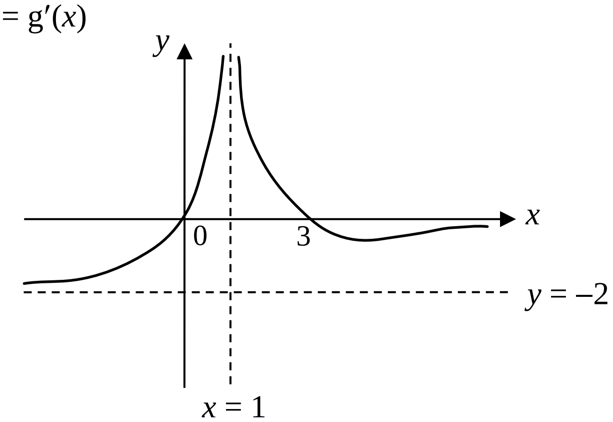
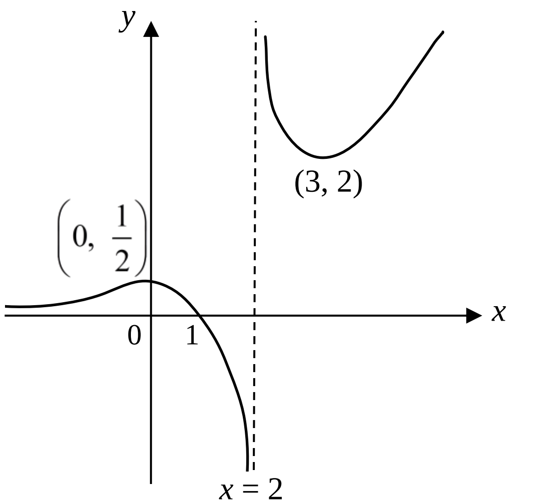


On separate diagrams, sketch the graphs of

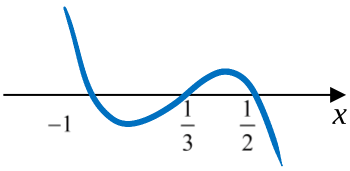
(i) $y = \mathrm{g}'(x)$, [2]

(ii) $y = \dfrac{1}{\mathrm{g}(x)}$, [2]

showing clearly in each case, the equations of the asymptotes and the coordinates of the turning points and axial intercepts, where applicable.

<table>
<tr><td>2i</td><td>(i) $y = \mathrm{g}'(x)$
</td></tr>
<tr><td>2ii</td><td>$y = \dfrac{1}{\mathrm{g}(x)}$
</td></tr>
</table>

3 Without using a calculator, solve the inequality $\frac{3x^2}{1-2x}>1$. Hence solve $\frac{3x^2}{1-2|x|}>1$. [5]

3	$\frac{3x^2}{1-2x}>1$ $\frac{3x^2+2x-1}{1-2x}>0$ $(3x-1)(x+1)(1-2x)>0$ $x<-1$ or $\frac{1}{3}<x<\frac{1}{2}$ $\frac{3\|x\|^2}{1-2\|x\|}>1$ Replace x by $\|x\|$, $\|x\|<-1$ (no real solution) or $\frac{1}{3}<\|x\|<\frac{1}{2}$ $\therefore -\frac{1}{2}<x<-\frac{1}{3}$ or $\frac{1}{3}<x<\frac{1}{2}$ 

4 The sequence of real numbers $u_1, u_2, u_3, \ldots$ is defined by

$$u_{n+1} = \frac{n+2}{n+4} u_n \text{ and } u_1 = a, \text{ where } n \geq 1 \text{ and } a \in \mathbb{R}.$$

(i) Prove by mathematical induction that $u_n = \dfrac{12a}{(n+2)(n+3)}$ for $n \geq 1$. [4]

(ii) Determine the limit of $n(n+2)\dfrac{u_n}{u_1}$ as $n \to \infty$. [2]

4i	Let P_n be the statement $u_n = \dfrac{12a}{(n+2)(n+3)}$ for $n \geq 1$. When n = 1, LHS = $u_1 = a$ (given) RHS = $\dfrac{12a}{(1+2)(1+3)} = a$ = LHS $\therefore$ P_1 is true. Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \dfrac{12a}{(k+2)(k+3)}$. When $n = k + 1$, LHS $= u_{k+1} = \dfrac{k+2}{k+4} u_k$ $= \dfrac{k+2}{k+4} \dfrac{12a}{(k+2)(k+3)}$ $= \dfrac{12a}{(k+4)(k+3)}$ $= \dfrac{12a}{((k+1)+2)((k+1)+3)}$ = RHS $\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
4ii	As $n \to \infty$, $n(n+2)\dfrac{u_n}{u_1} = n(n+2)\dfrac{1}{a}\dfrac{12a}{(n+2)(n+3)} = \dfrac{12n}{n+3} = \dfrac{12}{1+\frac{3}{n}} \to 12$

