RAFFLES INSTITUTION
H3 Mathematics 9820

Chapter 9 Calculus

The topics for Calculus, namely Differentiation, Integration, Power Series and Differential Equations, should be familiar from your study of H2 Mathematics. In this chapter we collect together some additional tips and techniques that might be useful in H3 Mathematics.

SYLLABUS INCLUDES

- Knowledge of the following topics (differentiation, Maclaurin series, integration techniques and differential equations) and suitable extensions
- Proving statements involving derivatives and integrals

CONTENT

1 Derivatives

It is important to understand that the derivative is defined via a limiting process.

Definition 1 Let I be an open interval and x_0 be a point in I. We say that a function defined on I is differentiable at I if the limit $\lim_{x \to x_0} \frac{\mathrm{f}(x) - \mathrm{f}(x_0)}{x - x_0}$ exists. In this case, the limit is called the derivative of f at the point x_0, and is written as $\mathrm{f}'(x_0)$. If this limit exists for every point in I, we say that f is differentiable on I and its derivative is written as $\mathrm{f}'(x)$.

Example 1 Let us use the definition above to find the derivative of $\mathrm{f}(x) = x^2$. For any real number *x*, we have

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} (2x+h) = 2x.$$

This derivation is also known as differentiation from first principles.

Exercise 1 Find from first principles the derivatives of the following:

(i) $\mathrm{f}(x) = x^n$, where n is a positive integer;

(ii) $\mathrm{f}(x) = \sin x$.

In similar fashion, we can work out the derivatives of simple functions and then use familiar results like the product, quotient and chain rules to work out derivatives of more complicated functions. Such techniques will be assumed to have been mastered and will not be pursued here.

Higher Order Derivatives

If a function is such that $\mathrm{f}'(x)$ is still differentiable, we can form the second derivative $\mathrm{f}''(x)$ and similarly for higher order derivatives. When it exists, the n^{th} derivative of $y = \mathrm{f}(x)$ with respect to x is denoted by $\mathrm{f}^{(n)}(x)$ or $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$.

Using the product rule and induction, we can prove the following generalized product rule:

If f and g are n times differentiable, then $(\mathrm{fg})^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \mathrm{f}^{(k)} \mathrm{g}^{(n-k)}$ for all positive integers n, where $\mathrm{f}^{(n)}$ denotes the n^{th} derivative of f. This result was an exercise in Tutorial 1, and the proof is basically done via Mathematical Induction.

Here, let us just interest ourselves in finding a formula for the n^{th} derivative of certain functions.

Exercise 2 Find a formula for the n^{th} derivative of the following functions

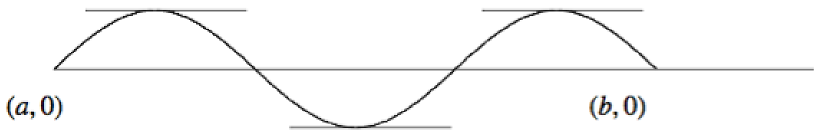
(i) $y = x^5 e^{2x}$;

(ii) $y = e^x \sin x$.

Let us end the discussion on derivatives by introducing two important results in real analysis involving derivatives. There is no need to know how to prove them, but you should be able to have an intuitive understanding of why they are true.

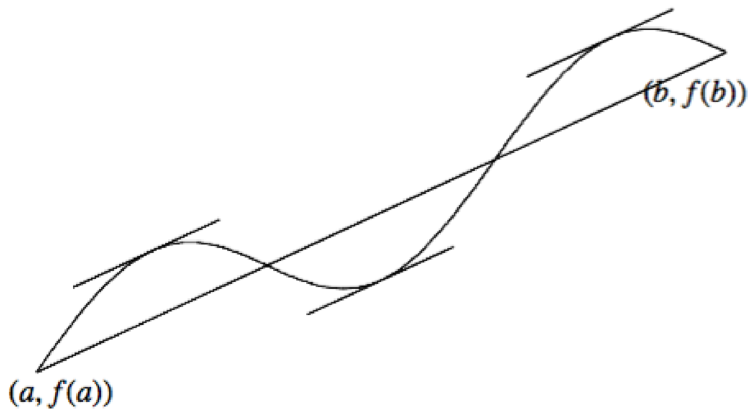
Rolle's Theorem

Rolle's Theorem states that if f is a continuous function on an interval [a, b] and differentiable on (a, b), given that f(a) = f(b) = 0, then there exists a c in the interval (a, b) such that $f'(c) = 0$.



