(Pure Mathematics) Chapter 1: Equations

Objectives

At the end of the chapter, you should be able to:

(a) solve quadratic equations using factorisation, completing the square, formula, sketching of graph and a GC;

(b) understand and use the conditions for a quadratic equation to have (i) two real and distinct roots, (ii) two real and equal roots, (iii) no real roots;

(c) understand and use the conditions for a quadratic equation to be always positive (or always negative);

(d) formulate a quadratic equation from a problem situation and interpret the solution in the context of the problem;

(e) solve a pair of simultaneous equations, one linear and one quadratic, by substitution;

(f) formulate a system of linear equations from a problem situation;

(g) find the solution of a system of linear equations using a Graphing Calculator.

Content

References

1. New Additional Mathematics, Ho Soo Thong (Msc, Dip Ed), Khor Nyak Hiong (Bsc, Dip Ed)

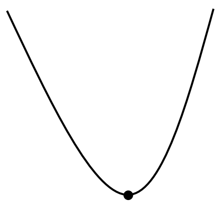
2. New Syllabus Additional Mathematics (7^{th} Edition), Shinglee Publishers Ptd Ltd

1.1 Quadratic Functions

The expression $f(x) = ax^2 + bx + c$, where $a \neq 0$, is called a quadratic function.

When the graph of the function $y = ax^2 + bx + c$ is drawn, two types of graphs are obtained, depending on the value of *a*.

$a > 0$

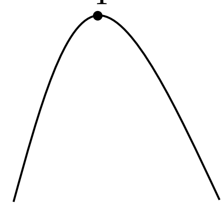


X

U shape

$a < 0$

Y



Inverted – U shape

Note:

1. *a* is the coefficient of x^2
   b is the coefficient of x
   c is the coefficient of x^0 (also known as the constant term or the term independent of *x*)

2. If $a > 0$, the curve has a minimum point at X.
   For example, in $y = 2x^2 - 7x + 4$, $a = 2$, $b = -7$, $c = 4$. Since $a = 2 > 0$, the curve is U shape and has a minimum point.

   If $a < 0$, the curve has a maximum point at Y.
   For example, in $y = -x^2 - 6x + 9$, $a = -1$, $b = -6$, $c = 9$. Since $a = -1 < 0$, the curve is Inverted – U shape and has a maximum point.

3. The shape of a quadratic function is symmetrical about its minimum or maximum point.

1.1.1 Solving Quadratic Equations

The general form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

There are five methods of solving a quadratic equation, namely:

(a) Factorisation
(b) Completing the Square
(c) Formula
(d) Graphical Method
(e) GC

(a) Factorisation

Method
1. Bring all the terms to one side of the equation such that the other side is 0.
2. Use Cross Factorisation Method to factorize the equation.

Example 1
Solve the quadratic equation $3x^2 + x - 2 = 0$ using factorization.

Solution:

$3x^2 + x - 2 = 0$

$(3x-2)(x+1) = 0$

$3x - 2 = 0$ or $x + 1 = 0$

$x = \frac{2}{3}$ or $x = -1$

True or False?

$(3x-2)(x+1) = 2$

$\Rightarrow 3x - 2 = 2$ or $x + 1 = 2$

Ans: False

Note:
1. The solution of the quadratic equation, i.e. $x = \frac{2}{3}$ or $x = -1$ are also called the **roots** of the quadratic equation.

2. $(3x-2)$ and $(x+1)$ are called **factors** of the quadratic expression $3x^2 + x - 2$.

(b) Completing the Square

A quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ can be expressed to the form $a(x-p)^2 + q = 0$ by completing the square.

Note:
The coefficient of x^2 must be 1 before completing the square method can be carried out.

Example 2
Solve the quadratic equation $2x^2 + 8x + 1 = 0$ by completing the square.

Solution:

$2x^2 + 8x + 1 = 0$ $2(x^2 + 4x) + 1 = 0$ $2[x^2 + 4x + 2^2 - 2^2] + 1 = 0$ $2[(x+2)^2 - 2^2] + 1 = 0$ $2(x+2)^2 = 7$	$x + 2 = \pm\sqrt{\frac{7}{2}}$ $x = -2 \pm \sqrt{\frac{7}{2}}$

(c) Formula

The general formula of solving a quadratic equation of the form $ax^2+bx+c=0$, $a \neq 0$ is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Example 3
Solve the quadratic equation $x^2+7x-3=0$ using formula.

Solution:

$x^2+7x-3=0$

$$\begin{aligned} x &= \frac{-7\pm\sqrt{7^2-4(1)(-3)}}{2} \\ &= \frac{-7\pm\sqrt{61}}{2} \\ &= \frac{-7+\sqrt{61}}{2} \text{ or } \frac{-7-\sqrt{61}}{2} \end{aligned}$$

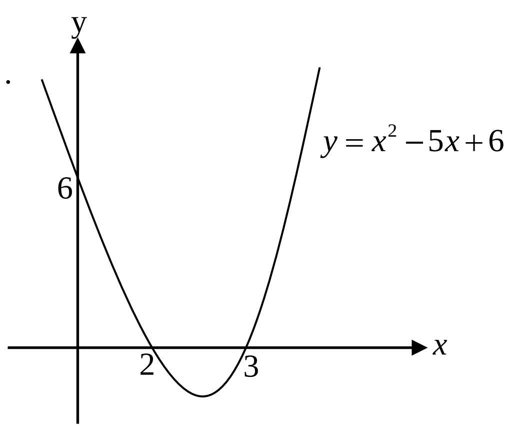
Note:
The expression b^2-4ac in the general formula is known as the **discriminant** of the quadratic equation as it determines the nature (type) of roots that a quadratic equation has.

(d) Graphical Method

Example 4
Solve the quadratic equation $x^2-5x+6=0$ using graphical method.

Solution:

From the graph, $x=2$ or $x=3$.



Why are the x-intercepts the solution for $x^2-5x+6=0$?

Ans: The x-intercepts are the points of intersection between the line $y=0$ and the curve $y=x^2-5x+6$.

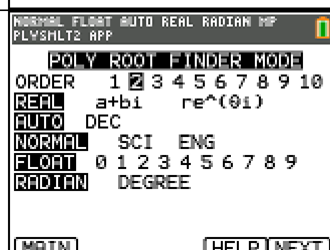
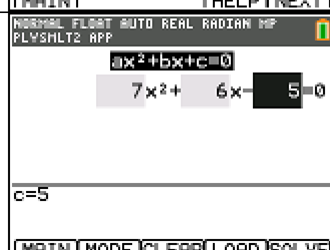
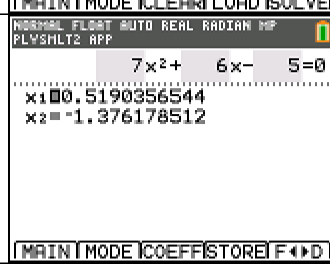
Note:
The x – intercepts of the curve $y=x^2-5x+6$ are the solutions of the quadratic equation $x^2-5x+6=0$.

(e) GC

Example 5

Solve the quadratic equation $7x^2+6x-5=0$ using a calculator.

Solution:

Steps	Screenshot	Remarks
Press  Select PlySmlt2		
Press [1] to select POLYNOMIAL ROOT FINDER		
Select 2 in the ORDER option Press [GRAPH] (NEXT) to go to the next screen		ORDER prefers to the order of the polynomial that you are solving. (i.e. highest degree of the polynomial)
Enter the coefficients of the quadratic equation (i.e. a, b and c) and select " + " or " – " according to the equation that you are solving.		
Press [GRAPH] (SOLVE) to solve the equation Note: Press F◀ ▶D will sometimes convert decimal to fraction		The solutions are given by x_1 and x_2.

From GC, $x=0.519$ or $x=-1.38$

Note:
GC may not give exact values. If the question requires exact answer, do not use GC but you can always use GC to check your answer.

Exercise 1

1. Solve the following equations using the stated method in the bracket.
   (a) $2x^2-11x=-12$ (factorisation)
   (b) $3x^2-8x+1=0$ (completing the square)
   (c) $-x^2-6x+1=0$ (formula)
   (d) $2x^2-13x+14=0$ (GC)

Solution:

(a) $2x^2-11x=-12$ $2x^2-11x+12=0$ $(2x-3)(x-4)=0$ $2x-3=0$ or $x-4=0$ $x=\frac{3}{2}$ or $x=4$	(b) $3x^2-8x+1=0$ $3\left[x^2-\frac{8}{3}x\right]+1=0$ $3\left[x^2-\frac{8}{3}x+\left(-\frac{4}{3}\right)^2-\left(-\frac{4}{3}\right)^2\right]+1=0$ $3\left(x-\frac{4}{3}\right)^2-\frac{16}{3}+1=0$ $\left(x-\frac{4}{3}\right)^2=\frac{13}{9}$ $x-\frac{4}{3}=\pm\sqrt{\frac{13}{9}}$ $x=\frac{4\pm\sqrt{13}}{3}$
(c) $-x^2-6x+1=0$ $x=\frac{6\pm\sqrt{(-6)^2-4(-1)(1)}}{2(-1)}$ $x=\frac{6\pm\sqrt{40}}{-2}$ $x=\frac{6\pm2\sqrt{10}}{-2}$ $x=-3-\sqrt{10}$ or $x=-3+\sqrt{10}$	(d) Using GC, $x=5.14$ or $x=1.36$

Answer

1	(a) $x=1\frac{1}{2}$ or $x=4$, (b) $x=\frac{4\pm\sqrt{13}}{3}$ (c) $x=-(3\pm\sqrt{10})$ (d) $x=5.14$ or $x=1.36$

1.1.2 Nature of Roots of a Quadratic Equation

Recall that the solutions of a quadratic equation $ax^2+bx+c=0$, $a \neq 0$ are given by

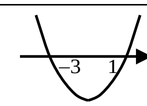
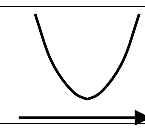
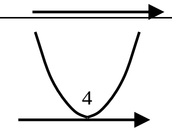
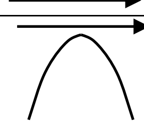
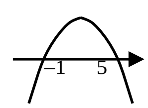
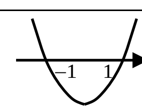
$$x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

The expression b^2-4ac in the general formula is known as the **discriminant** of the equation as it determines the nature (type) of roots that a quadratic equation has.

Learning Experience

Purpose: To understand the connection between the discriminant and the nature of roots of an equation and to develop reasoning skills.

(1) Complete the table below.

	Quadratic equation	Discriminant b^2-4ac	Quadratic formula to determine the roots $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Roots	Nature of roots	Sketch of curve
(i)	$x^2+2x-3=0$	16	$x=\frac{-2\pm\sqrt{(2)^2-4(1)(-3)}}{2(1)}$	−3, 1	2 real and distinct roots	
(ii)	$x^2-3x+5=0$	−11	$x=\frac{3\pm\sqrt{(-3)^2-4(1)(5)}}{2(1)}$	NA	No real roots	
(iii)	$x^2-8x+16=0$	0	$x=\frac{8\pm\sqrt{(-8)^2-4(1)(16)}}{2(1)}$	4	2 real and equal roots	
(iv)	$-2x^2+7x-9=0$	−23	$x=\frac{-7\pm\sqrt{(7)^2-4(-2)(-9)}}{2(-2)}$	NA	No real roots	
(v)	$10+8x-2x^2=0$	144	$x=\frac{-8\pm\sqrt{(8)^2-4(-2)(10)}}{2(-2)}$	−1, 5	2 real and distinct roots	
(vi)	$x^2-1=0$	4	$x=\frac{-0\pm\sqrt{(0)^2-4(1)(-1)}}{2(1)}$	−1, 1	2 real and distinct roots	

(2) From the attempt to fill in the table above, what can you say about the discriminant and nature of roots? Write down a mathematical statement describing the relationship.