5 The complex number z satisfies the equation $\dfrac{1+z^3}{1-z^3} = \sqrt{3}\,\mathrm{i}$.

Without the use of a graphing calculator, express z^3 in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r \geq 0$ and $-\pi < \theta \leq \pi$. Hence find the roots of the equation. [6]

5

$$\frac{1+z^3}{1-z^3} = \sqrt{3}\mathrm{i}$$

$$1+z^3 = \sqrt{3}\mathrm{i} - \sqrt{3}z^3\mathrm{i}$$

$$z^3(1+\sqrt{3}\mathrm{i}) = \sqrt{3}\mathrm{i} - 1$$

$$z^3 = \frac{\sqrt{3}\mathrm{i}-1}{1+\sqrt{3}\mathrm{i}}$$

$$= \frac{2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}}{2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}} = \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$$

<u>Alternative</u>

$$z^3 = \frac{\sqrt{3}\mathrm{i}-1}{\sqrt{3}\mathrm{i}+1}$$

$$= \frac{\sqrt{3}\mathrm{i}-1}{\sqrt{3}\mathrm{i}+1} \times \frac{\sqrt{3}\mathrm{i}-1}{\sqrt{3}\mathrm{i}-1}$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

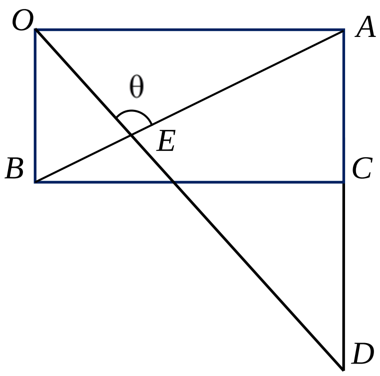
$$= \mathrm{e}^{\frac{\pi}{3}\mathrm{i}}$$

$$z^3 = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}+2n\pi\right)}, \qquad n \in \mathbb{Z}$$

$$z = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{9}+\frac{2n\pi}{3}\right)}, \qquad n = 0, \pm 1$$

$$z = \mathrm{e}^{\mathrm{i}\frac{\pi}{9}}, \mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}, \mathrm{e}^{\mathrm{i}\frac{7\pi}{9}}$$

6 The figure below shows a rectangle $OACB$ where $OA = 2OB$. Point D is on AC produced such that $AD : AC = \lambda : 1$ where λ is a constant. The lines OD and AB intersect at point E. It is given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\angle OEA = \theta$.



Find $\overrightarrow{OD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, and show that $\overrightarrow{OD} \cdot \overrightarrow{AB} = (\lambda - 4)|\mathbf{b}|^2$. [4]

In the case when E is the foot of perpendicular from A to OD, deduce the value of λ. [2]

Using this value of λ and given that $\mathbf{a} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, find $\left|\overrightarrow{OE}\right|$. [2]

6	$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \overrightarrow{OA} + \lambda\overrightarrow{AC}$ $= \mathbf{a} + \lambda\mathbf{b}$ $\overrightarrow{OD} \cdot \overrightarrow{AB} = (\mathbf{a} + \lambda\mathbf{b}) \cdot (\mathbf{b} - \mathbf{a})$ $= \mathbf{a} \cdot \mathbf{b} - \lvert\mathbf{a}\rvert^2 + \lambda\lvert\mathbf{b}\rvert^2 - k(\mathbf{b} \cdot \mathbf{a})$ $= \lambda\lvert\mathbf{b}\rvert^2 - \lvert\mathbf{a}\rvert^2$ (since $\mathbf{a} \cdot \mathbf{b} = 0 = \mathbf{b} \cdot \mathbf{a}$) $= \lambda\lvert\mathbf{b}\rvert^2 - 4\lvert\mathbf{b}\rvert^2$ (since $\lvert\mathbf{a}\rvert = 2\lvert\mathbf{b}\rvert$) $= (\lambda - 4)\lvert\mathbf{b}\rvert^2$ (shown)
	E is foot of perpendicular from A to OD (i.e. $AB \perp OD$) $\Rightarrow \overrightarrow{OD} \cdot \overrightarrow{AB} = 0$ $\Rightarrow (\lambda - 4)\lvert\mathbf{b}\rvert^2 = 0$ $\therefore \lambda = 4$ (since $\lvert\mathbf{b}\rvert \neq 0$)