Mean Value Theorem

The Mean Value Theorem states that if f is a continuous function on an interval [a, b] and differentiable on (a, b), there exists a c in the interval (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



We can understand this result geometrically in the following manner: Typically one draws the graph of a function y = f(x) which happens to make several turns (i.e. has inflection points) on an interval [a, b] and the straight line connecting the points (a, f(a)), (b, f(b)) on the same set of axes. Then one observes that the slope of this line is equal to the slope of the tangent to the graph of the function at least at one value of x in (a, b).

Applications of the Mean Value Theorem

Example 2 The function $\mathrm{f}(x)=\mathrm{e}^x$ has derivative $\mathrm{f}'(x)=\mathrm{e}^x$ for all real *x*. We will now show that $\mathrm{e}^x \geq 1+x$ for all real *x*.

If $x = 0$, the inequality is an equality. If $x > 0$, applying the Mean Value Theorem, we know that there is a real number c such that $0<c<x$ and $\mathrm{e}^x-\mathrm{e}^0=\mathrm{e}^c(x-0)$. Since $\mathrm{e}^c>1$ for $x>0$, $\mathrm{e}^x-\mathrm{e}^0=\mathrm{e}^c(x-0)>x \Rightarrow \mathrm{e}^x>x+1$. Repeat a similar argument for $x < 0$ and we are done.

Exercise 3 If $0<a\leq b$, show the following:

(i) $\dfrac{b-a}{1+b^2} \leq \tan^{-1} b - \tan^{-1} a \leq \dfrac{b-a}{1+a^2}$;

(ii) $1-\dfrac{a}{b} \leq \ln\left(\dfrac{b}{a}\right) \leq \dfrac{b}{a}-1$;

(iii) $a^2 \leq \dfrac{a^2+ab+b^2}{3} \leq b^2$.

2 Integration

What is commonly referred to as Integration in the A-level syllabus is actually two separate concepts, involving the computation of areas and the inverse of differentiation, i.e. finding which functions, when differentiated, will result in the given function. That there is an intimate connection between these two concepts is the statement of a seminal theorem, known appropriately as the Fundamental Theorem of Calculus.

2.1 Riemann Sums

Let $\mathrm{f}:[a,b]\to\mathbb{R}$ be a continuous function. One way to compute the area bounded by the curve $y = \mathrm{f}(x)$, the *x*-axis, the lines $x = a$ and $x = b$ is as the limit of Riemann sums.

Definition 2 A **partition** of $[a, b]$ is a finite set of points $\{x_k\}$, $k = 0, 1, \ldots, n$ such that

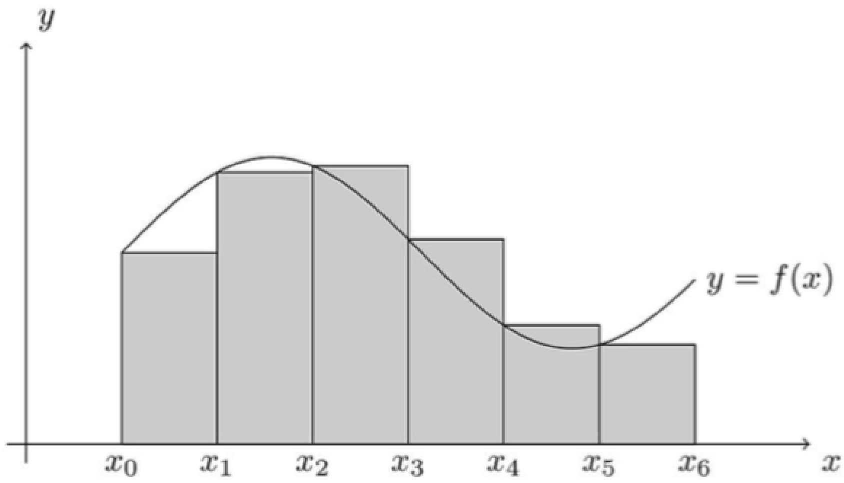
$$a=x_0<x_1<\ldots<x_n=b.$$

We will only use partitions in which the points $\{x_k\}$ are evenly spaced, that is $x_k-x_{k-1}=\frac{b-a}{n}$ for all $k = 1, 2, \ldots, n$. Thus $x_k=a+\frac{k}{n}(b-a)$. Such partitions are also known as **regular** partitions.