$b^2 - 4ac > 0 \Leftrightarrow$ Two real and distinct roots

$b^2 - 4ac < 0 \Leftrightarrow$ No real roots

$b^2 - 4ac = 0 \Leftrightarrow$ Two real and equal (repeated/coincident) roots

$b^2 - 4ac \geq 0 \Leftrightarrow$ Two real roots

<u>Case 1</u>

When $b^2 - 4ac > 0$, there are two values for $\pm\sqrt{b^2 - 4ac}$. Therefore the solutions are $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. Thus the equation has two real and distinct (different) roots.

Graphically, the curve $y = ax^2 + bx + c$ cuts the x-axis at two different points.

Consider the following equations,

<table>
<tr>
<td>

(i) $x^2 - 5x + 6 = 0$

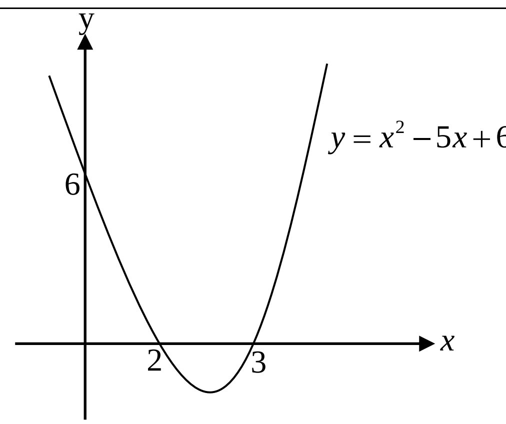
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{1}}{2}$$

$$= \frac{5+1}{2} \text{ or } \frac{5-1}{2}$$

$$= 3 \text{ or } 2$$

</td>
<td>



</td>
</tr>
<tr>
<td>

(ii) $-x^2 + 5x + 3 = 0$

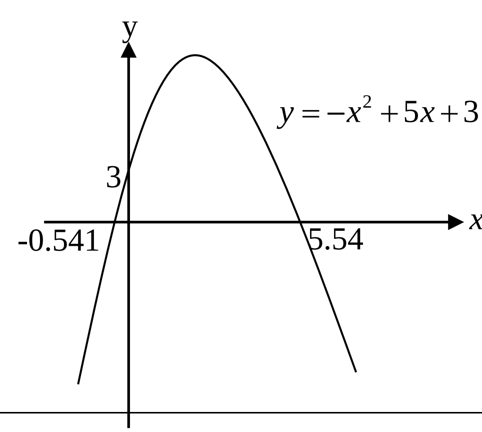
$$x = \frac{-(5) \pm \sqrt{(5)^2 - 4(-1)(3)}}{2(-1)}$$

$$= \frac{-5 \pm \sqrt{37}}{-2}$$

$$= \frac{-5 + \sqrt{37}}{-2} \text{ or } \frac{-5 - \sqrt{37}}{-2}$$

$$= -0.541 \text{ or } 5.54$$

</td>
<td>



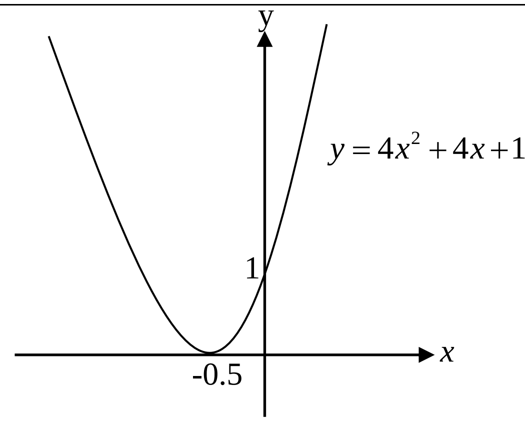
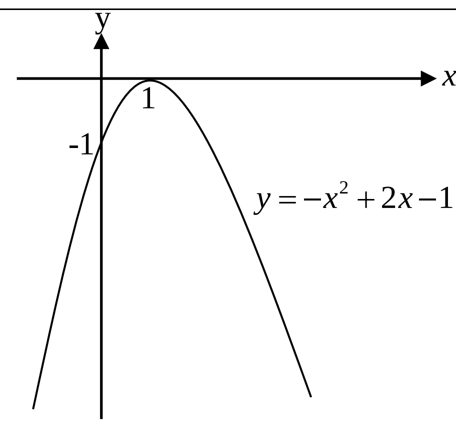
</td>
</tr>
</table>

Case 2

When $b^2-4ac=0$, $\sqrt{b^2-4ac}=0$. Therefore the solutions are $x=\frac{-b+0}{2a}$ and $x=\frac{-b-0}{2a}$. Thus the equation has two real and equal roots.

Graphically, the curve $y=ax^2+bx+c$ touches the x-axis at $x=-\frac{b}{2a}$.
i.e. the x-axis is a tangent to the curve.

Consider the following equations,

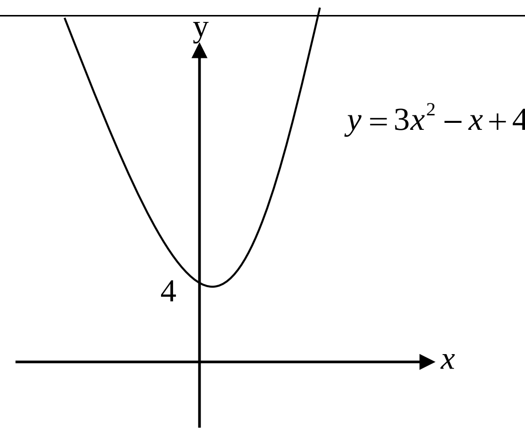
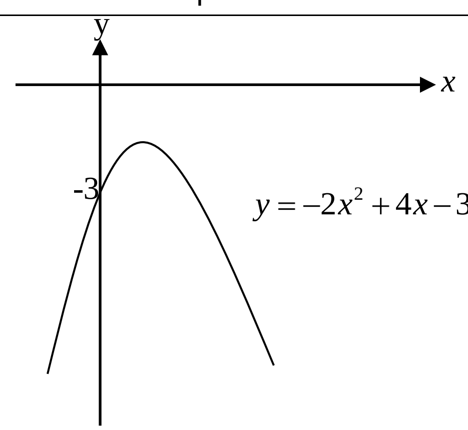
(i) $4x^2+4x+1=0$ $x=\frac{-(4)\pm\sqrt{(4)^2-4(4)(1)}}{2(4)}$ $=\frac{-4\pm\sqrt{0}}{8}$ $=\frac{-4+0}{8}$ or $\frac{-4-0}{8}$ $=-\frac{1}{2}$	
(ii) $-x^2+2x-1=0$ $x=\frac{-(2)\pm\sqrt{(2)^2-4(-1)(-1)}}{2(-1)}$ $=\frac{-2\pm\sqrt{0}}{-2}$ $=\frac{-2+0}{-2}$ or $\frac{-2-0}{-2}$ $=1$	

Case 3

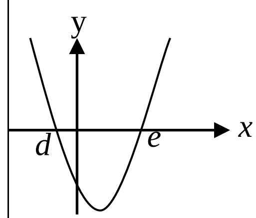
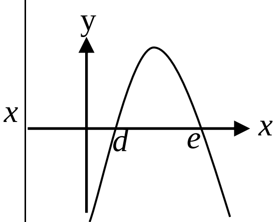
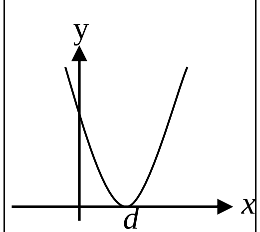
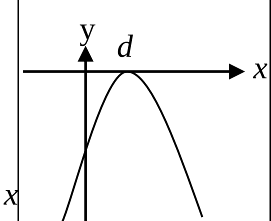
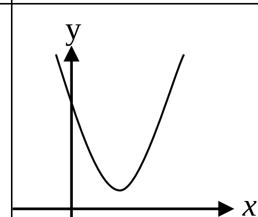
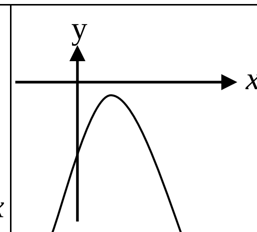
When $b^2 - 4ac < 0$, $\sqrt{b^2 - 4ac}$ has no real solution. There is no real value for the square root of a negative number. Thus the equation has two complex or imaginary roots. i.e. The equation has no real roots.

Graphically, the curve $y = ax^2 + bx + c$ does not cut or touch the *x*-axis. It lies entirely above or below the *x*-axis.

Consider the following equations,

(i) $3x^2 - x + 4 = 0$ $x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(4)}}{2(3)}$ $= \dfrac{1 \pm \sqrt{-47}}{6}$ $= \dfrac{1+\sqrt{-47}}{6}$ or $\dfrac{1-\sqrt{-47}}{6}$ There are no real solutions.	
(ii) $-2x^2 + 4x - 3 = 0$ $x = \dfrac{-(4) \pm \sqrt{(4)^2 - 4(-2)(-3)}}{2(-2)}$ $= \dfrac{-4 \pm \sqrt{-8}}{-4}$ $= \dfrac{-4+\sqrt{-8}}{-4}$ or $\dfrac{-4-\sqrt{-8}}{-4}$ There are no real solutions.	

The above 3 cases are summarised in the table below:

Case	Discriminant	Nature of Roots of $ax^2+bx+c=0$	Graph of $y=ax^2+bx+c$: coefficient of x^2, $a>0$	Graph of $y=ax^2+bx+c$: coefficient of x^2, $a<0$	Solutions
1.	Discriminant >0	2 real and distinct roots			$x=d$ or $x=e$
			The curve cuts the *x*-axis at 2 distinct points.		
2.	Discriminant $=0$	2 real and equal roots. (repeated / coincident roots)			$x=d$
			The curve touches the *x*-axis at 1 point. The *x*-axis is tangent to the curve.		
3.	Discriminant <0	2 imaginary or complex roots. No real roots.			No real solutions
			The curve lies entirely above the *x*- axis.	The curve lies entirely below the *x*- axis.	
			The curve does not cut or touch the *x*- axis.		

Note:
From case 1 and case 2, we can further summarise them into

Discriminant ≥ 0 $\Leftrightarrow$ Discriminant >0 or Discriminant $=0$
$\Leftrightarrow$ roots are real

Example 8
Describe the nature of the roots for each of the following equations.
(a) $x^2+6x+9=0$ (b) $x^2-3x+7=0$ (c) $x^2-22x+57=0$

Solution:

(a) $x^2+6x+9=0$
Discriminant
$=(6)^2-4(1)(9)$
$=0$
$\Rightarrow$ 2 real and equal roots

(b) $x^2-3x+7=0$
Discriminant
$=(-3)^2-4(1)(7)$
$=-19<0$
$\Rightarrow$ no real roots

(c) $x^2-22x+57=0$
Discriminant
$=(-22)^2-4(1)(57)$
$=256>0$
$\Rightarrow$ 2 real and distinct roots

Example 9
Find the possible values of k for which the quadratic equation $2x^2-4x=3k-1$ where k is a constant has
(i) two real and distinct roots,
(ii) two real and equal roots,
(iii) real roots,
(iv) no real roots.

