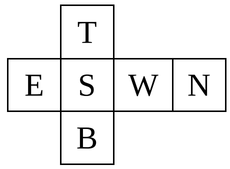
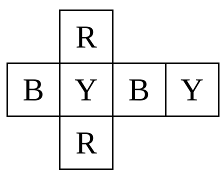
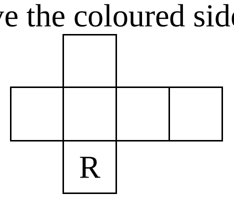
Q1	Solutions	Comments
(i)	$\frac{1}{\left(N+\sqrt{N^2-1}\right)^k} = \frac{\left(N-\sqrt{N^2-1}\right)^k}{\left(N+\sqrt{N^2-1}\right)^k\left(N-\sqrt{N^2-1}\right)^k}$ $= \frac{\left(N-\sqrt{N^2-1}\right)^k}{\left(N^2-\left(N^2-1\right)\right)^k}$ $= \frac{\left(N-\sqrt{N^2-1}\right)^k}{1^k}$ $= \left(N-\sqrt{N^2-1}\right)^k$ $\left(N+\sqrt{N^2-1}\right)^k + \frac{1}{\left(N+\sqrt{N^2-1}\right)^k}$ $= \left(N+\sqrt{N^2-1}\right)^k + \left(N-\sqrt{N^2-1}\right)^k$ $= \sum_{i=0}^{k}\binom{k}{i}N^{k-i}\left(N^2-1\right)^{\frac{i}{2}} + \sum_{i=0}^{k}\binom{k}{i}N^{k-i}\left(-\left(N^2-1\right)^{\frac{1}{2}}\right)^i$ $= \sum_{i=0}^{k}\binom{k}{i}N^{k-i}\left(N^2-1\right)^{\frac{i}{2}} + \sum_{i=0}^{k}\binom{k}{i}N^{k-i}\left(-1\right)^i\left(N^2-1\right)^{\frac{i}{2}}$ $= \sum_{i=0}^{k}\binom{k}{i}\left(1+\left(-1\right)^i\right)N^{k-i}\left(N^2-1\right)^{\frac{i}{2}}$ $= \begin{cases} 2\binom{k}{0}N^k + 2\binom{k}{2}N^{k-2}\left(N^2-1\right) + 2\binom{k}{4}N^{k-4}\left(N^2-1\right)^2 + \dots \\ +2\binom{k}{k}\left(N^2-1\right)^{\frac{k}{2}}, & \text{if } k \text{ is even} \\ 2\binom{k}{0}N^k + 2\binom{k}{2}N^{k-2}\left(N^2-1\right) + 2\binom{k}{4}N^{k-4}\left(N^2-1\right)^2 + \dots \\ +2\binom{k}{k-1}N\left(N^2-1\right)^{\frac{k-1}{2}}, & \text{if } k \text{ is odd} \end{cases}$	Many students tried to show this using MI. This is a valid method, but either requires binomial expansion similar to the solution here or uses $\left[x^k+x^{-k}\right]\left(x^1+x^{-1}\right)$ $= x^{k+1}+x^{k-1}+x^{-(k+1)}+x^{-(k+1)}$ which requires 2 predecessors and therefore 2 base cases.
	For k even, $\frac{k}{2}$ is an integer, hence $2\binom{k}{0}N^k + 2\binom{k}{2}N^{k-2}\left(N^2-1\right) + 2\binom{k}{4}N^{k-4}\left(N^2-1\right)^2 + \dots$ $+2\binom{k}{k}\left(N^2-1\right)^{\frac{k}{2}}$ is an integer since N and $\binom{k}{r}$ are also integers	