Method 1

$$\overrightarrow{OD}=\begin{pmatrix}4\\4\\-2\end{pmatrix}+4\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}12\\0\\6\end{pmatrix}=6\begin{pmatrix}2\\0\\1\end{pmatrix}$$

$$\left|\overrightarrow{OE}\right|=\frac{\overrightarrow{OA}\cdot\overrightarrow{OD}}{\left|\overrightarrow{OD}\right|}\ \text{(length of projection of } \overrightarrow{OA} \text{ on } \overrightarrow{OD}\text{)}$$

$$=\frac{1}{6\sqrt{5}}\left[6\begin{pmatrix}4\\4\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}\right]$$

$$=\frac{6\sqrt{5}}{5}$$

Method 2

Area of $\Delta OAB = \frac{1}{2}(OB)(OA)=\frac{1}{2}(OE)(AB)$

$$\therefore\quad \left|\overrightarrow{OE}\right|=\frac{\left|\overrightarrow{OA}\right|\times\left|\overrightarrow{OB}\right|}{\left|\overrightarrow{AB}\right|}$$

$$=\frac{\sqrt{16+16+4}\times\sqrt{4+1+4}}{\sqrt{4+25+16}}=\frac{(6)(3)}{\sqrt{45}}=\frac{6\sqrt{5}}{5}$$

Method 3

$$\overrightarrow{OD}=\begin{pmatrix}4\\4\\-2\end{pmatrix}+4\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}12\\0\\6\end{pmatrix}=6\begin{pmatrix}2\\0\\1\end{pmatrix}$$

$$\left|\overrightarrow{OD}\right|=6\sqrt{5}.$$

Since $\triangle OBE$ is similar to $\triangle DAE$

$$\frac{OE}{DE}=\frac{OB}{DA}=\frac{1}{4}$$

$$OE=\frac{1}{5}OD=\frac{6}{\sqrt{5}}.$$

7 The function f is defined by

$$\mathrm{f} : x \mapsto \left|\frac{1-4x}{x+1}\right|,\ x \in \mathbb{R},\ x \geq k.$$

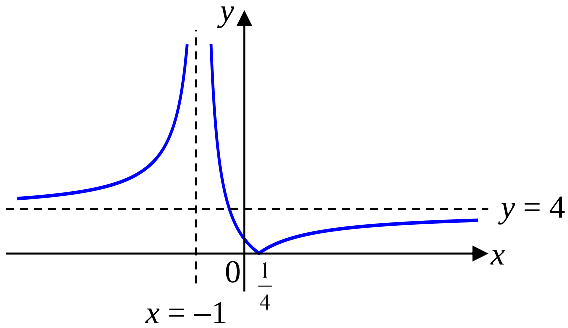
(i) With the aid of a graph, find the least value of k such that f has an inverse. [2]

(ii) Using the least value of k found in **(i)**,

(a) find $\mathrm{f}^{-1}(x)$ and state its domain, [3]

(b) find the exact solution(s) of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

Describe a sequence of two transformations which would transform the graph of $y = \mathrm{f}(x)$ onto the graph of $y = \left|\frac{2+4x}{2-x}\right|$. [2]

7i	Using GC,  The least value of k is $\frac{1}{4}$.
7iia	Let $y = -\left(\frac{1-4x}{x+1}\right) = \frac{4x-1}{x+1}$ $x = \frac{y+1}{4-y}$ $\mathrm{f}^{-1}(x) = \frac{x+1}{4-x}$ $\mathrm{D}_{\mathrm{f}^{-1}} = [0, 4)$
7iib	Since the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ intersect on the line $y = x$ $\mathrm{f}(x) = x$ $-\left(\frac{1-4x}{x+1}\right) = x$ $x^2 - 3x + 1 = 0$ $x = \frac{3 \pm \sqrt{5}}{2}$ $\left(\text{both values } \in \mathrm{D}_{\mathrm{f}} \text{ and } \mathrm{D}_{\mathrm{f}^{-1}}\right)$
	$y = \left|\frac{1-4x}{x+1}\right| \xrightarrow{x \to -x} y = \left|\frac{1+4x}{1-x}\right| \xrightarrow{x \to \frac{1}{2}x} y = \left|\frac{1+2x}{1-\frac{1}{2}x}\right| = \left|\frac{2+4x}{2-x}\right|$ The transformations (in either order) are - A reflection about the y-axis. - A scaling of factor 2 parallel to the x-axis.