Solution:

$2x^2-4x=3k-1$

$2x^2-4x-(3k-1)=0$

Discriminant $=(-4)^2-4(2)[-(3k-1)]$
$=16+24k-8$
$=24k+8$

(i) Since the quadratic equation has two real and distinct roots,
Discriminant >0
$24k+8>0$
$k>-\dfrac{8}{24}$
$k>-\dfrac{1}{3}$

(ii) Since the quadratic equation has two real and equal roots,
Discriminant = 0
$24k+8=0$
$k=-\dfrac{1}{3}$

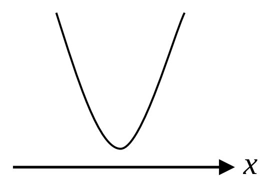
(iii) Since the quadratic equation has real roots,
Discriminant ≥ 0
$24k+8\geq 0$
$k\geq -\frac{1}{3}$

(iv) Since the quadratic equation has no real roots,
Discriminant < 0
$24k+8<0$
$k<-\frac{1}{3}$

1.1.3 Conditions for Quadratic Equations to be always positive (or always negative)

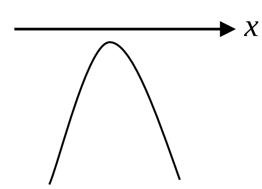
(i) When ax^2+bx+c is always positive (i.e. $ax^2+bx+c>0$ for all real values of x), the graph of $y = ax^2 + bx + c$ lies entirely above the x–axis. The equation $ax^2+bx+c=0$ has no real roots.

Conditions: Coefficient of x^2, $a>0$ and Discriminant $=b^2-4ac<0$.



(ii) When ax^2+bx+c is always negative (i.e. $ax^2+bx+c<0$ for all real values of x), the graph of $y = ax^2 + bx + c$ lies entirely below the x–axis. The equation $ax^2+bx+c=0$ has no real roots.

Conditions: Coefficient of x^2, $a<0$ and Discriminant $= b^2-4ac<0$.



In summary,

The function $y = ax^2 + bx + c$ is always positive	$\Leftrightarrow$	Coefficient of $x^2 > 0$ and Discriminant < 0.
The function $y = ax^2 + bx + c$ is always negative	$\Leftrightarrow$	Coefficient of $x^2 < 0$ and Discriminant < 0.

This gives an alternative method to show that a given quadratic expression is always positive or always negative besides completing the square.

Example 10
Show that the expression $2x^2+6x+12$ is always positive for all $x\in\mathbb{R}$.

Solution:
Method 1: By using Discriminant

Since the coefficient of $x^2 = 2 > 0$ and the discriminant $=6^2-4(2)(12)=-60<0$, the expression $2x^2+6x+12$ is always positive for all $x\in\mathbb{R}$.

Method 2: By completing the square

$$\begin{aligned}2x^2+6x+12 &= 2\left(x^2+3x\right)+12\\ &= 2\left[\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]+12\\ &= 2\left(x+\frac{3}{2}\right)^2-\frac{9}{2}+12\\ &= 2\left(x+\frac{3}{2}\right)^2+\frac{15}{2}\end{aligned}$$

Since $\left(x+\frac{3}{2}\right)^2\geq 0,\ 2\left(x+\frac{3}{2}\right)^2+\frac{15}{2}\geq\frac{15}{2}>0.$

Therefore, $2x^2+6x+12$ is always positive for all $x\in\mathbb{R}$.

Example 11
Show that the roots of the equation $x^2+x=(1+x)p$ are real for all real values of *p*.

Solution:
Aim: to show roots are real $\Rightarrow$ to show discriminant ≥ 0

$x^2+x=(1+x)p$

$x^2-p-px+x=0$

$x^2+(1-p)x-p=0$

$$\begin{aligned}\text{Discriminant} &= (1-p)^2-4(1)(-p)\\ &= 1-2p+p^2+4p\\ &= p^2+2p+1\\ &= (1+p)^2\geq 0 \text{ for all real values of } p\end{aligned}$$

Therefore, the equation has real roots for all real values of *p*. (shown)

What does this phrase "real values of *p*" mean?
Ans: *p* is a real value

Example 12

Determine the range of values of k for which the function

(a) x^2+5x+k is always positive for all real values of x.

(b) kx^2+5x+1 is always negative for all real values of x.

Solution:

(a) x^2+5x+k is always positive, i.e. $x^2+5x+k>0$,

Discriminant <0 and coefficient of $x^2=1>0$

$5^2-4(1)(k)<0$

$4k>25$

$k>\frac{25}{4}$

$k>6\frac{1}{4}$

(b) kx^2+5x+1 is always negative, i.e. $kx^2+5x+1<0$,

Discriminant <0 and Coefficient of $x^2=k<0\cdots(2)$

$(5)^2-4(k)(1)<0$

$25-4k<0$

$k>\frac{25}{4}$

$k>6\frac{1}{4}\cdots(1)$

Combining (1) and (2), there is no solution.

Exercise 2

1. Describe the nature of the roots for each of the following equations.
   (a) $3x^2+5x+4=0$
   (b) $3-4x-x^2=0$
   (c) $2x^2-9x+16=x(x-1)$
   (d) $\frac{1}{x}-\frac{1}{x+1}=\frac{1}{5}$

Solution:

(a) Discriminant $=(5)^2-4(3)(4)=-23<0$ (no real roots)

(b) Discriminant $=(-4)^2-4(-1)(3)=28>0$. (2 real and distinct roots)

(c) $x^2-8x+16=0$
Discriminant $=(-8)^2-4(1)(16)=0$. (2 real and equal roots)

(d) $5(x+1-x)=x(x+1)$
$x^2+x-5=0$
Discriminant $=(1)^2-4(1)(-5)=21>0$ (2 real and distinct roots)

2. Find the values of k for which the equation $2x^2-kx+8=0$ has coincident roots.

Solution:
Coincident roots means "Repeated Real Roots".
Discriminant $=0$

$k^2-4(2)(8)=0$

$k=\pm 8$

3. The roots of the equation $ax^2-4ax+4a-5=0$ are real and distinct, find the range of values of a.

Solution:
Since the equation has distinct real roots,
Discriminant > 0

$(-4a)^2-4(a)(4a-5)>0$

$16a^2-16a^2+20a>0$

$20a>0$

$a>0$

4. Find the range of values of k for which the equation $x^2+(1-k)x=k$ has real roots.

Solution:

$x^2+(1-k)x-k=0$

Real roots $\Rightarrow$ Discriminant ≥ 0

$(1-k)^2-4(1)(-k)\geq 0$

$1-2k+k^2+4k\geq 0$

$k^2+2k+1\geq 0$

$(k+1)^2\geq 0$ for all real values of x

$k\in\mathbb{R}$

5. Show that the equation $3x^2+kx+k-5=0$ has real and distinct roots for all real values of k.

Solution:

Discriminant $=k^2-4(3)(k-5)$

$=k^2-12k+60$

$=(k-6)^2+24$

Since $(k-6)^2\geq 0$, $(k-6)^2+24\geq 24>0$ for all real values of k.

$\therefore$ Discriminant >0 for all real values of k.

Hence the equation has real and distinct roots. (shown)

6. Find the range of possible values of the constant k if the quadratic function kx^2-2x+7 is always (i) positive, (ii) negative.

If the graph $y=kx^2-2x+7$ just touches the x-axis, write down the value of k.

Solution:

(i) Since kx^2-2x+7 is always positive,

Discriminant < 0 and Coefficient of x^2, $k>0$ … (1)

$(-2)^2-4(k)(7)<0$

$4-28k<0$

$k>\frac{1}{7}$ ... (2)

Combining (1) and (2) gives $k>\frac{1}{7}$

(ii) Since kx^2-2x+7 is always negative, Discriminant < 0 and Coefficient of x^2, $k < 0$ … (1)

$(-2)^2 - 4(k)(7) < 0$

$4 - 28k < 0$

$k > \frac{1}{7}$ ... (2)

Combining (1) and (2) gives no solution.

If the graph $y = kx^2 - 2x + 7$ just touches the *x*-axis, discriminant = 0, therefore $k = \frac{1}{7}$.

Answer:

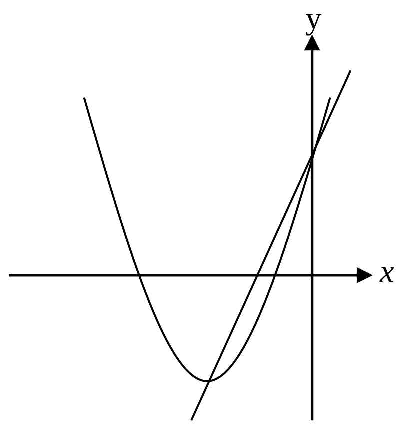
1.	(a) No real roots (b) 2 real and distinct roots (c) 2 real and equal roots (d) 2 real and distinct roots	2.	$k = \pm 8$
3.	$a > 0$	4.	$k \in \mathbb{R}$
5.	-	6.	(i) $k > \frac{1}{7}$, (ii) no solution; $k = \frac{1}{7}$

1.1.4 Intersection Problems Leading to Quadratic Equations

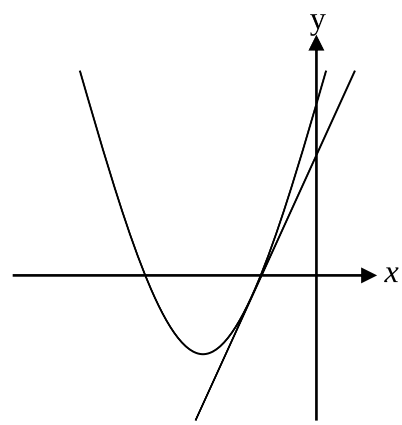
Consider the following two equations:

$$y = 2x + 1$$
$$y = x^2 + 4x + k$$

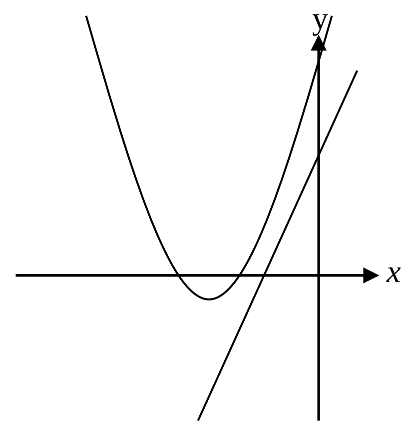
The following diagrams show the three possible cases with regard to their intersection:



Case 1



Case 2



Case 3

To find out the values of *k* which will correspond to the cases, we can solve the equations simultaneously:

$$y = 2x + 1 \cdots (1)$$
$$y = x^2 + 4x + k \cdots (2)$$

(1) = (2), $x^2 + 4x + k = 2x + 1$

$$x^2 + 2x + (k-1) = 0 \cdots (3)$$

The discriminant of the quadratic equation (3) will determine the nature of roots, and thus will indicate whether the line intersects the curve or not.