Definition 3 The Riemann sum associated to f and a regular partition with n subintervals, denoted by S(f, n) is given by

$$S(\mathrm{f}, n) = \sum_{k=1}^{n}(x_k - x_{k-1})\mathrm{f}(x_{k-1}) = \frac{b-a}{n}\sum_{k=1}^{n}\mathrm{f}\left(a + \frac{k-1}{n}(b-a)\right).$$

S(f, n) can be taken to be an approximation to the area under the curve, with the case for n = 6 illustrated below:



To get a better approximation to the area, it seems natural to increase the number of rectangles used, so we can attempt to let $n \to \infty$ to compute the exact area. Unfortunately, it is not always the case that a limit will exist. In those cases in which the limit exists, it is *defined* to be the (Riemann) definite integral $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$. In other words,

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n}\mathrm{f}\left(a + \frac{k-1}{n}(b-a)\right),$$

provided the limit exists. For the above Riemann sums, we used the left end point of the rectangles to compute the height $\mathrm{f}(x_{k-1})$. We could equally well have used the right end point, giving the following equivalent result

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n}\mathrm{f}\left(a + \frac{k}{n}(b-a)\right).$$

For the A-levels, you do not need to worry about whether the limit exists. Rather, focus on understanding how the approximation works. In particular if we restrict ourselves to the domain [0, 1], we have

$$\int_0^1 \mathrm{f}(x)\,\mathrm{d}x = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{f}\left(\frac{k-1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{f}\left(\frac{k}{n}\right).$$

Example 3 Let us evaluate $\int_0^1 x^2 \, dx$ using Riemann sums.

You may use the result $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Using a regular partition on [0, 1], we have

$$\begin{aligned}\int_0^1 x^2 \, dx &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \\ &= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\ &= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}.\end{aligned}$$

Sometimes the table is turned, and Riemann sums are used to evaluate limits of sums. This, of course, assumes we have some other means of computing definite integrals.

Example 4 To evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}$, we try to write it as a Riemann sum of some function.

In fact we have $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{1+x} \, dx = \ln 2$.

Note that this provides another method to evaluate some series which we have briefly discussed in Chapter 7.

Exercise 4 Evaluate $\int_0^2 x^3 \, dx$ using Riemann sums. You may use the result $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

Exercise 5 Evaluate the following:

(i) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \sin \frac{k\pi}{n}$,

(ii) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{(n+k)^2}$.

2.2 Integration Techniques

Even with a list of anti-derivatives at hand, we are still far from being able to evaluate most integrals. In this section, we will look at two main techniques to augment our H2 artillery, integration using reduction formula as well as the use of suitable substitutions. However, it is important to appreciate that there are certain anti-derivatives that cannot be expressed as a finite operation consisting of addition, multiplication and composition of "usual" functions. For example, we cannot "evaluate" anti-derivatives like $\int e^{x^2}\ \mathrm{d}x$.

2.2.1 Reduction Formula

Some integration methods are applicable to functions involving a power n where n is a small positive integer. For instance $\int \cos^n x\ \mathrm{d}x$ can be found when n = 4 by using trigonometric identities, but the same method will be unwieldy for $\int \cos^{20} x\ \mathrm{d}x$. Systematically reducing the value of n is called a reduction method, and is usually based on the technique of integration by parts.

Example 5 If $I_n = \int \cos^n x\ \mathrm{d}x$, applying integration by parts we have

$$\begin{aligned} I_n &= \int \cos^n x\ \mathrm{d}x = \int \cos x \cos^{n-1} x\ \mathrm{d}x \\ &= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x\ \mathrm{d}x \\ &= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\ \mathrm{d}x \\ &= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\ \mathrm{d}x - (n-1)\int \cos^n x\ \mathrm{d}x \\ &= \sin x \cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \end{aligned}$$

Rearranging we have $I_n = \dfrac{\sin x \cos^{n-1} x}{n} + \dfrac{n-1}{n} I_{n-2}$. This gives us a recurrence relation for I_n, so that starting with I_0 and I_1 (which are easily found), we can systematically work out I_n for larger values of n.

If I_n is a definite integral with limits of, for example 0 and $\dfrac{\pi}{2}$, we have $I_n = \dfrac{n-1}{n} I_{n-2}$. From $I_0 = \dfrac{\pi}{2}$ and $I_1 = 1$, we can obtain $I_{2m} = \dfrac{\pi}{2}\prod_{k=1}^{m}\dfrac{2k-1}{2k}$ and $I_{2m+1} = \prod_{k=1}^{m}\dfrac{2k}{2k+1}$. These results are also valid for $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\ \mathrm{d}x$.

Some reduction formulae cause the value of n to fall by 2 in each step, while in other cases n falls only by 1 every step. Also, it is not always required to use integration by parts.

Exercise 6

(i) Establish a reduction formula that could be used to find $I_n = \int x^n e^x \, dx$ and use it to find I_4.

(ii) If $J_n = \int \tan^n x \, dx$, find a reduction formula and use it to evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x \, dx$.

2.2.2 Useful Substitutions

In the H2 syllabus, if a substitution is required to evaluate an integral, it will be provided. However, for H3, this is not necessarily the case. So there is a need to discuss how we can come up with some useful substitutions.

General Tips

When encountering expressions like $\int \frac{1}{\sqrt{x}+1} \, dx$, the idea is to simplify the integrand into something we know how to deal with. $u = \sqrt{x}$ is a natural choice and it does work. (Check!)