8 **(i)** Use the substitution $x=\sin^2\theta$, where $0<\theta<\dfrac{\pi}{2}$ and $0<x<1$, to show that

$$\int\sqrt{\frac{x}{1-x}}\,\mathrm{d}x=\sin^{-1}\sqrt{x}-\sqrt{x(1-x)}+c \qquad \text{where } c \text{ is an arbitrary constant.} \qquad [5]$$

(ii) The region R is bounded by the curve $y=\left(\dfrac{x}{1-x}\right)^{\frac{1}{4}}$ and the lines $y = 4x - 1$ and $x=\dfrac{1}{4}$. Find the volume of revolution formed when R is rotated completely about the x-axis, giving your answer in exact form. [5]

8i

$$x=\sin^2\theta \;\Rightarrow\; \frac{\mathrm{d}x}{\mathrm{d}\theta}=2\sin\theta\cos\theta$$

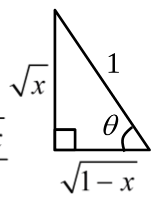
$$\begin{aligned}\int\sqrt{\frac{x}{1-x}}\mathrm{d}x &= \int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\left(2\sin\theta\cos\theta\right)\mathrm{d}\theta\\ &=\int 2\sin^2\theta\,\mathrm{d}\theta\\ &=\int(1-\cos 2\theta)\,\mathrm{d}\theta\\ &=\theta-\frac{1}{2}\sin 2\theta+c\\ &=\theta-\sin\theta\cos\theta+c\end{aligned}$$

For $0<\theta<\dfrac{\pi}{2}$:

$$x=\sin^2\theta \Rightarrow \sin\theta=\sqrt{x}$$
$$\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x}$$

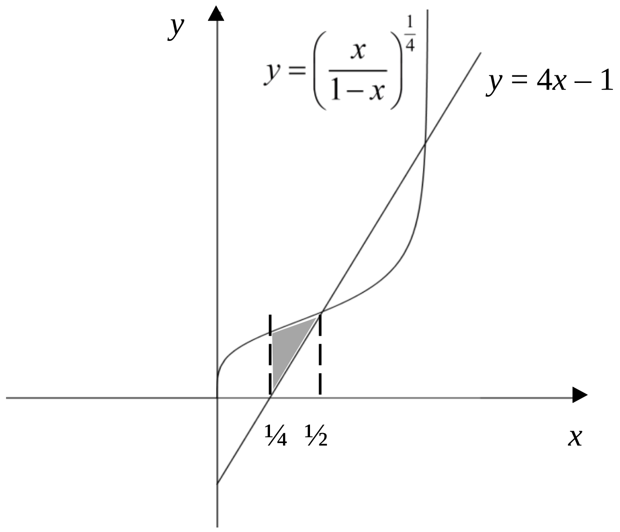
Alternative

$$\sin\theta=\frac{\sqrt{x}}{1}$$
$$\cos\theta=\frac{\sqrt{1-x}}{1}$$



$$\therefore\int\sqrt{\frac{x}{1-x}}\mathrm{d}x = \sin^{-1}\sqrt{x}-\sqrt{x(1-x)}+c \quad \text{(shown)}$$

8ii



From GC, $y=\left(\dfrac{x}{1-x}\right)^{\frac{1}{4}}$ and $y = 4x - 1$ intersect at $x=\dfrac{1}{2}$

	Required volume $= \pi\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\left(\frac{x}{1-x}\right)^{\frac{1}{4}}\right)^2 \mathrm{d}x - \frac{1}{3}\pi(1)^2\frac{1}{4}$ $= \pi\int_{\frac{1}{4}}^{\frac{1}{2}}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x - \frac{\pi}{12}$ $= \pi\left[\sin^{-1}\sqrt{x} - \sqrt{x(1-x)}\right]_{\frac{1}{4}}^{\frac{1}{2}} - \frac{\pi}{12}$ $= \pi\left[\left(\sin^{-1}\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2}(1-\frac{1}{2})}\right) - \left(\sin^{-1}\sqrt{\frac{1}{4}} - \sqrt{\frac{1}{4}(1-\frac{1}{4})}\right)\right] - \frac{\pi}{12}$ $= \pi\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right] - \frac{\pi}{12}$ $= \pi\left[\frac{\pi}{12} + \frac{3\sqrt{3}-7}{12}\right]$

9 Two planes p_1 and p_2 have equations $2x+z=2$ and $\mathbf{r}\cdot\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ respectively.