Case	Discriminant	Nature of Roots	Types of Intersection
1.	Discriminant > 0	2 real and distinct roots	The line intersects the curve at 2 distinct points.
2.	Discriminant = 0	2 real and equal (repeated / coincident) roots.	The line is tangent to the curve. The line touches the curve at 1 point.
3.	Discriminant < 0	2 imaginary or complex roots. No real roots.	The line does not intersect the curve.

Method

1. Combine the equation of the line and the curve into 1 equation in terms of x.
2. Manipulate the equation to the form of $ax^2+bx+c=0$.
3. Decide on the case required by the question
   Case 1: the line intersects the curve at 2 distinct points $\Leftrightarrow$ Discriminant >0
   Case 2: the line is tangent to the curve $\Leftrightarrow$ Discriminant $=0$
   Case 3: the line does not intersect the curve $\Leftrightarrow$ Discriminant <0
4. Solve!

Example 13

Find the range of values of k for which the line $y=2x+1$ intersects the curve $y=x^2+4x+k$ at two distinct points. Also state the range of values of k for the following cases:

(i) the line $y=2x+1$ is tangent to the curve $y=x^2+4x+k$

(ii) the line $y=2x+1$ does not intersect the curve $y=x^2+4x+k$

(iii) the line $y=2x+1$ intersects the curve $y=x^2+4x+k$ at most once

Solution:

$y=2x+1\cdots(1)$

$y=x^2+4x+k\cdots(2)$

(1) = (2), $x^2+4x+k=2x+1$

$x^2+2x+(k-1)=0\cdots(3)$

Since the line intersects the curve at two distinct points,
Discriminant > 0

$(2)^2-4(1)(k-1)>0$

$4-4k+4>0$

$4k<8$

$k<2$

(i) $k=2$

(ii) $k>2$

(iii) When the line $y=2x+1$ intersects the curve $y=x^2+4x+k$ at most once, there will be either 1 (repeated) real root or no real root.

Discriminant ≤ 0

$\Rightarrow$ $k\geq 2$

Exercise 3

1. Find the values of k for which the line $y = kx + 5$ is tangent to the curve $x^2 + y^2 = 9$.

Solution:

$y = kx + 5 \cdots (1)$

$x^2 + y^2 = 9 \cdots (2)$

Sub. (1) into (2)

$x^2 + (kx+5)^2 = 9$

$x^2 + k^2x^2 + 10kx + 25 - 9 = 0$

$(1+k^2)x^2 + 10kx + 16 = 0$

Since the line is tangent to the curve, Discriminant = 0

$(10k)^2 - 4(1+k^2)(16) = 0$

$100k^2 - 64 - 64k^2 = 0$

$36k^2 - 64 = 0$

$9k^2 - 16 = 0$

$(3k-4)(3k+4) = 0$

$k = -1\frac{1}{3}$ or $k = 1\frac{1}{3}$

2. Given the curve $y = 4x^2 + mx + m - 3$, find the possible values of m for which x-axis is a tangent to the curve.

Solution:

Since x-axis is a tangent to the curve, the curve touches the x-axis at one point.
The two roots are real and repeated.
Discriminant = 0

$(m)^2 - 4(4)(m-3) = 0$

$m^2 - 16m + 48 = 0$

$(m-4)(m-12) = 0$

$m = 4$ or $m = 12$

3. The line $y = mx + c$ is a tangent to the curve $x^2 + y^2 = 4$. Prove that $4m^2 = c^2 - 4$.

Solution:

$y = mx + c \cdots (1)$

$x^2 + y^2 = 4 \cdots (2)$

Sub (1) into (2),

$x^2+\left(mx+c\right)^2=4$

$x^2+m^2x^2+2mcx+c^2-4=0$

$\left(1+m^2\right)x^2+2mcx+c^2-4=0$

Since the line is a tangent to the curve, discriminant = 0

$(2mc)^2-4(1+m^2)(c^2-4)=0$

$4m^2c^2-4c^2+16-4m^2c^2+16m^2=0$

$16m^2=4c^2-16$

$4m^2=c^2-4$ (Proven)

Answer:

1.	$k=-1\frac{1}{3}$ or $k=1\frac{1}{3}$	2.	4, 12

1.2 Simultaneous Linear and Quadratic Equations in Two Unknowns

Recall that a linear equation is an equation between two variables that gives a straight line when plotted on a graph.

Example 14

Is this a linear equation?		
(i)	$3x=8$	Yes/No
(ii)	$3x-6y-2=0$	Yes/No
(iii)	$5^2x=6y^2$	Yes/No
(iv)	$5^2x-6y=5$	Yes/No
(v)	$5^2x-\frac{y}{2}=5$	Yes/No
(vi)	$5^2x-\frac{2}{y}=5$	Yes/No

Any other equations that are NOT linear are called non-linear equations.

1.2.1 Solving Simultaneous Linear and Quadratic Equations in Two Unknowns

A pair of simultaneous equations, one linear and one quadratic, can be solved by using method of substitution.

Method

1. From the linear equation, express one variable in terms of the other.
2. Substitute the new equation into the non-linear equation to obtain a quadratic equation with only 1 variable.
3. Solve the quadratic equation.

Example 15

Without using a calculator, solve the simultaneous equations

$$3x^2 + y^2 = 28$$
$$3x + y = 8$$

Which equation is easier to make x or y the subject?
Ans: Equation (2)

Solution:

$$3x^2 + y^2 = 28 \cdots (1)$$
$$3x + y = 8 \quad \cdots (2)$$

Note that Equation (1) is a non linear equation. Equation (2) is a linear equation.

Begin by making y the subject in (2): $y = 8 - 3x \cdots (3)$

Next substitute this expression for y into (1):

$$3x^2 + (8-3x)^2 = 28$$

Expand $(8-3x)^2$ to get: $3x^2 + 64 - 48x + 9x^2 = 28$

Make RHS 0: $3x^2 + 64 - 48x + 9x^2 - 28 = 0$

Simplify: $12x^2 - 48x + 36 = 0$

$$x^2 - 4x + 3 = 0 \cdots (4)$$
$$(x-1)(x-3) = 0$$

Solve: $x - 1 = 0$ or $x - 3 = 0$

$x = 1$ $\qquad$ $x = 3$

Substitute $x = 1$ into (3): $y = 8 - 3(1)$

$= 5$

Substitute $x = 3$ into (3): $y = 8 - 3(3)$

$= -1$

Example 16
Without using a calculator, solve the simultaneous equations

$$\frac{x^2}{6} - \frac{y}{4} = 1$$
$$2x + 3y = 5$$

Leaving your answer in exact form.

Which equation is easier to make x or y the subject?
Ans: Equation 2

Solution:

$$\frac{x^2}{6} - \frac{y}{4} = 1 \cdots (1)$$
$$2x + 3y = 5 \cdots (2)$$

From (2) we can make y the subject: $3y = 5 - 2x$

$$y = \frac{5-2x}{3} \cdots (3)$$

Substitute this expression for y into (1): $\frac{x^2}{6} - \frac{1}{4}\left(\frac{5-2x}{3}\right) = 1$

$$\frac{x^2}{6} - \frac{5-2x}{12} = 1$$

Take common denominator: $\frac{2x^2 - (5-2x)}{12} = 1$

Multiply throughout by 12: $2x^2 - (5-2x) = 12$

Make RHS '0': $2x^2 - (5-2x) - 12 = 0$

$$2x^2 + 2x - 17 = 0$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(2)(-17)}}{2(2)}$$

$$x = \frac{-2-\sqrt{140}}{4} \quad \text{or} \quad x = \frac{-2+\sqrt{140}}{4}$$

$$x = \frac{-1-\sqrt{35}}{2} \quad \text{or} \quad x = \frac{-1+\sqrt{35}}{2}$$

Substitute $x = \frac{-1-\sqrt{35}}{2}$ into (3): $y = \frac{5 - 2\left(\frac{-1-\sqrt{35}}{2}\right)}{3} = \frac{5-\left(-1-\sqrt{35}\right)}{3} = \frac{6+\sqrt{35}}{3}$

Substitute $x = \frac{-1+\sqrt{35}}{2}$ into (3): $y = \frac{5 - 2\left(\frac{-1+\sqrt{35}}{2}\right)}{3} = \frac{5-\left(-1+\sqrt{35}\right)}{3} = \frac{6-\sqrt{35}}{3}$

1.2.3 Application Questions

Example 17

The straight line $2x+y=2$ meets the curve $x^2+xy+3=0$ at the points *A* and *B*. Find points *A* and *B* and then calculate the co-ordinates of the mid-point of *AB* without using a calculator.

Solution:

Recap and Thinking Process:

Given two points *A* and *B* with coordinates (x_1, y_1) and (x_2, y_2) respectively, the

$$\text{Mid-point of } AB = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Therefore to find the mid-point of *AB*, we have to find the co-ordinates of *A* and *B* first.

The co-ordinates of *A* and *B* are the points of intersection of the line and curve.
To find the points of intersection, we solve the two equations simultaneously.

$$2x+y=2 \quad \cdots(1)$$
$$x^2+xy+3=0 \cdots(2)$$

From (1): $y=2-2x \cdots (3)$

Substitute (3) into (2):

$$x^2+x(2-2x)+3=0$$
$$x^2+2x-2x^2+3=0$$
$$-x^2+2x+3=0$$
$$(-x+3)(x+1)=0$$
$$-x+3=0 \quad \text{or} \quad x+1=0$$
$$x=3 \qquad\qquad x=-1$$

Substitute $x=3$ into (3): $y=2-2(3)$
$=-4$

Substitute $x=-1$ into (3): $y=2-2(-1)$
$=4$

Therefore $A=(3,-4)$ and $B=(-1,4)$.

Midpoint of *AB* $=\left(\frac{3-1}{2}, \frac{-4+4}{2}\right)=(1,0)$

Example 18

The price, P in dollars, of a good is related to the quantity, Q in thousands, by its demand and supply in the market. A demand curve shows that as price decreases, the quantity demanded of the product increases and follows the equation: $P+\frac{1}{2}Q=760$.

A supply curve shows that as price increases, quantity of the product supplied increases and follows the equation : $P=(Q+6)^2+74$.

The point at which supply equals demand is the equilibrium price and quantity. Find the equilibrium price and quantity.