Expressions involving $\sqrt{1 \pm x^2}$, $\sqrt{x^2 \pm 1}$.

The main idea is to use a suitable trigonometric substitution, and using one of the following trigonometric identities $\sin^2\theta + \cos^2\theta = 1$ or $1 + \tan^2\theta = \sec^2\theta$ to simplify the expression in the surd to a single trigonometric function.

For example, if we encounter an integral like $\int \frac{1}{\sqrt{x^2 - 4}} \, dx$, we could use the substitution $x = 2\sec\theta$. Then $\int \frac{1}{\sqrt{x^2-4}} \, dx = \int \frac{2\sec\theta\tan\theta}{\sqrt{4\sec^2\theta - 4}} \, d\theta = \int \frac{2\sec\theta\tan\theta}{2\tan\theta} \, d\theta = \int \sec\theta \, d\theta$.

Expressions of the form $\int R(\sin x, \cos x)\ \mathrm{d}x$, where R is a rational function of two variables. The substitution that works in all cases is $t = \tan\dfrac{x}{2}$ which also means that we have

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}$$

and we can hence reduce the original expression to a rational function, which we can then apply the technique for rational functions to evaluate.

Obviously, applying this method requires the use of trigonometric identities (which can turn out to be very ugly). But in theory, we can be sure of obtaining the answer (though we do not know very well at the end of how many pages...)

Example 6 Let us evaluate $\displaystyle\int \frac{1}{2+\sin x}\ \mathrm{d}x$.

Using the substitution $t = \tan\dfrac{x}{2}$, we have

$$\begin{aligned}
\int \frac{1}{2+\sin x}\ \mathrm{d}x &= \int \frac{1}{2+\frac{2t}{1+t^2}} \frac{2}{1+t^2}\ \mathrm{d}t \\
&= \int \frac{2}{2(1+t^2)+2t}\ \mathrm{d}t \\
&= \int \frac{1}{t^2+t+1}\ \mathrm{d}t \\
&= \int \frac{1}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\ \mathrm{d}t \\
&= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C \\
&= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right) + C
\end{aligned}$$

Of course, one should not always turn to this substitution when we see a rational function of sine and cosine. In some cases, we can always try to see if the integral is of the form $\displaystyle\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\ \mathrm{d}x$, such as $\displaystyle\int \frac{\sin x}{2+\cos x}\ \mathrm{d}x$ (of course the substitution will still work).

A particular case

In the case where the rational function of sine and cosine is of the form $\int \cos^m x \sin^n x \, \mathrm{d}x$, the substitution used can be made simpler:

- If m is odd, we can use the substitution $t = \sin x$;
- If n is odd, we can use the substitution $t = \cos x$;
- If m and n are even, we can 'linearize' the expression using the double angle formula.

Try to understand why these substitutions work!

Exercise 7 Evaluate the following integrals:

(i) $\int \cos^3 x \sin^4 x \, \mathrm{d}x$;

(ii) $\int \cos^4 x \sin^2 x \, \mathrm{d}x$;

(iii) $\int \frac{\cos x - \sin x}{1 + \cos^2 x} \, \mathrm{d}x$.

3 Maclaurin Series

For H3 Mathematics, instead of working with just the first few terms in the series expansion, you should be at ease with working with the general term.

Techniques to find Series Expansions

In H2 Mathematics, you have seen how to use standard series or repeated differentiation to find Maclaurin series. Here we consider a few more approaches.

A power series can be differentiated or integrated term by term to get new series, as the following example illustrates.

Example 7 To find the series expansion of $y = \tan^{-1} x$, we start with

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - \ldots$$

and integrate to obtain

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C$$

where C is an arbitrary constant. To determine C, we note that when $x = 0$, $y = \tan^{-1} 0 = 0$ and thus $C = 0$.

Another approach is to form a differential equation and differentiate that repeatedly. You should already be familiar with this in H2 Mathematics, but by making use of the generalized product rule $(\mathrm{fg})^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \mathrm{f}^{(k)} \mathrm{g}^{(n-k)}$ we can do it even more efficiently.

Example 8

Returning to $y = \tan^{-1} x$, we have $(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 1$. Differentiating with respect to x gives

$$(1+x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Using the generalized product rule we thus have

$$(1+x^2)\frac{\mathrm{d}^{n+2} y}{\mathrm{d}x^{n+2}} + \binom{n}{1} 2x \frac{\mathrm{d}^{n+1} y}{\mathrm{d}x^{n+1}} + \binom{n}{2} 2 \frac{\mathrm{d}^{n} y}{\mathrm{d}x^{n}} + 2x\frac{\mathrm{d}^{n+1} y}{\mathrm{d}x^{n+1}} + \binom{n}{1} 2 \frac{\mathrm{d}^{n} y}{\mathrm{d}x^{n}} = 0 .$$

Substituting $x = 0$ and simplifying, we get $y^{(n+2)}(0) = -n(n+1)y^{(n)}(0)$. Using this relation and the fact that $y(0) = 0, y'(0) = 1$ and $y''(0) = 0$, we have $y^{(2m)}(0) = 0, y^{(2m+1)}(0) = (-1)^m (2m)!$. This again gives the same result in Example 7, that $\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$.