(i) Obtain a vector equation of the line of intersection, l, between p_1 and p_2. [2]

(ii) A third plane p_3 contains l and is perpendicular to p_1. Find a vector equation of p_3, in scalar product form. [3]

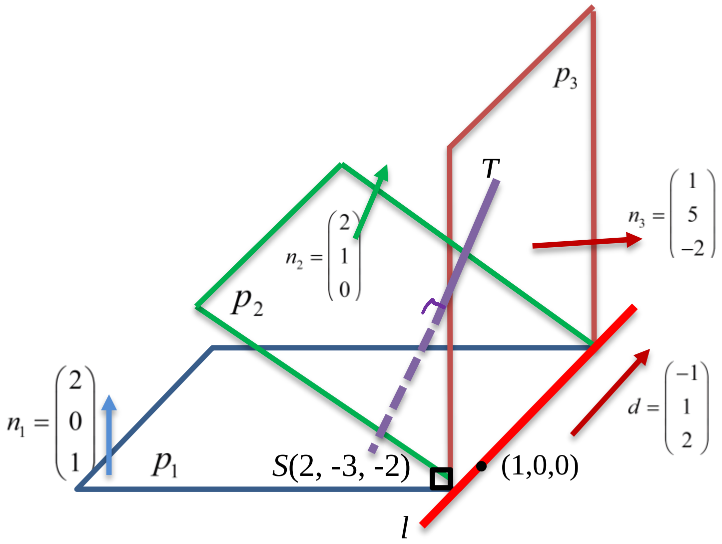
(iii) The point S lies on p_1 and the point T lies on p_3 such that the line ST is perpendicular to p_2. If the coordinates of S are $(2,-3,-2)$, find the coordinates of T. [4]

Find the acute angle between ST and p_1. [2]

9i	$p_1:\ 2x+z=2$ …..(1) $p_2:\ 2y-z=0$ …..(2) Using GC, solve (1) and (2) to get $x=1-\frac{1}{2}\lambda$ $y=\frac{1}{2}\lambda$ $z=\lambda$ $\therefore$ equation of l is $\mathbf{r}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\alpha\begin{pmatrix}-1\\1\\2\end{pmatrix},\ \alpha\in\mathbf{R}$ **<u>Alternative method</u>** $\begin{pmatrix}2\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}-2\\2\\4\end{pmatrix}=2\begin{pmatrix}-1\\1\\2\end{pmatrix}$. $\quad\therefore\begin{pmatrix}-1\\1\\2\end{pmatrix}$ is a direction vector of l. Let a common point of p_1 and p_2 be $(x, y, 0)$. Substitute $(x, y, 0)$ into eq (1) & (2): we have $x = 1$, $y = 0$. $\therefore(1, 0, 0)$ lies on l. $\therefore$ equation of l is $\mathbf{r}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\alpha\begin{pmatrix}-1\\1\\2\end{pmatrix},\ \alpha\in\mathbf{R}$

<table>
<tr><td>9ii</td><td>

Plane p_3 contains l and $\perp p_1$. S lies on p_1, T lies on p_3, $ST \perp p_2$.



Plane p_3 contains l $\Rightarrow$ $p_3 // \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and p_3 contains point $(1,0,0)$

Plane p_3 is perpendicular to p_1 $\Rightarrow$ $p_3 // \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$$\therefore\ \mathbf{n_3} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$$

Equation of plane p_3:

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \qquad \text{i.e. } \mathbf{r} \cdot \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} = 1$$

</td></tr>
<tr><td>9iii</td><td>

$$ST \perp p_2 \Rightarrow \overrightarrow{ST} // \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \Rightarrow \overrightarrow{ST} = k \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

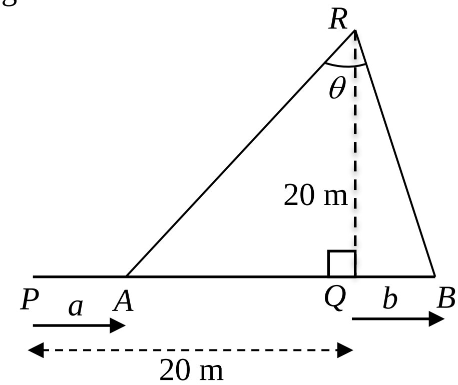
$$\Rightarrow \overrightarrow{OT} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} + k \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