Solution:

$P+\frac{1}{2}Q=760$

$P=-\frac{1}{2}Q+760.....(1)$

Substitute P from (1) into equation $P=(Q+6)^2+74$

We get $-\frac{1}{2}Q+760=(Q+6)^2+74$

Expand: $-Q+1520=2(Q^2+12Q+36+74)$

Simplify we get: $2Q^2+25Q-1300=0$

$(Q-20)(2Q+65)=0$

$Q=20$ or $Q=-(65/2)$ (reject since Q cannot be negative)

Then P = 750

Equilibrium price is \$750 and quantity is 20,000. (the quantity is 20 thousands)

Exercise 4

1. Without using a calculator, solve the following pairs of simultaneous equations

(a) $2x+y=7$

$x^2-xy+y^2=7$

(b) $x+y=4$

$(x+1)^2+(y+1)^2=25$

(c) $2x-5y=13$

$3x-\frac{4}{y}=16$

Solution:

(a) $2x+y=7$ -----(1) $x^2-xy+y^2=7$ -------------(2) From (1): $y=7-2x$ --------(3) Substitute (3) into (2): $x^2-x(7-2x)+(7-2x)^2=7$ $x^2-7x+2x^2+49-28x+4x^2=7$ $7x^2-35x+42=0$ $x^2-5x+6=0$ $(x-3)(x-2)=0$ $x-3=0$ or $x-2=0$ $x=3$ $x=2$ Substitute $x=3$ into (3): $y=7-2(3)=1$ Substitute $x=2$ into (3): $y=7-2(2)=3$	(b) $x+y=4$ -----------(1) $(x+1)^2+(y+1)^2=25$ ---------(2) From (1): $x=4-y$ ---(3) Substitute (3) into (2): $(4-y+1)^2+(y+1)^2=25$ $(5-y)^2+(y+1)^2=25$ $25-10y+y^2+y^2+2y+1=25$ $2y^2-8y+1=0$ $y=\frac{8\pm\sqrt{(-8)^2-4(2)(1)}}{2(2)}$ $=\frac{8\pm\sqrt{56}}{4}$ $=\frac{4\pm\sqrt{14}}{2}$ Substitute $y=\frac{4+\sqrt{14}}{2}$ into (3): $x=\frac{4-\sqrt{14}}{2}$ Substitute $y=\frac{4-\sqrt{14}}{2}$ into (3): $x=\frac{4+\sqrt{14}}{2}$
(c) $2x-5y=13$ $x=\frac{13+5y}{2}$ ......(1) $3x-\frac{4}{y}=16$ ....(2) Substitute (1) into (2): $3\left(\frac{13+5y}{2}\right)-\frac{4}{y}=16$ $\frac{3y(13+5y)-8}{2y}=16$ $39y+15y^2-8=32y$	

$15y^2+7y-8=0$ $(15y-8)(y+1)=0$ $y=\frac{8}{15}$ or $y=-1$ When $y=\frac{8}{15}$, $x=\frac{47}{6}$ When $y=-1$, $x=4$	

2. If (2,1) is a solution of the simultaneous equation $x^2+xy+ay=2ax+3y=b$, find the value of *a* and *b*. Find the other solution of the set of simultaneous equations.

Solution:

Note that $x^2+xy+ay=2ax+3y=b$

$$\Rightarrow \quad x^2+xy+ay=2ax+3y$$

$$x^2+xy+ay=b\cdots(1)$$

$$2ax+3y=b\cdots(2)$$

Substitute $x=2$, $y=1$ into $x^2+xy+ay=b$:

$$4+2+a=b$$

$$6+a=b\cdots(3)$$

Substitute $x=2$, $y=1$ into $2ax+3y=b$:

$$4a+3=b\cdots(4)$$

Using GC, $a=1$, $b=7$

Substitute $a=1$ and $b=7$ into (1) and (2):

$$x^2+xy+y=7\cdots(5)$$

$$2x+3y=7\cdots(6)$$

From (6):

$$y=\frac{7-2x}{3}\cdots(7)$$

Substitute (7) into (5):

$$x^2+x\left(\frac{7-2x}{3}\right)+\frac{7-2x}{3}=7$$

$$\frac{3x^2+7x-2x^2+7-2x}{3}=7$$

$$x^2+5x+7=21$$

$$x^2+5x-14=0$$

$$(x+7)(x-2)=0$$

$$x+7=0 \quad \text{or} \quad x-2=0$$

$$x=-7 \qquad x=2 \text{ (Reject)}$$

Substitute $x=-7$ into (7): $y=\dfrac{7-2(-7)}{3}$

$=7$

Therefore the other solution is $(-7,7)$.

3. [2008 'A' Level Qn 2]
The sum of two numbers *x* and *y* is 20 and the sum of their squares is 300. Given that $x>y$, find the exact values of *x* and *y*.

Solution:

$$x+y=20\cdots(1)$$

$$x^2+y^2=300\cdots(2)$$

From (1): $y=20-x\cdots(3)$

Substitute (3) into (2): $x^2+(20-x)^2=300$

$$x^2+400-40x+x^2=300$$

$$2x^2-40x+100=0$$

$$x^2-20x+50=0$$

$$x=\frac{-(-20)\pm\sqrt{(-20)^2-4(1)(50)}}{2(1)}$$

$$=\frac{20\pm\sqrt{200}}{2}$$

$$=\frac{20\pm10\sqrt{2}}{2}$$

$$=10\pm5\sqrt{2}$$

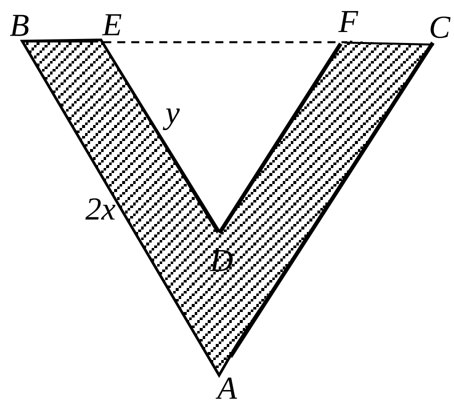
Substitute $x=10-5\sqrt{2}$ into (3): $y=20-\left(10-5\sqrt{2}\right)$

$=10+5\sqrt{2}$ (Reject as $x>y$)

Substitute $x=10+5\sqrt{2}$ into (3): $y=20-\left(10+5\sqrt{2}\right)$

$=10-5\sqrt{2}$

4. [2016 A levels H1 Qn 5]



The diagram shows a V-shape which is formed by removing the equilateral triangle *DEF*, in which $DE = y$ cm, from an equilateral triangle *ABC*, in which $AB = 2x$ cm. The points *E* and *F* are on *BC* such that $BE = FC$. The area of the V-shape *ABEDFCA* is $2\sqrt{3}$ cm^2.

(i) Show that $4x^2 - y^2 = 8$.

(ii) Given that the perimeter of *ABEDFCA* is 10 cm, find the values of *x* and *y*.

Examiner's Report: Candidates who performed well in this part demonstrated an ability to apply their knowledge to an unfamiliar situation. Finding the area of the V-shape involved subtracting the areas of two equilateral triangles. Some candidates first found the height of an equilateral triangle, and then its area, while others obtained the areas of the triangles directly by using the formula Area of triangle = ½ absinC. Some responses to this part showed errors in calculations involving the use of Pythagoras' theorem.

Solution:

(i) Since DEF is an equilateral triangle, $\measuredangle EFD = 60^o$

Area of the remaining shape *ABEDFCA*

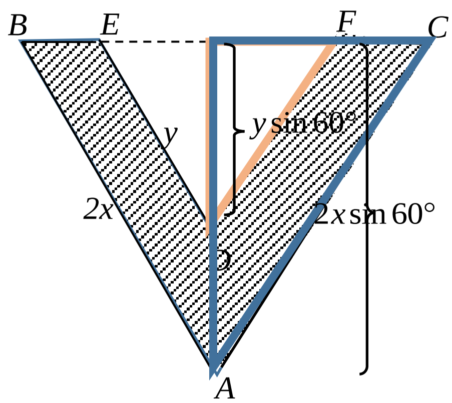
Area of triangle = ½ *ab*sin*C*

= Area of triangle *ABC* – Area of triangle *EDF*

$= \frac{1}{2}(2x)^2 \sin 60° - \frac{1}{2}(y)^2 \sin 60°$ (properties of equilateral triangles)

$= \sqrt{3}x^2 - \frac{\sqrt{3}}{4}y^2$

Alternative



Area of the remaining shape *ABEDFCA*
= Area of triangle *ABC* – Area of triangle

Area of triangle = ½ (base)(height)

$=\frac{1}{2}(2x)(2x\sin 60°)-\frac{1}{2}(y)(y\sin 60°)$

$=\sqrt{3}x^2-\frac{\sqrt{3}}{4}y^2$

Given that Area of the remaining shape *ABEDFCA* = $2\sqrt{3}$ cm^2.

$\sqrt{3}x^2-\frac{\sqrt{3}}{4}y^2=2\sqrt{3}$

$x^2-\frac{1}{4}y^2=2$

$4x^2-y^2=8\ (\text{shown})$

(ii) From (i), $4x^2-y^2=8\ -(1)$

Given that perimeter of the shape *ABEDFCA* = 10 cm

$2(2x)+2(y)+2x-y=10$

$4x+2y+2x-y=10$

$6x+y=10$

Substitute $y=10-6x$ into (1):

$4x^2-(10-6x)^2=8$

$4x^2-\left(100-120x+36x^2\right)=8$

$-32x^2+120x-108=0$

$32x^2-120x+108=0$

$8x^2-30x+27=0$

$(4x-9)(2x-3)=0$

$x=1.5$ or $x=2.25$

For $x=1.5$, $y=1$

For $x=2.25$, $y=-3.5$ (reject as length y cannot be less than 0)

The values of x and y are $x=1.5$ and $y=1$

Answer

1(a)	$x=3,\ y=1$ or $x=2,\ y=3$
1(b)	$x=\frac{4-\sqrt{14}}{2},\ y=\frac{4+\sqrt{14}}{2}$ or $x=\frac{4+\sqrt{14}}{2},\ y=\frac{4-\sqrt{14}}{2}$
1(c)	$x=7\frac{5}{6},\ y=\frac{8}{15}$ or $x=4,\ y=-1$
2	$a=1,\ b=7;\ (-7,7)$
3	$x=10+5\sqrt{2},\ y=10-5\sqrt{2}$
4	$x=1.5$ and $y=1$

1.3 Systems of Linear Equation (SOLE)

You learnt that a linear equation is an equation between two variables that gives a straight line when plotted on a graph and how to solve a pair of simultaneous equation where one equation is linear and the other equation is non-linear. In this section, you will learn to solve a system of linear equations.

A system of linear equations is a collection of linear equations with the same set of variables.

For example,

$$\left.\begin{aligned} 2x + y &= 3 \\ 4x - 3y &= 0 \end{aligned}\right\}$$ is a system of linear equations in TWO variables *x* and *y*.

$$\left.\begin{aligned} 2x + y - z &= 3 \\ 4x - 3y + 4z &= 0 \\ -3x + 9y + z &= 4 \end{aligned}\right\}$$ is a system of linear equations in THREE variables *x*, *y* and *z*.

$$\left.\begin{aligned} x^2 + y^2 &= 1 \\ 2x^2 - 3y^2 &= -2 \end{aligned}\right\}$$ is NOT a system of linear equations of *x* and *y*.