4 Differential Equations

In your H2 Mathematics syllabus you are very much restricted to solving differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(y)$ and those reducible to these two via substitutions. For the H3 syllabus, you should still be able to use substitutions to reduce the differential equations, as well as formulating a differential equation from novel problem situations.

We shall equip you with 2 additional simple tools to tackle solving of first order differential equations. You need not know them, but you can be easily guided to solve problems using these techniques.

Variable Separable Equations (in H3 syllabus)

A first order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y), \qquad \textbf{(1)}$$

where f(x) is a function of x and g(y) is a function of y, is said to be **separable** or to have **separable variables**.

Rewriting **(1)**, we have

$$\frac{1}{\mathrm{g}(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x) \qquad \textbf{(2)}$$

provided that g(y) ≠ 0.

Integrating **(2)** with respect to x, we have

$$\int \left(\frac{1}{\mathrm{g}(y)}\frac{\mathrm{d}y}{\mathrm{d}x} \right) \mathrm{d}x = \int \mathrm{f}(x)\,\mathrm{d}x\,.$$

Hence, the general solution of the above variable separable first order differential equation is given by

$$\int \frac{1}{\mathrm{g}(y)}\,\mathrm{d}y = \int \mathrm{f}(x)\,\mathrm{d}x$$

which will introduce an arbitrary constant.

Note that this is not very much different from solving $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(y)$; the whole idea is to separate the x and y variables on both sides of the expression which we eventually integrate.

Example 9

Find the general solution of the differential equation $y(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x}-1=y^2$.

Rearranging the expression such that all the y appears on the LHS and the x on the RHS,

$$\int \frac{y}{y^2+1}\ \mathrm{d}y=\int \frac{1}{1+x^2}\ \mathrm{d}x$$

$\frac{1}{2}\ln(y^2+1)=\tan^{-1}x+C$, where C is an arbitrary constant.

$$\ln(y^2+1)=2\tan^{-1}x+2C$$

$$y^2+1=\mathrm{e}^{2\tan^{-1}x+2C}$$

$y^2+1=A\mathrm{e}^{2\tan^{-1}x}$, where $A=\mathrm{e}^{2C}$.

Integrating Factor (not in H3 syllabus)

The idea of the integrating factor is to make one side of a differential equation an exact differential. For example, consider the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x}+y=2$. If you recall the product rule, you might recognize that the expression $x\frac{\mathrm{d}y}{\mathrm{d}x}+y$ can be written as $\frac{\mathrm{d}}{\mathrm{d}x}(xy)$. Thus the differential equation integrates trivially to $xy = 2x + c$.

Differential equations of this type are called *exact differential equations*. Of course, we will not always get such nice differential equations. But we will discuss how we can introduce an 'integrating factor' which makes the differential equation exact.

Example 10

Find the general solution of the differential equation

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}+y^2=\mathrm{e}^x.$$

By observation, if we multiply this equation throughout by e^x**,** we obtain

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^x+y^2\mathrm{e}^x=\mathrm{e}^{2x}.$$

Recall the product rule of differentiation, this is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\mathrm{e}^x\right)=\mathrm{e}^{2x}.$$

Integrating on both sides, the general solution is $y^2\mathrm{e}^x=\frac{1}{2}\mathrm{e}^{2x}+C$.

The term e^x is also known as the integrating factor, and this approach works for linear first order differential equations. Let us detail this method.

Method of Integrating Factor (not in H3 syllabus)

1. Write the first order linear differential equation in the standard form

$$\frac{dy}{dx} + p(x)y = q(x)$$

   In particular, the coefficient of $\frac{dy}{dx}$ must be 1.

2. Find the integrating factor $u(x) = e^{\int P(x)dx}$.
3. Multiply the differential equation by the integrating factor

$$\left(e^{\int p(x)dx}\right)\frac{dy}{dx} + p(x)\left(e^{\int p(x)dx}\right)y = q(x)\left(e^{\int p(x)dx}\right).$$

4. Write the result obtained above as

$$\frac{d}{dx}\left(y\,e^{\int p(x)dx}\right) = q(x)\left(e^{\int p(x)dx}\right)$$

   (Since $\frac{d}{dx}\left(y\,e^{\int p(x)dx}\right) = \left(e^{\int p(x)dx}\right)\frac{dy}{dx} + p(x)\left(e^{\int p(x)dx}\right)y$)

5. Integrate both sides with respect to x to obtain the solution as

$$y\,e^{\int p(x)dx} = \int q(x)\left(e^{\int p(x)dx}\right)dx.$$

Question:
Why is it not necessary to include any arbitrary constant in finding the integrating factor?