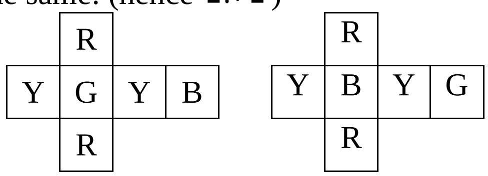
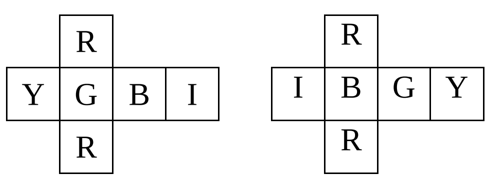
	For k odd, $\frac{k-1}{2}$ is an integer, hence $2\binom{k}{0}N^k+2\binom{k}{2}N^{k-2}\left(N^2-1\right)+2\binom{k}{4}N^{k-4}\left(N^2-1\right)^2+\ldots$ $+2\binom{k}{k-1}N\left(N^2-1\right)^{\frac{k-1}{2}}$ is an integer since N and $\binom{k}{r}$ are also integers. Therefore $\left(N+\sqrt{N^2-1}\right)^k+\frac{1}{\left(N+\sqrt{N^2-1}\right)^k}$ is an integer for all positive integer k.	
(ii)	Let $I=\left(N+\sqrt{N^2-1}\right)^k+\frac{1}{\left(N+\sqrt{N^2-1}\right)^k}$, where I is the integer from part (i). Since $\frac{1}{\left(N+\sqrt{N^2-1}\right)^k}$ is positive and $\frac{1}{\left(N+\sqrt{N^2-1}\right)^k}<\frac{1}{N^k}\le\frac{1}{2^k}\le\frac{1}{2}$ for $N\ge 2$, $I>\left(N+\sqrt{N^2-1}\right)^k>I-\frac{1}{2}$. Therefore the integer closest to $\left(N+\sqrt{N^2-1}\right)^k$ is I.	Some students neglected to show that what the closest integer is. Alternatively, students should at least mention that the closest integer is AT MOST $\left(N+\sqrt{N^2-1}\right)^k$ away.
	Notice that for any real number $x>\frac{1}{2}$, we have $2x-\frac{1}{2}<x+\sqrt{x^2-1}$ $\Leftrightarrow x-\frac{1}{2}<\sqrt{x^2-1}$ $\Leftrightarrow \left(x-\frac{1}{2}\right)^2<x^2-1,$ $\Leftrightarrow x^2-x+\frac{1}{4}<x^2-1$ $\Leftrightarrow x>\frac{5}{4}$	Most students were able to show this part of the inequality.

	Hence for $N \geq 2 > \frac{5}{4}$, we have $2N - \frac{1}{2} < N + \sqrt{N^2 - 1}$ $\Rightarrow \frac{1}{N + \sqrt{N^2 - 1}} < \frac{1}{2N - \frac{1}{2}}$ $\Rightarrow \frac{1}{\left(N + \sqrt{N^2 - 1}\right)^k} < \left(2N - \frac{1}{2}\right)^{-k}$ Hence $\left(N + \sqrt{N^2 - 1}\right)^k$ differs from I by less than $\left(2N - \frac{1}{2}\right)^{-k}$.	

Q2	Solutions	Comments
(i)	$(1+ax)(1+bx)(1+cx)$ $=1+(a+b+c)x+(ac+bc+ab)x^2+abcx^3$ $=1+(ac+bc+ab)x^2+abcx^3$ Therefore, $q=ac+bc+ab,\ r=abc$ and $a+b+c=0.$	
(ii)	$\ln(1+qx^2+rx^3)$ $=(qx^2+rx^3)-\dfrac{(qx^2+rx^3)^2}{2}+\dots$ $=qx^2+rx^3-\dfrac{q^2}{2}x^4-qrx^5+\dots$	Some students did not use the Maclaurin expansion in MF26, and instead went through differentiation.
(iii)	$\ln(1+qx^2+rx^3)=\ln(1+ax)+\ln(1+bx)+\ln(1+cx)$ Coefficient of $x^n=\dfrac{(-1)^{n+1}a^n}{n}+\dfrac{(-1)^{n+1}b^n}{n}+\dfrac{(-1)^{n+1}c^n}{n}=(-1)^{n+1}T_n$ where $T_n=\dfrac{a^n+b^n+c^n}{n}$.	Most students were able to use the correct logarithm law. They should directly look for the general term in the expansion (given in MF26), instead of generalizing from the coefficients of $x, x^2, x^3, \dots$
(iv)	$(-1)^{2+1}T_2=q\Rightarrow T_2=-q$ $(-1)^{3+1}T_3=r\Rightarrow T_3=r$ $(-1)^{5+1}T_5=-qr\Rightarrow T_5=-qr$ Hence $\dfrac{(a^2+b^2+c^2)(a^3+b^3+c^3)}{6}$ $=\dfrac{(a^2+b^2+c^2)}{2}\dfrac{(a^3+b^3+c^3)}{3}$ $=T_2T_3$ $=-qr$ $=T_5$ $=\dfrac{a^5+b^5+c^5}{5}$	The question states that the results in (ii) and (iii) should be used. Students need to follow this instruction strictly.