1.3.1 Using Graphing Calculator to solve SOLE

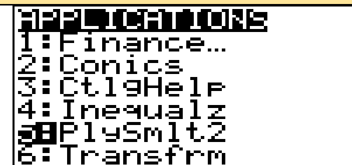
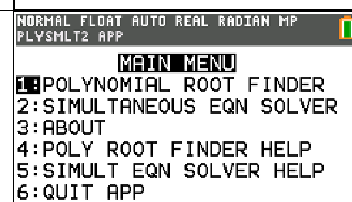
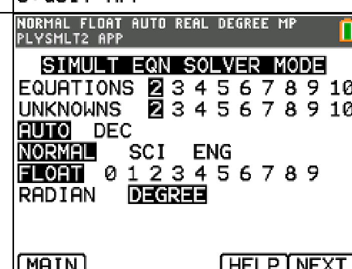
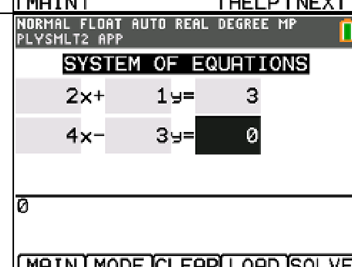
In this section, you will learn to solve a system of equations using a Graphing Calculator.

Example 19

Solve, using GC, the system of linear equations

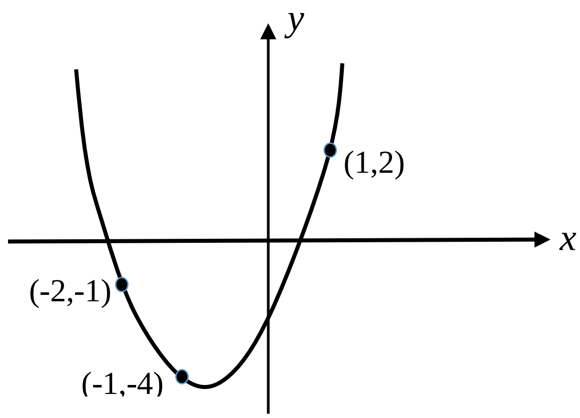
$$\begin{aligned} 2x + y &= 3 \\ 4x - 3y &= 0 \end{aligned}.$$

Solution:

Steps	Screenshot
Press [APPS] You should see this screen. Press [5] or whichever option that gives **PlySmlt2**	
You should see this screen. Press [2].	
You should see this screen. Select the number of equations and unknowns Since this question has 2 equations and 2 unknowns, Press NEXT	
Key in all the information. Press SOLVE.	
Presentation of answer: From GC, the solutions are $x = \frac{9}{10}, y = \frac{6}{5}$.	

Example 20

The curve of the parabola is shown in the figure below. The curve has equation of the form $y = ax^2 + bx + c$ and passes through the points $(-2,-1)$, $(-1,-4)$ and $(1,2)$. Find a, b and c.



Solution:

Substitute the *x*-values and *y*-values of each points into the equation and simplify, we obtain the following system of equations.

$(-2,-1)$: $a(-2)^2 + b(-2) + c = -1$

$4a - 2b + c = -1$ ...(1)

$(-1,-4)$: $a(-1)^2 + b(-1) + c = -4$

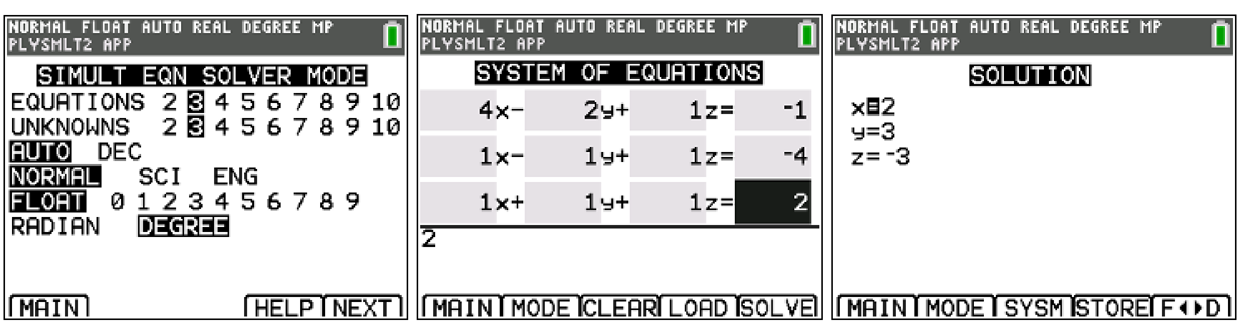
$a - b + c = -4$ …(2)

$(1,2)$: $a(1)^2 + b(1) + c = 2$

$a + b + c = 2$ …(3)

From GC, *a* = 2, *b* = 3, *c* = - 3.

GC Screenshots Using **PlySmlt2**



1.3.2 Modelling a System of Linear Equations to Solve Practical Problems

Step 1: Identify and define the variables.
Step 2: Formulate the problem into a system of linear equations with the given variables.
Step 3: Solve the system of equations using the Graphic Calculator.
Step 4: Check the reasonableness of your answer.

Example 21

A farmer needs to mix the two types of food mixtures to feed the chickens, so that they have 100g of protein and 480g of carbohydrates. Food mixture P has 25% protein and 70% carbohydrates and food mixture Q has 10% protein and 60% carbohydrates. How many grams of the two types of food mixtures should the farmer mix for his chickens?

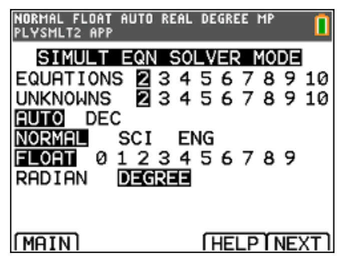
Solution:
Let p be be the amount in grams of food mixture P and
q be the amount in grams of food mixture Q.

Since 100g of proteins must be mixed from food mixture P and Q and P has 25% proteins and Q has 10% proteins means:

$$\frac{25}{100}p+\frac{10}{100}q=100$$

$$0.25p+0.1q=100 \quad \ldots.(1)$$

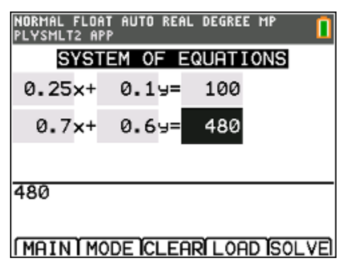
GC Screenshots using **PlySmlt2**



Since 480g of carbohydrates must be mixed from food mixture P and Q and P has 70% carbohydrates and Q has 60% carbohydrates means:

$$\frac{70}{100}p+\frac{60}{100}q=480$$

$$0.7p+0.6q=480 \quad \ldots.(2)$$



From GC, $p=150,\ q=625$

Therefore the farmer should mix 150g of mixture P and 625g of mixture Q.

Example 22

ABC Company manufactures 3 types (small, medium and large sizes) of water bottles for a school to sell. The cost price of a small water bottle is \$3, medium is \$5 and large is \$6, and the water bottles are selling for \$5, \$8 and \$12 respectively. The school sold 120 water bottles daily which cost \$520 to manufacture and the daily revenue is \$900. How many of each type are manufactured by this ABC Company?

Solution:

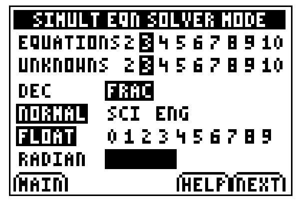
Let x, y, z represent the number of small, medium and large water bottles manufactured by ABC company respectively.

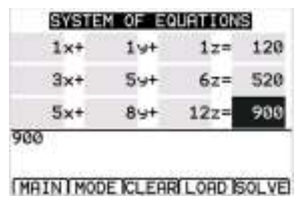
Number of bottles sold daily: $x+y+z=120$ ….(1)
Cost of manufacturing: $3x+5y+6z=520$ …(2)
Revenue: $5x+8y+12z=900$ …(3)

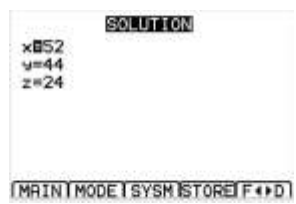
From GC, x =52, y =44, z =24.

The company manufactured 52 small bottles, 44 medium bottles and 24 large bottles

GC Screenshots Using **PlySmlt2**







Example 23 [Self Reading Example: Application of Closed Leontief Model]

The Leontief closed model is a model for the economics of a whole country or region. In the model there are n industries producing n different products such that the input equals the output. In other words, consumption equals production.

An economy has 3 sectors: Coal, Electric, and Steel. The table below shows how the output of each sector is distributed among the various sectors. For example, in the first row, to produce 7 units of coal needs 40% Electric and 60% Steel. How much must each of the three sectors produce to meet the output demands and still ensure no wastage of any of the sectors?

Input			
Coal	**Electric**	**Steel**	**Output**
0	0.4	0.6	7 units of Coal
0.6	0.1	0.2	3 units of Electric
0.4	0.5	0.2	4 units of Steel

Solution:

Let the output for coal, electric and steel be x, y and z respectively.

$$0.4y + 0.6z = 7 \ldots (1)$$
$$0.6x + 0.1y + 0.2z = 3 \ldots (2)$$
$$0.4x + 0.5y + 0.2z = 4 \ldots (3)$$

Using GC, $x = \frac{9}{7}$, $y = \frac{22}{7}$, $z = \frac{67}{7}$

Exercise 5

1. [2017/SAJC H2 Math CT/Q1]
Four friends went to a fruit seller to buy durians. The weight of the different varieties of durian and the total amount paid by each person are shown in the table below:

	Adam	Bryan	Charlie	Wayne
D13 (kg)	4.2	3.7	3.3	2.9
Golden Phoenix (kg)	3.1	0	3.7	4.2
Musang King (kg)	1.5	4.8	3.0	1.2
Total amount paid ($)	98.60	118.20	131.60	k

Assuming that, for each variety of durian, the price per kilogram paid by each of the friends is the same, find the price per kg of each variety. Hence calculate the total amount, $ k, paid by Wayne. [4]

Solution
Let the price per kg of D13, Golden Phoenix, Musang King be x, y and z dollars respectively.

$4.2x+3.1y+1.5z=98.6 --(1)$
$3.7x+4.8z=118.2 --(2)$
$3.3x+3.7y+3z=131.6 --(3)$
Using GC, solving (1), (2) and (3),
$x=6, y=14, z=20$

Total amount Wayne Paid = 2.9(6)+4.2(14)+1.2(20)
= $100.20

2. Sum of the digits of a positive three-digit number is 19. The tens digit is four less than twice the hundreds digit. The number decreases by 99 when the digits are reversed. Find the number.
[Answer: 685]

Solution
Let the hundreds digit be x
Let the tens digit be y
Let the ones digit be z
The sum of the individual digits is 19

$$x+y+z=19 \ldots(1)$$

The tens digit is four less than twice the hundreds digit

$$y=2x-4$$
$$2x-y=4 \ldots(2)$$

The number decreased by 99 when the digits are reversed.

$$100x+10y+z=100z+10y+x+99$$
$$99x-99z=99$$
$$x-z=1\ldots(3)$$

From GC, $x=6, y=8, z=5$. The number is 685.

3. A new brand of plant fertilizer is to be made from three different types of chemicals (A, B, C). The mixture includes 80% of chemicals A and B. Chemical B and C must be in ratio of 3 to 4 by weight. How much of each type of chemical is needed to make 600kg of plant fertilizer. [Answer: $a=390$ kg, $b=90$ kg, $c=120$ kg]

Solution

Let the amount in kg of A be a

Let the amount in kg of B be b

Let the amount in kg of C be c

The 3 chemicals must add up to 600kg

$$a+b+c=600\ldots.(1)$$

Since chemical A and B must make up 80%

$$a+b=80\%\times 600$$
$$a+b=480\ldots(2)$$

The ratio of $b:c$ is $3:4$. Hence

$$\frac{b}{c}=\frac{3}{4}$$
$$4b=3c$$
$$4b-3c=0 \quad ....(3)$$

Using the GC to solve $a=390$ kg, $b=90$ kg, $c=120$ kg.