Since point T lies on p_3, it satisfies the equation of plane p_3,

</td></tr>
</table>

	i.e. $\left[\begin{pmatrix}2\\-3\\-2\end{pmatrix}+k\begin{pmatrix}0\\2\\-1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\5\\-2\end{pmatrix}=1$ $\Rightarrow (2-15+4)+k(0+10+2)=1$ $\Rightarrow k=\frac{5}{6}$ $\therefore \overrightarrow{OT}=\begin{pmatrix}2\\-3\\-2\end{pmatrix}+\frac{5}{6}\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}2\\-4/3\\-17/6\end{pmatrix}$ Coordinates of T are $\left(2,-\frac{4}{3},-\frac{17}{6}\right)$.
9iv	$\overrightarrow{ST} // \begin{pmatrix}0\\2\\-1\end{pmatrix}$. Let the acute angle between ST and n_1 be θ. $\theta=\cos^{-1}\left\lvert\frac{\begin{pmatrix}0\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}}{\sqrt{4+1}\sqrt{4+1}}\right\rvert=\cos^{-1}\frac{1}{5}$ Hence, the acute angle between ST and p_1 $=90^\circ-\cos^{-1}\frac{1}{5}$ $=11.5^o$ **<u>Alternative method</u>** Acute angle between ST and $p_1=\sin^{-1}\left\lvert\frac{\begin{pmatrix}0\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}}{\sqrt{4+1}\sqrt{4+1}}\right\rvert=\sin^{-1}\frac{1}{5}=11.5^o$

10 P and Q are two points lying 20 m apart on a horizontal straight line. Two particles A and B are initially located at P and Q respectively. A begins to move towards Q and B begins to move away from Q. At time t s, the distance travelled by A and B are a m and b m respectively where $0 \le a < 20$. The fixed point R is located 20 m vertically above point Q such that angle $ARB = \theta$.



By considering θ as the sum of two acute angles, show that

$$\tan\theta = \frac{20(20-a+b)}{400-20b+ab}.$$ [3]

(a) On day 1, A and B move in such a way that the distance of B from Q is always twice the distance of A from P, that is, $b = 2a$. Find, using differentiation, the value of a when θ is maximum. [4]

[You do not need to show that θ is maximum.]

(b) On day 2, A and B resume their starting positions at P and Q, and move such that θ remains a constant.

(i) Show that $b = \dfrac{20a}{40-a}$. [2]

If A moves at a constant speed of 0.5 ms^{-1}, find the speed of B at $t = 30$. [3]

10	At time t, $AQ = 20 - a$ and $BQ = b$ $\tan\angle ARQ = \dfrac{20-a}{20}$ and $\tan\angle QRB = \dfrac{b}{20}$ $\tan\theta = \tan(\angle ARQ + \angle QRB)$ $= \dfrac{\tan ARQ + \tan QRB}{1 - \tan ARQ \tan QRB}$ $= \dfrac{\frac{20-a}{20} + \frac{b}{20}}{1 - \frac{20-a}{20}\left(\frac{b}{20}\right)}$ $= \dfrac{20(20-a+b)}{400-20b+ab}$ (shown)

10a	When $b = 2a$, $\tan\theta = \dfrac{20(20+a)}{400-40a+2a^2}$ Differentiate with respect to a, $\sec^2\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}a} = \dfrac{20[(400-40a+2a^2)-(-40+4a)(20+a)]}{(400-40a+2a^2)^2}$ $= \dfrac{20(1200-80a-2a^2)}{(400-40a+2a^2)^2}$ At stationary point, $\dfrac{\mathrm{d}\theta}{\mathrm{d}a} = \dfrac{20(1200-80a-2a^2)}{\sec^2\theta(400-4a+2a^2)^2} = 0$ $\cos^2\theta(1200-80a-2a^2) = 0$ Since $\cos\theta \neq 0$ ($\theta \neq \pi/2$) $a = 11.6$ (3 s.f.) or $a = -51.6$ (rejected since $a > 0$) $\therefore$ θ is maximum when $a = 11.6$
10bi	Since $\tan\theta = 1$ when $t = 0$, $\dfrac{20(20-a+b)}{400-20b+ab} = 1$ $400-20a+20b = 400-20b+ab$ $b = \dfrac{20a}{40-a}$
10bii	$b = \dfrac{20a}{40-a}$ $\dfrac{\mathrm{d}b}{\mathrm{d}t} = \dfrac{800}{(40-a)^2}\dfrac{\mathrm{d}a}{\mathrm{d}t}$ At $t = 30$, $a = 0.5 \times 30 = 15$, $\dfrac{\mathrm{d}b}{\mathrm{d}t} = \dfrac{800}{(40-15)^2}(0.5) \approx 0.64\ \mathrm{ms}^{-1}$

11 **(a)** By using small angle approximations, where x is small enough for x^3 and higher powers of x to be neglected, show that $\dfrac{\sin\left(2x-\dfrac{\pi}{4}\right)}{2-\sin x} \approx \sqrt{2}\left(-\dfrac{1}{4}+px+qx^2\right)$, where p and q are constants to be determined. [5]

(b) A curve has equation $y^2 - xy = 4 - \sin x$.