Q3	Solutions	Comments
	Using $u = x: 0 + \mathrm{a}(x)(1) + \mathrm{b}(x)x = 0$ Using $u = \mathrm{e}^x$: $\mathrm{e}^x + \mathrm{a}(x)(\mathrm{e}^x) + \mathrm{b}(x)\mathrm{e}^x = 0$, which implies $1 + \mathrm{a}(x) + \mathrm{b}(x) = 0$ since $\mathrm{e}^x > 0$. Subtracting the two equations, we obtain $1 + (1 - x)\mathrm{b}(x) = 0$. Therefore, $\mathrm{b}(x) = \dfrac{1}{x-1}$ and $\mathrm{a}(x) = -\dfrac{x}{x-1}$.	
(i)	From $y = \dfrac{1}{u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$, we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{u}\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \dfrac{1}{u^2}\left(\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)^2$ $\Leftrightarrow \dfrac{1}{u}\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \dfrac{1}{u^2}\left(\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \dfrac{1}{u^2}\left(\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \dfrac{x}{1-x}\left(\dfrac{1}{u}\dfrac{\mathrm{d}u}{\mathrm{d}x}\right) = \dfrac{1}{1-x}$ $\Leftrightarrow \dfrac{1}{u}\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} + \dfrac{x}{1-x}\left(\dfrac{1}{u}\dfrac{\mathrm{d}u}{\mathrm{d}x}\right) = \dfrac{1}{1-x}$ $\Leftrightarrow \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} + \dfrac{x}{1-x}\dfrac{\mathrm{d}u}{\mathrm{d}x} + \dfrac{1}{x-1}u = 0$	Many students wrongly wrote $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{u}\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \dfrac{1}{u^2}\left(\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)$ (without the power 2 in the last term).
(ii)	Using (1), the general solution is $u = Ax + B\mathrm{e}^x$. Therefore, $y = \dfrac{1}{u}\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{Ax + B\mathrm{e}^x}\left(A + B\mathrm{e}^x\right)$. When $x = 0, y = 2$, $2 = \dfrac{A+B}{B}$ $2B = A + B$ $B = A$. Hence, $y = \dfrac{1}{Ax + A\mathrm{e}^x}\left(A + A\mathrm{e}^x\right) = \dfrac{1+\mathrm{e}^x}{x+\mathrm{e}^x}$.	Some students found the values of $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ and/or $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 0, y = 2$. They are not required in the question. The approach should be to first find the general solution to (2), with the help of the general solution of (1) given in the question, and then substitute the initial condition to find the particular solution.

Q4	Solutions	Comments
	For convenience, we will draw all diagrams using a net diagram with sides Top, Bottom, East, South, West, North: 	Symmetry for this question is difficult. H2 syllabus deals with rotational symmetry. There is a need to consider other axis of rotation here.
(i)	3 colours. Since a vertex shares 3 faces adjacent to each other, they must be all different in colour. The only way a cube can be coloured in 3 colours is when opposite sides are coloured in the same colour. Hence, there are ${}^7C_3 = 35$ such cubes. In 3 colours, this is the only possible result up to rotation:  There are 35 such cubes.	This part was generally well done.
(ii)	Since 6 colours are used, we have ${}^7C_6 = 7$ ways to choose the colours used. With an uncoloured cube, there are too many rotational symmetry in 3D. Hence, we start by colouring in 1 side. If red is available, we will paint red. (o.w. use orange) Since the cube will end with 1 coloured side and 5 uncoloured, there is $6 \div 6 = 1$ way to colour it in. Rotate the cube to have the coloured side bottom.  This reduces the symmetry to 2D, similar to a square table question. Since only the top side is unique, there's 5 colour choice, and the remaining 4 sides is a square rotational symmetry.	Alternative method: $\dfrac{{}^7C_6 \times 6!}{{}^6C_1 \times 4}$ ${}^7C_6$ for choosing the colours, $6!$ for arranging 6 colours, ${}^6C_1$ to choose the bottom side (locking it in to reduce 3D to 2D), $\div 4$ for rotational symmetry.