4. [2009 HCI CT]

A housewife bought a total of 6.3kg of meat from each of the supermarkets A, B and C. The table below shows the price per kg of the different kinds of meat and the total amount spent in each supermarket. The price per kg of fish in supermarket C is unknown.

	supermarket A	supermarket B	supermarket C
Fish	\$ 10.50/kg	\$ 10.80/kg	
Pork	\$ 6.20/kg	\$ 6.50/kg	\$6.10/kg
Chicken	\$ 5.40/kg	\$ 5.50/kg	\$5.60/kg
Total amount spent	\$ 46.95	\$ 48.34	\$ 48.45

Given that the housewife bought an equal amount of the same type of meat in each supermarket, find the price per kg of fish she paid in supermarket C. [Answer: \$11]

Solution

Let the amount in kg of fish, pork and chicken bought be x, y and z. Then

$x + y + z = 6.3$ ….(1)

$10.50x + 6.20y + 5.40z = 46.95$ …(2)
$10.80x + 6.50y + 5.50z = 48.34$ …(3)

From GC, $x = 2.3$, $y = 1.5$, $z = 2.5$
She bought 2.3kg of fish, 1.5 kg of pork and 2.5 kg of chicken.

The price per kg of fish in Supermarket C is

$$\frac{48.45-(6.10)(1.5)-(5.60)(2.5)}{2.3}=\$11.00$$

5. [2009 NJC CT] At NJUC supermarket, a discount is offered on pork, chicken and beef if more than a certain weight of it is bought. On a particular grocery shopping trip, Mrs Low bought a total of 12kg of meat as follows:

Types of Meat	Price per kg ($)	Discount offered
Pork	14	20% for more than 6kg bought
Chicken	9	15% for more than 3kg bought
Beef	11	10% for more than 2kg bought

She bought at least 3.5kg of each type of meat and spent $124.03 after an overall discount of $10.17.

Find the weight of each type of meat Mrs Low bought from the supermarket.

Solution
Let P, C and B be the weight of pork, chicken and beef Mrs Low bought respectively.

$P+C+B=12$ --- (1)
$14P+9C+11B=124.03+10.17=134.20$ --- (2)

Since Mrs Low bought at least 3.5kg of each type of meat, she must have bought less than 6kg of pork.

$14P+0.85\times 9C+0.90\times 11B=124.03$
$14P+7.65C+9.9B=124.03$ --- (3)

Using GC, $P=3.8; C=4.6; B=3.6$
Mrs Low bought 3.8kg of pork, 4.6kg of chicken and 3.6kg of beef at the supermarket.

6. [2009 JJC CT]
A store sells large, medium and small sizes of white, yellow and brown toy bears. The selling price of a large, medium and small bear of any colour is $\$x$, $\$y$ and $\$z$ respectively. The number of bears of each type that were sold in one day are given in the following table.

	Large	Medium	Small
white	5	3	1
yellow	2	7	5
brown	6	4	2

The total sale of white toy bears per day was \$593, from yellow toy bears was \$829 and from brown toy bears was \$778.
Mr Tan bought 3 large white toy bears, 1 medium brown toy bear and 5 small yellow toy bears. How much did he pay altogether?

Solution
The selling price of a large, medium and small toy bear of any colour is \$*x*, \$*y* and \$*z* respectively.

$$5x+3y+z=593\ldots.(1)$$
$$2x+7y+5z=829\ \ldots(2)$$
$$6x+4y+2z=778\ \ldots.(3)$$

Using GC, $x = 72$, $y = 60$, $z = 53$
He paid $3x + 1y + 5z = 3(72)+60 + 5(53) = \$541$

7. A quadratic curve passes through the points with coordinates $(2,7.5)$, $(11,12)$ and $(17,30)$. Find the equation of the quadratic curve.

Solution:
Let the equation of the quadratic curve be $y=\mathrm{f}(x)=ax^2+bx+c$

$$\mathrm{f}(2)=7.5\Rightarrow\quad 7.5=2^2a+2b+c$$
$$4a+2b+c=7.5\ \ldots.(1)$$

$$\mathrm{f}(11)=12\Rightarrow\quad 12=11^2a+11b+c$$
$$121a+11b+c=12\ldots(2)$$

$$\mathrm{f}(17)=30\Rightarrow 30=17^2a+17b+c$$
$$289a+17b+c=30\ldots\quad(3)$$

From GC, $a=\frac{1}{6},b=-\frac{5}{3},c=\frac{61}{6}=10\frac{1}{6}$

The equation of the quadratic curve is $y=\frac{x^2}{6}-\frac{5x}{3}+\frac{61}{6}$.

Practice Questions

1 [MI/2016/Promo Q1]
Without the use of a calculator, solve the simultaneous equations

$$x + 5y = 3,$$
$$x^2 + xy = 2.$$

Solution:

$x + 5y = 3 \cdots\cdots\cdots (1)$

$x^2 + xy = 2 \cdots\cdots (2)$

From (1), $x = 3 - 5y \cdots (3)$

Substitute (3) into (2):

$(3-5y)^2 + (3-5y)y = 2$

$9 - 30y + 25y^2 + 3y - 5y^2 = 2$

$20y^2 - 27y + 7 = 0$

$(20y-7)(y-1) = 0$ or

$y = \dfrac{27 \pm \sqrt{729 - 4(20)(7)}}{40}$

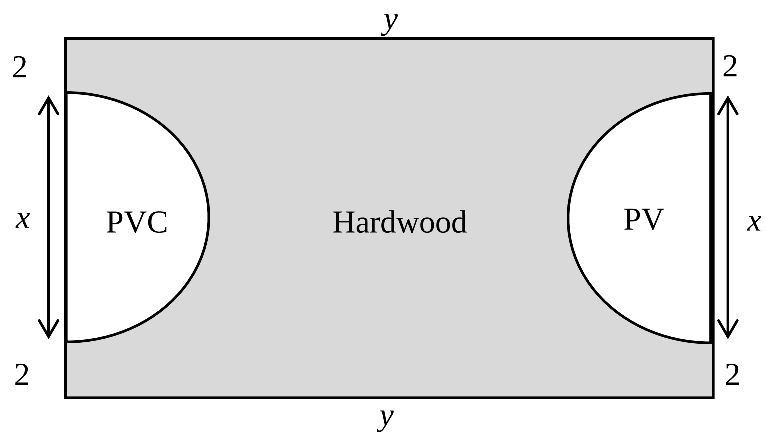
$y = 1$ or $y = \dfrac{14}{40} = \dfrac{7}{20}$

when $y = 1,\ x = -2$

when $y = \dfrac{7}{20},\ x = \dfrac{5}{4}$

2 [JJC/2015/Promo Q2]

An architect is designing a basketball court. The two semicircles at the side of the court is made up of PVC material with diameter x metres, while the remaining court (shaded region) is made up of hardwood.



Given that the perimeter of the basketball court is 90 m, and the area made up of hardwood is 350 m^2. Find the values of x and y.

Solution:

$$2y + 2x + 4 + 4 = 90$$

$$x + y = 41$$

$$(x+4)y - \pi\left(\frac{1}{2}x\right)^2 = 350$$

$$(x+4)(41-x) - \pi\left(\frac{1}{2}x\right)^2 = 350$$

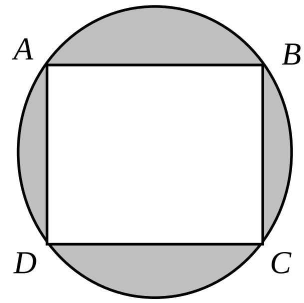
$$-x^2 + 37x + 164 - \frac{1}{4}\pi x^2 = 350$$

$$\left(\frac{1}{4}\pi + 1\right)x^2 - 37x + 186 = 0$$

$x = 12.148 = 12.1$ or $x = 8.5760 = 8.58$

$y = 28.852 = 28.9$ or $y = 32.424 = 32.4$

3 [TJC/2015/Promo Q4]



The figure above shows a rectangular flowerbed *ABCD* being ploughed within a circular garden. A fence is then constructed around the flowerbed *ABCD*. Given that the total length of fence used is 56 m and the total area of the flowerbed is $192\ \text{m}^2$, find the exact area of the unused land in the garden, indicated by the shaded region in the diagram.

Solution:
Let x and y be the length and breadth of the flowerbed, in m.

Then

$$2(x+y)=56$$
$$xy=192$$

Solving simultaneously gives

$$x=16 \quad \text{and} \quad y=12$$

So radius of circular garden is

$$r=\sqrt{\left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2}=\sqrt{8^2+6^2}=10$$

Thus the area of unused land in the garden is $\pi(10)^2-(16)(12)=\left(100\pi-192\right)\text{m}^2$

4 [ACJC/H1/Promo/2017/Q1]

Find, algebraically, the set of values of k for which

$$(2-k)+2kx+x^2>0$$

for all real values of x.

Solution:

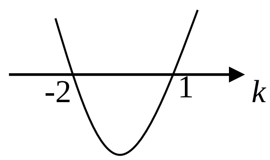
Since $x^2 + 2kx + (2-k) > 0$,

Discriminant < 0 and coefficient of $x^2 = 1 > 0$

$(2k)^2 - 4(1)(2-k) < 0$

$4k^2 - 8 + 4k < 0$

$k^2 + k - 2 < 0$



$(k-1)(k+2) < 0$

$-2 < k < 1$

$\therefore \{k \in \mathbb{R} : -2 < k < 1\}$

5 [NYJC/H1/Promo/2017/Q1]

Find the range of values of k for which $kx^2 + (k+1)x + (k+1)$ is never negative for all real values of x.

Solution:

Since $kx^2 + (k+1)x + (k+1) \geq 0$,

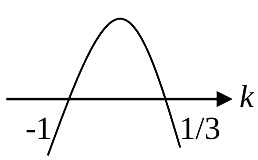
Discriminant ≤ 0 and coefficient of $x^2 > 0$

$(k+1)^2 - 4(k)(k+1) \leq 0$ and $k > 0$

$(k+1)(k+1-4k) \leq 0$

$(k+1)(1-3k) \leq 0$

$k \leq -1$ or $k \geq \frac{1}{3}$



Combining $k \leq -1$ or $k \geq \frac{1}{3}$ and $k > 0$, gives $k \geq \frac{1}{3}$

6 [EJC/H1/Promo/2017/Q2]

Show that there are no real values of k such that $x^2+(k-2)x-2k$ is always positive.

Solution:

If $x^2+(k-2)x-2k>0$,

Discriminant <0 and coefficient of $x^2>0$

$(k-2)^2-4(1)(-2k)<0$ and $1>0$

$k^2-4k+4+8k<0$

$k^2+4k+4<0$

$(k+2)^2<0$

However $(k+2)^2\geq 0$ for all real values of k.

Therefore there are no real values of k such that $x^2+(k-2)x-2k$ is always positive.

7 [DHS/H1/Promo/2017/Q2]

The line $y=(2k+1)x+k$ intersects the curve $y=1-\dfrac{k}{x}$, where k is a non-zero constant. Without using a calculator, find the exact set of values of k.