Although the method of integrating factor is not in the H3 syllabus, you can be guided to do questions which involves them, as the following exercise shows.

Exercise 8

Show, by means of the substitution $y = \frac{1}{z}$, that the differential equation

$$(1+x)\frac{dy}{dx} - 2y + (1+x)y^2 = 0 \qquad (*)$$

reduces to

$$\frac{dz}{dx} + \left(\frac{2}{1+x}\right)z = 1.$$

By considering $\frac{d}{dx}\left((1+x)^2 z\right)$ or otherwise, solve the differential equation (*) for $x > -1$, given that $y = 1$ when $x = 0$.

Exercise 9 [9810/2009/3]

Air pressure, P pascals, reduces with height above sea level, h metres. At any height, the rate of change of air pressure with respect to height is proportional to the air pressure. On the summit of Mount Everest, at a height of 8848 metres above sea level, air pressure is approximately one third of its value at sea level. Show that air pressure halves for approximately **every** 5600 metres of height gained.

Tutorial

1. [RI/2014/Lecture Test 2/Q5]

Let f be a function on $[0,\infty)$, defined by $\mathrm{f}(x)=\ln(\mathrm{e}^{2x}+2\mathrm{e}^{-x})$.
Let C denote the curve $y=\mathrm{f}(x)$.

(i) Show that $\mathrm{f}(x)=2x+\ln(1+2\mathrm{e}^{-3x})$ [1]

(ii) Show that f is strictly increasing on $[0,\infty)$. [2]

(iii) Hence, sketch the curve C and the curve $y = 2x$ on the same diagram. [2]

(iv) Show that $\ln(1+u)\le u$ for all $u\ge 0$ [3]

Let $A(\alpha)$ denote the area bounded between the curves C and $y = 2x$, and the lines $x = 0$ and $x=\alpha$.

(v) Show that $A(\alpha)\le\dfrac{2}{3}$ for all real α. [4]

2. [RI/9824/Prelims/2]

A sequence of polynomials $P_n(x)$ is defined recursively by $P_1(x)=x-\frac{1}{2}$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n(x)=nP_{n-1}(x), \qquad P_n(0)=P_n(1)$$

for $n\ge 2$.

(i) By considering $P_3(0)=P_3(1)$, show that $P_2(x)=x^2-x+\frac{1}{6}$. [2]

(ii) Find $P_3(x)$ and $P_4(x)$. [3]

(iii) Prove, using Mathematical Induction, that

$$P_{n+1}(x+1)=P_{n+1}(x)+(n+1)x^n$$

for $n\ge 0$. [4]

(iv) Use the results in **(ii)** and **(iii)** to obtain a formula for $\sum_{x=0}^{m}x^3$. [3]

3. [9824/2015/3]

(i) Prove that $\int\tan^{n+2}x\ \mathrm{d}x+\int\tan^n x\ \mathrm{d}x=\dfrac{\tan^{n+1}x}{n+1}+c$, for $n\ge 0$, where c is an arbitrary constant. [2]

(ii) Hence find $\int\tan^5 x\ \mathrm{d}x$. [3]

(iii) Let $\mathrm{f}(x)$, $\mathrm{g}(x)$ and $\mathrm{h}(x)$ be functions of x such that $\mathrm{f}'(x)=x\mathrm{g}'(x)$ and $\mathrm{h}'(x)=(1+x^2)\mathrm{g}'(x)\mathrm{g}(x)$. Use integration by parts to find the following. [Your answers must be given in terms of one or more of $\mathrm{f}(x)$, $\mathrm{g}(x)$ and $\mathrm{h}(x)$.]

(a) $\int\mathrm{g}(x)\ \mathrm{d}x$ [2]

(b) $\int\mathrm{f}(x)\mathrm{g}(x)\ \mathrm{d}x$ [4]

(iv) Hence find $\int\ln(1+x^2)\tan^{-1}x\ \mathrm{d}x$. [3]

A graphing calculator must not be used for this question.

4 [RI/9820/2017/Prelims/Q6]

(i) Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{f}\left(\frac{1}{n}\right)+\mathrm{f}\left(\frac{2}{n}\right)+...+\mathrm{f}\left(\frac{n}{n}\right)\right\}$$

is $\int_0^1 \mathrm{f}(x)\ \mathrm{d}x$. [2]

(ii) Hence evaluate $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{(3n^2+k^2)}}$, leaving your answer in exact form. [4]

(iii) **(a)** By considering the function $\mathrm{g}(x)=x-\sin x$, show that $\sin x\le x$ for $x\ge 0$.