	Number of ways $=7\times5\times\frac{4!}{4}=210$	
(iii)	If 4 colours are used, 2 colours must be used twice and painted opposite each other. There are ${}^{7}C_{2}=21$ ways these colour can be chosen and painted. The remaining 2 colours have ${}^{5}C_{2}=10$ choices and can be painted in $2!\div2$ ways. These are the same: (hence $2!\div2$)  Number of ways $=21\times10\times2!\div2=210$	Most students have some idea of tackling this question, but may have missed out some order of rotational symmetries or considered identical cases.
	If 5 colours are used, 1 colour must be used twice and painted opposite each other. There are 7 ways this colour can be chosen and painted. The remaining 4 colours have ${}^{6}C_{4}=15$ choices and can be painted in $\frac{4!}{4}\div2=3$ ways. ($\div2$ when the cube is flipped upside down) Eg. (similar when top/bottom flipped)  Number of ways $=7\times15\times3=315$	
	3-colours: 35 4-colours: 210 5-colours: 315 6-colours: 210 Total no. of ways $=35+210+315+210=770$	

Q5	Solutions	Comments
(i)	$\int_a^b x^2 \mathrm{d}x = \dfrac{b^3-a^3}{3} = \dfrac{(b-a)(b^2+ab+a^2)}{3}$ $\left(\int_a^b x\ \mathrm{d}x\right)^2 = \left(\dfrac{b^2-a^2}{2}\right)^2 = \dfrac{(b+a)^2(b-a)^2}{4}$ $\dfrac{(b-a)(b^2+ab+a^2)}{3} = \dfrac{(b+a)^2(b-a)^2}{4}$ Since $b-a\neq 0,$ we have $4(b^2+ab+a^2) = 3(b+a)^2(b-a)$ $4\left[(b+a)^2-ab\right] = 3(b+a)^2(b-a)$ $4(b+a)^2-4ab = 3(b+a)^2(b-a)$ $4p^2-4ab = 3p^2q$ Note that $4ab=(b+a)^2-(b-a)^2 = p^2-q^2$. Hence $4p^2-(p^2-q^2)=3p^2q$ $3p^2+q^2=3p^2q.$	Most students had no problem integrating and equating the two integrals, but some faced difficulties when trying to replace in terms of *p* and *q*.
(ii)	Since $b>a=1,$ by part **(i)**, $3(b+1)^2+(b-1)^2 = 3(b+1)^2(b-1)$ $3b^2+6b+3+b^2-2b+1=3(b+1)^2(b-1)$ $4(b^2+b+1)=3(b+1)^2(b-1)$ $3b^3-b^2-7b-7=0$	
(iii)	$3p^2(q-1)=q^2$ $p^2=\dfrac{q^2}{3(q-1)}$ $p^2-q^2=\dfrac{q^2}{3(q-1)}-q^2=\dfrac{q^2(1-3q+3)}{3(q-1)}=\dfrac{q^2(4-3q)}{3(q-1)}$ Since $p^2-q^2=4ab\geq 0,$ we have $\dfrac{q^2(4-3q)}{3(q-1)}\geq 0,$ implying $\dfrac{4-3q}{q-1}\geq 0$. Solving, we obtain $1<q\leq\dfrac{4}{3}.$	Many students were unable to do this part of the question.