(i) Show that there is no tangent to the curve that is parallel to the y-axis. [4]

(ii) Given that $y = 2$ when $x = 0$, find the Maclaurin's series for y up to and including the term in x^2. [3]

11a	$\dfrac{\sin\left(2x-\dfrac{\pi}{4}\right)}{2-\sin x} = \dfrac{\dfrac{1}{\sqrt{2}}(\sin 2x - \cos 2x)}{2-\sin x}$ $\approx \dfrac{\dfrac{1}{\sqrt{2}}(2x-1+2x^2)}{2-x}$ since x is small $= \dfrac{1}{\sqrt{2}}(2x^2+2x-1)(2-x)^{-1}$ $= \dfrac{1}{\sqrt{2}}(2x^2+2x-1)\ 2^{-1}(1-\dfrac{x}{2})^{-1}$ $= \dfrac{1}{2\sqrt{2}}(2x^2+2x-1)\ \left(1+\dfrac{x}{2}+\left(\dfrac{x}{2}\right)^2+...\right)$ $= \dfrac{\sqrt{2}}{4}\left(2x^2+2x-1+x^2-\dfrac{x}{2}-\left(\dfrac{x}{2}\right)^2+....\right)$ $= \sqrt{2}\left(-\dfrac{1}{4}+\dfrac{3x}{8}+\dfrac{11}{16}x^2+....\right)$
11bi	$y^2 - xy = 4 - \sin x$ ----- (1) Differentiating wrt x $2y\dfrac{dy}{dx} - \left(x\dfrac{dy}{dx}+y\right) = -\cos x$ $(2y-x)\dfrac{dy}{dx} = y - \cos x$ ----- (2) $\dfrac{dy}{dx} = \dfrac{y-\cos x}{(2y-x)}$ Tangent parallel to y-axis $\Rightarrow x - 2y = 0 \Rightarrow 2y = x$ ----- (3) Substitute (3) into (1): $y^2 - 2y^2 = 4 - \sin(2y)$ $\sin(2y) = 4 + y^2 \geq 4$ (no real solution) $\therefore$ there is no tangent to the curve that is parallel to the y-axis.

11bii	Differentiating (2) wrt x, $(2y-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\left(2\frac{\mathrm{d}y}{\mathrm{d}x}-1\right)=\frac{\mathrm{d}y}{\mathrm{d}x}+\sin x$ $(2y-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-2\frac{\mathrm{d}y}{\mathrm{d}x}=\sin x$ ---- (2) When $x = 0$, $y = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{4}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{3}{32}$ $y=2+\frac{1}{4}x+\frac{3}{64}x^2+\ldots$

12 A curve C has parametric equations

$$x = 7 - 4\sin^2 t\,,\ \ y = 4 + 3\sin^3 t$$

where $-\frac{\pi}{2} < t \le \frac{\pi}{2}$.

(i) Show that the equation of the tangent to the curve at the point with parameter t is

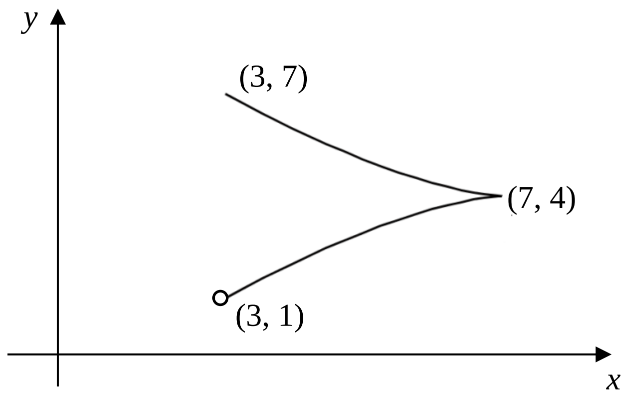
$$8y + 9x\sin t - 63\sin t + 12\sin^3 t - 32 = 0\,.$$

This tangent passes through a fixed point (X, Y). Give a brief argument to explain why there cannot be more than 3 tangents passing through (X, Y). [5]

(ii) Sketch the curve C. [2]

(iii) Show that the coordinates of the points of intersection between C and the line $8y + 9x - 83 = 0$ are (3, 7) and $\left(6,\ \frac{29}{8}\right)$. [3]

(iv) Find the area of the region bounded by C and the line $8y + 9x - 83 = 0$. [3]