Hence or otherwise, show that $x-\frac{x^3}{6}\le \sin x$ for $x\ge 0$. [4]

(b) Deduce that $\sum_{k=1}^{n}\left(\frac{k}{n^2}-\frac{k^3}{6n^6}\right)\sin\frac{k}{n}\le\sum_{k=1}^{n}\sin\frac{k}{n^2}\sin\frac{k}{n}\le\sum_{k=1}^{n}\frac{k}{n^2}\sin\frac{k}{n}$. [1]

(c) Hence determine the exact value of $\lim_{n\to\infty}\sum_{k=1}^{n}\sin\frac{k}{n^2}\sin\frac{k}{n}$. [3]

5. [RI/2013/Lecture Test 1/Q6]

Let g be the function defined on (–1, 1) such that g(0) = 0 and $\mathrm{g}'(x)=\frac{1}{\sqrt{1-x^2}}$. Let h be the composite function defined on $(-\pi,\ 0)$ by h(x) = g(cos x).

(i) Determine $\mathrm{h}'(x)$. [2]

(ii) Calculate $\mathrm{h}(-\frac{\pi}{2})$ and hence deduce an expression for h(x). [3]

6. [RI/2013/Lecture Test 1/Q7]

Let $\mathrm{f}:[0,1]\to\mathbb{R}_0^+$ be a continuous function. Suppose that there exists a real number a such that for all $t\in[0,1]$, $\mathrm{f}(t)\le a\int_0^t \mathrm{f}(u)\ \mathrm{d}u$. Let $\mathrm{F}(t)=\int_0^t \mathrm{f}(u)\ \mathrm{d}u$.

(i) Show that the function $\mathrm{g}(t)=\mathrm{F}(t)\mathrm{e}^{-at}$ is decreasing on [0, 1]. [2]

(ii) Hence or otherwise, show that f is identically zero. [3]

7. [RI/9824/2013/Prelims/Q2]

(a) Let $S_n = 1 - \frac{1}{3} + \frac{1}{5} - ... + (-1)^n \frac{1}{2n+1}$.

(i) State the sum of the series $1 - x^2 + x^4 - ... + (-1)^n x^{2n}$. [1]

(ii) Hence show that $S_n = \frac{\pi}{4} + (-1)^n \int_0^1 \frac{x^{2n+2}}{1+x^2}\, \mathrm{d}x$. [2]

(iii) Deduce that S_n differs from $\frac{\pi}{4}$ by at most $\frac{1}{2n+3}$. [2]

(b) (i) Given that $I_n = \int_0^{\frac{\pi}{4}} \frac{\sin(2n+1)x}{\sin x}\, \mathrm{d}x$, show that, for any positive integer n,

$$I_n - I_{n-1} = \frac{1}{n}\sin\left(\frac{1}{2}n\pi\right)$$ [2]

(ii) Hence or otherwise find the exact value of $\int_0^{\frac{\pi}{4}} \frac{\sin 11x}{\sin x}\, \mathrm{d}x$. [2]

8. [9824/2016/1]

Do not use a calculator in answering this question.

By first using the substitution $u = x + \frac{1}{x}$, find the exact value of

$$\int_1^{\frac{1}{2}(3+\sqrt{5})} \frac{x^2 - 1}{(x^2+1)\sqrt{x^4+x^2+1}}\, \mathrm{d}x.$$ [10]

9. [Specimen Paper/9820/Q2]

(i) Find the exact value of $\int_4^9 \frac{u}{\sqrt{u}-1}\, \mathrm{d}u$. [5]

(ii) Show that the differential equation $\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}\left(\frac{y}{x}\right) + \frac{y}{x^2}$ can be transformed into the equation $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{f}(u)$ by the substitution $y = xu$. [3]

(iii) A solution curve of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\sqrt{\frac{x}{y}} + \frac{y}{x} - \frac{x^2}{y}$$

passes through the point $\left(\frac{1}{3}, \frac{4}{3}\right)$. Find the exact value of the x-coordinate of the point where this curves intersects the line $y = 9x$. [4]

10. [9824/2014/3]

(i) **(a)** For any positive integer n, prove that

$$\frac{1}{1+x^2}=1-x^2+x^4-...+(-1)^{n-1}x^{2n-2}+(-1)^n\frac{x^{2n}}{1+x^2}\ .$$ [2]

(b) Let D be the difference between $\frac{\pi}{4}$ and the sum of the first 1000 terms of the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...$