Solution:

At point(s) of intersection,

$(2k+1)x+k=1-\dfrac{k}{x}$

$\Rightarrow (2k+1)x^2+(k-1)x+k=0$

Since line and curve may intersect at least once,

Discriminant ≥ 0

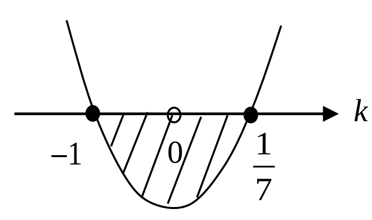
$(k-1)^2-4k(2k+1)\geq 0$

$k^2-2k+1-8k^2-4k\geq 0$

$-7k^2-6k+1\geq 0$

$7k^2+6k-1\leq 0$

$(7k-1)(k+1)\leq 0$



$\therefore$ set of values = $\{\ k\in\mathbb{R}: -1\leq k\leq \frac{1}{7}, k\neq 0\ \}$

8 [2016/FE/PJC/Q1]

The usual selling price of a facial toner, a moisturiser and a sun block sold by two companies *D* and *S* is \$365 in total. During the year-end sale, the two companies offered the following discounts to their customers:

Company	Discounts given for each item			Total price after the discount
	Facial Toner	Moisturiser	Sun Block	
D	10%	15%	15%	\$314.75
S	8%	25%	15%	\$301.55

(i) Find the usual selling prices of a facial toner, a moisturiser and a sun block.

(ii) The employees from Company *D* are offered a further 5% discount on the usual selling price of both the facial toner and the moisturiser. Determine if this additional discount will make it more attractive for the employees to purchase all the 3 items from their own company than from Company *S*.

Solution:

(i) Let \$$x$, \$$y$, \$$z$ be the usual selling prices of a facial toner, a moisturiser and a sun block respectively.

$x + y + z = 365$ -----------(1)

$0.9x + 0.85y + 0.85z = 314.75$ -----------(2)

$0.92x + 0.75y + 0.85z = 301.55$ ----------(3)

Using GC, $x = 90$, $y = 150$, $z = 125$

The usual selling prices of a facial toner, a moisturiser and a sun block are \$90, \$150 and \$125 respectively.

(ii) For the employees from Company *D*, the total amount to pay for the 3 items = \$ $(0.85\times 90 + 0.8\times 150 + 0.85\times 125)$ = \$302.75

The additional discount <u>does not make it more attractive</u> for the employees to purchase from their own company than from Company *S*.

9 [2017/Prelim/DHS/Q1]

Abel, Ben and Caleb travelled from Raffles Place to Bishan using the same route. They hired private transport from different car companies, Car X, Y and Z, respectively. Ben and Caleb had cash vouchers which could be used to offset their final fares. Ben's cash voucher was twice as much in value as Caleb's.

The fare comprises 3 components: A fixed base fare, the distance travelled and the time taken for the journey. The rates of the different companies are shown in the following table.

	Car X	Car Y	Car Z
Base fare ($)	3.20	3.00	3.00
Per kilometre ($)	0.55	0.80	0.45
Per minute ($)	0.29	0	0.20

Abel, Ben and Caleb paid $15.60, $6.60 and $9.40 respectively. Find the distance travelled and the time taken from Raffles Place to Bishan. Assume that the traffic is the same for all 3 journeys.

Solution:

Let a be the distance in km, b the time taken in minutes, c be the value of cash voucher that Caleb had.

$$\begin{cases} 3.2+0.55a+0.29b=15.6 \\ 3+0.8a=6.6+2c \\ 3+0.45a+0.2b=9.4+c \end{cases}$$

$$\begin{cases} 0.55a+0.29b=12.4 \\ 0.8a-2c=3.6 \\ 0.45a+0.2b-c=6.4 \end{cases}$$

Using GC,

$a=12, b=20, c=3$

Hence the time taken was <u>20 minutes</u> and the distance travelled was <u>12 km</u> .

Answer

1	$y=1,\ x=-2; y=\frac{7}{20},\ x=\frac{5}{4}$
2	$x=12.148=12.1$; $y=28.852=28.9$; $x=8.5760=8.58$; $y=32.424=32.4$
3	$(100\pi-192)\text{m}^2$
4	$\{k\in\mathbb{R}:-2<k<1\}$
5	$k\geq\frac{1}{3}$
6	-
7	$\{\ k\in\mathbb{R}:-1\leq k\leq\frac{1}{7},k\neq 0\ \}$
8	(i) x = 90, y = 150, z = 125 (ii) $302.75, does not make it more attractive
9	time taken = 20 minutes, distance travelled = 12 km

Summary

Simultaneous Equations

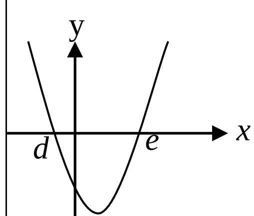
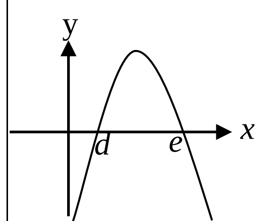
A pair of simultaneous equations, one linear and the other non-linear, can be generally solved by using method of substitution.

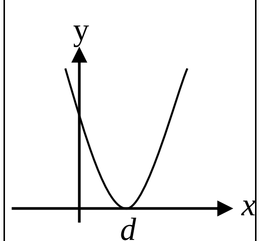
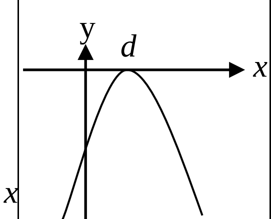
Method

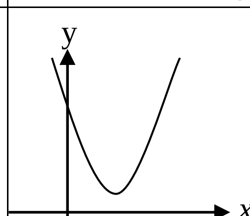
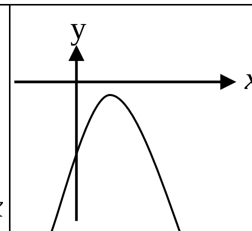
1. Express one unknown in terms of the other using the linear equation.
2. Substitute the new equation into the non-linear equation.

Quadratic Functions

Relationship between nature of roots and discriminant

<table>
<tr><th rowspan="2">Case</th><th rowspan="2">Discriminant</th><th rowspan="2">Nature of Roots of $ax^2+bx+c=0$</th><th colspan="2">Graph of $y=ax^2+bx+c$</th><th rowspan="2">Solutions</th></tr>
<tr><th>coefficient of x^2, $a>0$</th><th>coefficient of x^2, $a<0$</th></tr>
<tr><td rowspan="2">1.</td><td rowspan="2">Discriminant >0</td><td rowspan="2">2 real and distinct roots</td><td>
</td><td>
</td><td rowspan="2">$x=d$
or
$x=e$</td></tr>
<tr><td colspan="2">The curve cuts the x-axis at 2 distinct points.</td></tr>
<tr><td rowspan="2">2.</td><td rowspan="2">Discriminant $=0$</td><td rowspan="2">2 real and equal roots.

(repeated / coincident roots)</td><td>
</td><td>
</td><td rowspan="2">$x=d$</td></tr>
<tr><td colspan="2">The curve touches the x-axis at 1 point.
The x-axis is tangent to the curve.</td></tr>
<tr><td>3.</td><td>Discriminant <0</td><td>2 imaginary or complex roots.

No real roots.</td><td>
</td><td>
</td><td>No real solutions</td></tr>
</table>

			The curve lies entirely above the *x*- axis.	The curve lies entirely below the *x*- axis.	
			It does not cut or touch the *x*- axis.		

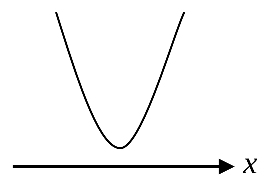
Note:
From case 1 and case 2, we can further summarise them into

Discriminant ≥ 0 $\Leftrightarrow$ Discriminant > 0 or Discriminant $= 0$
$\Leftrightarrow$ roots are real

<u>Conditions for Quadratic Equations to be always positive (or always negative)</u>

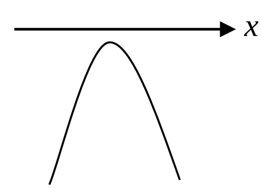
(i) When ax^2+bx+c is always positive (i.e. $ax^2+bx+c>0$ for all real values of *x*), the graph of $y = ax^2 + bx + c$ lies entirely above the *x*–axis. The equation $ax^2+bx+c=0$ has no real roots.

Conditions: Coefficient of x^2, $a>0$ and Discriminant $=b^2-4ac<0$.



(ii) When ax^2+bx+c is always negative (i.e. $ax^2+bx+c<0$ for all real values of *x*), the graph of $y = ax^2 + bx + c$ lies entirely below the *x*–axis. The equation $ax^2+bx+c=0$ has no real roots.

Conditions: Coefficient of x^2, $a<0$ and Discriminant $= b^2-4ac<0$.



In summary,

The function $y = ax^2 + bx + c$ is always positive $\Leftrightarrow$ Coefficient of $x^2 > 0$ and Discriminant < 0.

The function $y = ax^2 + bx + c$ is always negative $\Leftrightarrow$ Coefficient of $x^2 < 0$ and Discriminant < 0.

Relationship between discriminant and types of intersection between a line and a curve

Case	Discriminant	Nature of Roots	Types of Intersection
1.	Discriminant > 0	2 real and distinct roots	The line intersects the curve at 2 distinct points.
2.	Discriminant $= 0$	2 real and equal (repeated / coincident) roots.	The line is tangent to the curve. The line touches the curve at 1 point.
3.	Discriminant < 0	2 imaginary or complex roots. No real roots.	The line does not intersect the curve.

Solving Method

1. Combine the equation of the line and the curve into 1 equation in terms of x.
2. Manipulate the equation to the form of $ax^2 + bx + c = 0$.
3. Decide on the case required by the question
   Case 1: the line intersects the curve at 2 distinct points $\Leftrightarrow$ Discriminant > 0
   Case 2: the line is tangent to the curve $\Leftrightarrow$ Discriminant $= 0$
   Case 3: the line does not intersect the curve $\Leftrightarrow$ Discriminant < 0
4. Solve!

Systems of Linear Equation (SOLE)

Steps of modelling a system to solve practical problems:

Step 1: Identify and define the variables.
Step 2: Formulate equations with the given variables.
Step 3: Solve the system of equations using the Graphing Calculator.
Step 4: Check the reasonableness of your answer.

Checklist

I am able to:

- ☐ solve quadratic equations using factorisation, completing the square, formula, sketching of graph and a GC
- ☐ understand and use the conditions for a quadratic equation to have (i) two real and distinct roots, (ii) two real and equal roots, (iii) no real roots
- ☐ understand and use the conditions for a quadratic equation to be always positive (or always negative)
- ☐ use discriminant to solve problems involving intersection of a line and a curve
- ☐ formulate a quadratic equation from a problem situation and interpret the solution in the context of the problem
- ☐ Recognize a linear equation from a non-linear equation
- ☐ Solve a pair of simultaneous equations, one linear and one non linear, by substitution, without using a calculator
- ☐ Use a Graphing Calculator to solve SOLE
- ☐ Model a System of Linear Equations to Solve Practical Problems