RAFFLES INSTITUTION
2024 YEAR 6 LECTURE TEST 3
Higher 3

MATHEMATICS **9820**

22 August 2024

2 hours

Additional materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Write the number and part, for example 1a, you are responding to clearly.

1 Let n be a positive integer and denote by $\sigma(n)$ the sum of all the positive divisors of n. We say that a number n is perfect if $\sigma(n) = 2n$. For example, the 2 smallest perfect numbers are 6 and 28 since $\sigma(6) = 1 + 2 + 3 + 6 = 12$ and $\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56$.

(a) If p and $2^p - 1$ are prime numbers, show that $n = 2^{p-1}(2^p - 1)$ is a perfect number. [3]

(b) Write down 2 perfect numbers other than 6 and 28. [2]

(c) Show that the last digit of all perfect numbers of the form $n = 2^{p-1}(2^p - 1)$ is either 6 or 8. [5]

2 Let $\mathrm{S}(r, n)$ denote the number of ways of distributing r distinct objects into n identical boxes so that no box is empty.

(a) State the value of $\mathrm{S}(r, r-1)$. [1]

(b) Explain why $\mathrm{S}(r, 2) = 2^{r-1} - 1$. [2]

(c) Prove that $\mathrm{S}(r, n) = n\mathrm{S}(r-1, n) + \mathrm{S}(r-1, n-1)$ for $1 \leqslant n \leqslant r$. [3]

(d) Hence use mathematical induction to show that $\mathrm{S}(r, r-2) = \binom{r}{3} + 3\binom{r}{4}$ for $r \geqslant 3$. [5]

(e) Use a combinatorial argument to explain why the result in **(d)** is true. [3]

3 **(a)** Let $A = \{1, 2, ..., p\}$ and $\mathrm{f} : A \to A$. By considering the number of functions f that are surjective, use the principle of inclusion-exclusion to show that

$$p! = \sum_{r=0}^{p} (-1)^r \binom{p}{r} (p-r)^p. \qquad [5]$$

(b) Let p be a prime number. Hence show that

$$(p-1)! \equiv (-1)^{p+1} \sum_{r=1}^{p-1} r^{p-1} \text{ (modulo } p\text{)}. \qquad [3]$$

(c) Let p be a prime number and let n be a positive integer. Use mathematical induction to show that $n^p \equiv n$ (modulo p). [4]

(d) Deduce that $(p-1)! \equiv -1$ (modulo p) for a prime number p. [2]

4 You may assume that all integrals in this question converges.

(a) Show that

$$2\sum_{k=1}^{n}\cos(2kx)\sin x = \sin(2n+1)x - \sin x. \qquad [3]$$

(b) Deduce that for all positive integers n,

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\,\mathrm{d}x = \frac{\pi}{2}. \qquad [3]$$

Let f be a continuous function defined on $[a,b]$ such that it is also differentiable and its derivative f′ is also continuous on $[a,b]$.

(c) Use integration by parts to show that $\lim\limits_{n\to\infty}\displaystyle\int_a^b \mathrm{f}(x)\sin(nx)\,\mathrm{d}x = 0.$ [3]

Let g be defined on $[0, \frac{\pi}{2}]$ by

$$\mathrm{g}(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{\sin x} & \text{if } 0 < x \leqslant \dfrac{\pi}{2}, \\ 0 & \text{if } x = 0. \end{cases}$$

You may assume that the function g and its derivative g′ is continuous on $[0, \frac{\pi}{2}]$.

(d) Hence evaluate $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x.$ [3]

5 Let $\{x_1, x_2, ..., x_n\}$ and $\{y_1, y_2, ..., y_n\}$ be two sequences of real numbers. We say that the sequence $\{x_1, x_2, ..., x_n\}$ **majorizes** the $\{y_1, y_2, ..., y_n\}$, if the following conditions are fulfilled:

- $x_1 \geqslant x_2 \geqslant ... \geqslant x_n$;
- $y_1 \geqslant y_2 \geqslant ... \geqslant y_n$;
- $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$;
- $x_1 + x_2 + \cdots + x_k \geqslant y_1 + y_2 + \cdots + y_k$ for all $1 \leqslant k \leqslant n-1$.

For example, $\{3, 0, 0\}$ majorizes $\{2, 1, 0\}$ and $\{2, 1, 0\}$ majorizes $\{1, 1, 1\}$.

Let f be a convex function defined over the real numbers.

(a) Use a sketch to explain why if $x \leqslant y < z$ then

$$\frac{\mathrm{f}(z) - \mathrm{f}(x)}{z - x} \leqslant \frac{\mathrm{f}(z) - \mathrm{f}(y)}{z - y}.$$ [2]

Let $\{a_1, a_2, ..., a_n\}$ and $\{b_1, b_2, ..., b_n\}$ be two sequences of real numbers such that $\{a_1, a_2, ..., a_n\}$ **majorizes** the sequence $\{b_1, b_2, ..., b_n\}$, and let $c_i = \dfrac{\mathrm{f}(b_i) - \mathrm{f}(a_i)}{b_i - a_i}$. Define the sequences $\{A_i\}$ and $\{B_i\}$ by $A_0 = 0, A_k = \sum_{i=1}^{k} a_i$ and $B_0 = 0, B_k = \sum_{i=1}^{k} b_i$.

(b) **(i)** Show that $\sum_{i=1}^{n} (\mathrm{f}(a_i) - \mathrm{f}(b_i)) = \sum_{i=1}^{n-1} (c_i - c_{i+1})(A_i - B_i)$. [3]

(ii) Deduce that

$$\mathrm{f}(a_1) + \mathrm{f}(a_2) + \cdots + \mathrm{f}(a_n) \geqslant \mathrm{f}(b_1) + \mathrm{f}(b_2) + \cdots + \mathrm{f}(b_n).$$ [3]

(c) Let a, b, c be positive real numbers. Use the result in **(b)** to show that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leqslant \frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c}.$$ [3]

(d) Let $x_1, x_2, ..., x_n \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$. Show that

$$\cos(2x_1 - x_2) + \cos(2x_2 - x_3) + \cdots + \cos(2x_n - x_1) \leqslant \cos x_1 + \cos x_2 + \cdots + \cos x_n.$$ [4]