By considering $\int_0^1 \frac{x^{2n}}{1+x^2}\ \mathrm{d}x$, show that $\frac{1}{2001}>D>\frac{1}{4002}$. [7]

(ii) By expressing $1+x^4$ as $\left(1-x^2+x^4\right)+x^2$, find the exact value of

$$\int_0^1 \frac{1+x^4}{1+x^6}\ \mathrm{d}x\ .$$ [6]

11. [RI/9824/Prelims/3]

(i) Let a be a positive real number and f a function defined on $[0,a]$ satisfying $\mathrm{f}(x)=\mathrm{f}(a-x)$. Show that

$$\int_0^a x\mathrm{f}(x)\ \mathrm{d}x=\frac{a}{2}\int_0^a \mathrm{f}(a-x)\ \mathrm{d}x.$$

[2]

(ii) Hence, prove that for $n\in\mathbb{Z}^+$,

$$\int_0^\pi x\sin^n x\ \mathrm{d}x=\frac{\pi}{2}\int_0^\pi \sin^n x\ \mathrm{d}x=\pi\int_0^{\frac{\pi}{2}}\sin^n x\ \mathrm{d}x.$$

[2]

(iii) For nonnegative integers n, let $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\ \mathrm{d}x$. Show that $nI_n=(n-1)I_{n-2}$ where $n\geq 2$. [3]

(iv) Hence find the exact value of $\int_0^\pi x\sin^{11} x\ \mathrm{d}x$. [3]

(v) Show that the sequence defined by $x_n=nI_nI_{n-1}$ for $n\in\mathbb{Z}^+$ is constant. [1]

(vi) Hence show that, for $n\in\mathbb{Z}^+$, $\sqrt{\frac{\pi}{2(n+1)}}\leq I_n\leq\sqrt{\frac{\pi}{2n}}$. [3]

Assignment

1. (i) Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x}, \quad (x > 0, y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x}, \quad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x}, \quad (x > 0, y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x}, \quad (x > 0, y > 0)$$

that satisfies $y = 2$ when $x = 1$.

2 (i) It is given that A, B and C are real numbers. Show that the quadratic function $\mathrm{h}(t) = At^2 + 2Bt + C$ is non-negative for all real values of t if and only if $A > 0$ and $AC \geq B^2$.

(ii) Let f and g be continuous functions on the interval $[a, b]$. By considering $\int_a^b \left(t\mathrm{f}(x) + \mathrm{g}(x)\right)^2 \,\mathrm{d}x$, show that

$$\left(\int_a^b \left(\mathrm{f}(x)\right)^2 \mathrm{d}x\right)\left(\int_a^b \left(\mathrm{g}(x)\right)^2 \mathrm{d}x\right) \geq \left(\int_a^b \mathrm{f}(x)\mathrm{g}(x)\mathrm{d}x\right)^2$$

and determine when equality holds.

(iii) Hence show that $\int_0^1 \sqrt{1 + x^3} \,\mathrm{d}x < \frac{\sqrt{5}}{2}$.

(iv) Let k be a continuous and differentiable function on the interval $[0,1]$ satisfying $\mathrm{k}(1) = 0$. Show that

$$\left(\int_0^1 \mathrm{k}(x) \,\mathrm{d}x\right)^2 \leq \frac{1}{3}\left(\int_0^1 \left(\mathrm{k}'(x)\right)^2 \,\mathrm{d}x\right)$$

and determine when equality holds.

3 For every nonnegative integer n, define $I_n = \int_0^1 x^{2n+1}\mathrm{e}^{-x^2}\,\mathrm{d}x$.

(i) Show that for $n \geq 1$,

$$I_n = nI_{n-1} - \frac{1}{2\mathrm{e}}.$$ [3]

(ii) Show that for all positive integers n,

$$I_n = \frac{1}{2}(n!)\left(1 - \frac{1}{\mathrm{e}}S_n\right), \quad \text{where } S_n = \sum_{r=0}^{n}\frac{1}{r!}.$$ [4]

(iii) Show that $0 < I_n < \dfrac{1}{2n+2}$. [2]

(iv) Deduce that S_n differs from e by at most $\dfrac{\mathrm{e}}{(n+1)!}$. [2]

Additional Practice Questions

Refer to the compilation of 2010 to 2019 STEP I and II problems. A graphing calculator should not be used for all these questions.

1. **2010/STEP I/4**

2. **2010/STEP II/4**

3. **2011/STEP II/6**

4. **2012/STEP I/5**

5. **2013/STEP II/2**

6. **2014/STEP II/4**

7. **2015/STEP II/6**

8. **2016/STEP II/8**

9. **2017/STEP I/6**

10. **2017/STEP II/1**

11. **2018/STEP I/8**

12. **2019/STEP II/2**