12i	$\frac{dx}{dt} = -8\sin t\cos t \qquad \frac{dy}{dt} = 9\sin^2 t\cos t$ $\frac{dy}{dx} = -\frac{9}{8}\sin t$ Equation of tangent is: $y - 4 - 3\sin^3 t = -\frac{9\sin t}{8}\left(x - 7 + 4\sin^2 t\right)$ $8y - 32 - 24\sin^3 t = -9x\sin t + 63\sin t - 36\sin^3 t$ $8y + 9x\sin t - 63\sin t + 12\sin^3 t - 32 = 0$ (shown) Substitute (X, Y), we have $12\sin^3 t + (9X - 63)\sin t + 8Y - 32 = 0$. This is a cubic equation in terms of $\sin t$ and so there will at most be 3 distinct values of $\sin t$ that satisfies the cubic equation. Suppose $\sin t = m$ satisfies the cubic equation, there can only be one value of t that makes $\sin t = m$ since $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Hence there cannot be more than 3 tangents passing through (X, Y).
12ii	 End points at (3, 7) and (3, 1) (excluded) Symmetrical about $y = 7$ and zero gradient at (7, 4)

12iii

At intersection between C and $8y+9x-83=0$,

$$8\left(4+3\sin^3 t\right)+9\left(7-4\sin^2 t\right)-83=0$$

$$2\sin^3 t-3\sin^2 t+1=0$$

From GC,

$$\sin t=1 \text{ (repeated) or } \sin t=-\frac{1}{2}$$

When $\sin t=1$, $x = 7 - 4 = 3$; $y = 4 + 3 = 7$

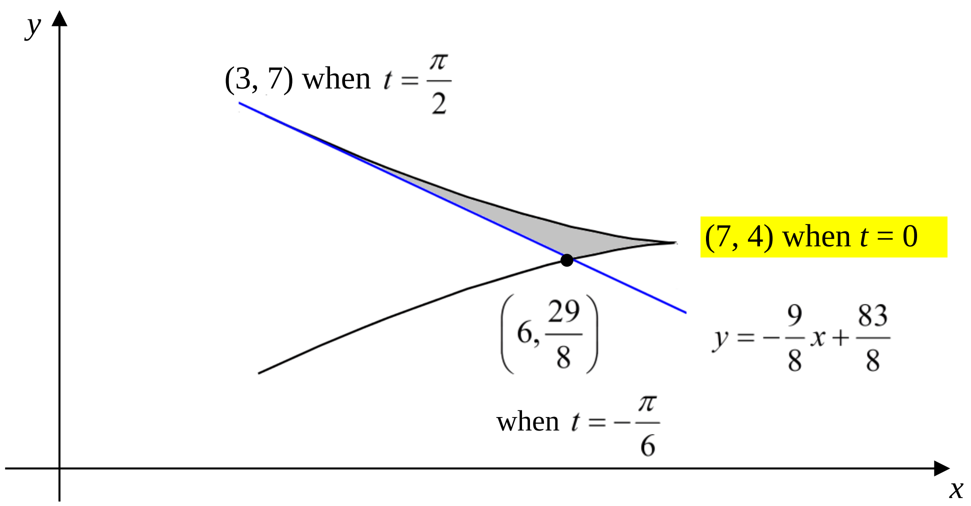
When $\sin t=-\frac{1}{2}$, $x = 7-4\left(-\frac{1}{2}\right)^2=6$; $y = 4+3\left(-\frac{1}{2}\right)^3=\frac{29}{8}$

<u>OR</u> $t=\frac{\pi}{2}$ or $t=-\frac{\pi}{6}$

When $t=\frac{\pi}{2}$, $x=7-4\left(\sin\frac{\pi}{2}\right)^2=3$ and $y = 7$

When $t=-\frac{\pi}{6}$, $x=7-4\left(\sin\left(-\frac{\pi}{6}\right)\right)^2=6$ and $y=\frac{29}{8}$

12iv



Area required

$$=\int_{\frac{29}{8}}^{7} x_{\text{curve } C}\,\mathrm{d}y-\int_{\frac{29}{8}}^{7} x_{\text{line}}\,\mathrm{d}y$$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}}\left(7-4\sin^2 t\right)\left(9\sin^2 t\cos t\right)\mathrm{d}t-\frac{1}{2}\left(\frac{27}{8}\right)(3+6)$$

= 1.0125 unit^2

<u>**Method 2**</u>

Area required

$$=\int_3^7 y_{\text{upper}}\,\mathrm{d}x-\int_6^7 y_{\text{lower}}\,\mathrm{d}x-\frac{1}{2}(3)(7+\frac{29}{8})$$

$$=\int_{\frac{\pi}{2}}^{0}\left(4+3\sin^3 t\right)\left(-8\sin t\cos t\right)\mathrm{d}t\;-\int_{-\frac{\pi}{6}}^{0}\left(4+3\sin^3 t\right)\left(-8\sin t\cos t\right)\mathrm{d}t-\frac{255}{16}$$

= 1.0125 unit^2