	Since $b=q+a=q+1,$ we have $2<b\le\frac{7}{3}.$	

Q6	Solutions	
(a)	For a word of length $n+2$, the last letter can be same as the penultimate letter or not. If the last letter is different, then we have $4\times A(n+1)$. If the last 2 letters are the same, then the third last letter will be same as the initial last letter. We can remove the last 2 letters and there would still be an odd number of consecutive letters in the word. $A(n+2)=4\times A(n+1)+A(n)$ $A(1)=5$ $\quad A(2)=5\times4=20$ $A(3)=4(20)+5=85$ $\quad A(4)=4(85)+20=360$ $A(5)=4(360)+85=1525$	In recurrence relations questions, students are strongly encouraged to list out the base cases explicitly. In this case, $A(1)$ and $A(2)$.
(b)	This is equivalent to distributing 7 identical balls into 5 distinct boxes. (+1 to each of the five, hence 12-5=7) The total number of ways such that each letter used once and no restrictions is given by $=\binom{7+4}{4}=330$. It is easy to see that only 1 letter can appear at least 5 times and other letters appearing at least once. This is equivalent to distributing 3 identical balls into 5 distinct boxes. (+5 to one, +1 to other four, hence 12-5-4=3) Hence, there are $=\binom{3+4}{4}\times5=175$ ways this happens. Therefore, there are $330-175=155$ ways.	Some students tried to do this by PIE or cases. These are possible methods, but listing cases is the last resort as it is difficult to score method/working marks.
(c)	Let $\lvert X\rvert$ be the number of ways in which only X is chosen. Similarly for other letters. No. of ways at least 1 chosen $=\lvert X \text{ or } Y \text{ or } Z \text{ or } J \text{ or } C\rvert$ $=\binom{5}{1}1^7-\binom{5}{2}2^7+\binom{5}{3}3^7-\binom{5}{4}4^7+\binom{5}{5}5^7$ $=16800$ Probability $=16800\div5^7=\frac{672}{3125}$ or 0.21504 (exact)	Some students misunderstood the question or considered the total cases wrongly. Denominator should be 5^7 instead of $\binom{7+4}{4}$.

		$\binom{7+4}{4}$ has it that $n(7X)=n(6X1Y)=1$, but $1=n(7X)\neq n(6X1Y)=7$
(d)	14 ways by listing. $\left\{\begin{array}{lll} 10, & & \\ 9+1, & & \\ 8+2, & 8+1+1, & \\ 7+3, & 7+2+1, & \\ 6+4, & 6+3+1, & 6+2+2, \\ 5+5, & 5+4+1, & 5+3+2, \\ 4+4+2, & & 4+3+3 \end{array}\right\}$	In identical to identical, listing is the only method. List as many cases as possible to get working marks.

Q7	Solutions	
(i)	$g(x)=\sin x$ $g'(x)=\cos x$ $g''(x)=-\sin x$ For $0<x<\pi$, $\sin x>0$. Hence $g''(x)=-\sin x<0$. Therefore g is concave downwards on its domain $0<x<\pi$. $h(x)=\ln x$ $h'(x)=\frac{1}{x}$ $h''(x)=-\frac{1}{x^2}<0$ for $x>0$ Therefore h is concave downwards on its domain $x>0$.	Generally, well done.
(ii)	$\alpha+\beta+\gamma=\pi$, and $0<\alpha,\ \beta,\ \gamma<\pi$ since $\alpha,\ \beta,\ \gamma$ are interior angles of a triangle. Since the function $g(x)=\sin x$, $0<x<\pi$ is a concave downwards function (from (i)), using Jensen’s inequality, $\frac{1}{3}(\sin\alpha+\sin\beta+\sin\gamma)\le\sin\left(\frac{1}{3}(\alpha+\beta+\gamma)\right)$ $\Rightarrow \sin\alpha+\sin\beta+\sin\gamma\le 3\sin\left(\frac{1}{3}\pi\right)$ $\Rightarrow \sin\alpha+\sin\beta+\sin\gamma\le\frac{3\sqrt{3}}{2}$ We note that equality holds if and only if $\alpha=\beta=\gamma$.	$g(x)=\sin x$ is concave for the domain $0<x<\pi$ only. Condition is not provided in question directly, hence it needs to be mentioned or shown.

(iii)	From (i), $\mathrm{h}(x)=\ln x$, $x>0$ is a concave downwards function. Hence for any positive numbers $x_1,\ x_2,\ \ldots,\ x_n$, $\frac{1}{n}\sum_{k=1}^{n}\ln(x_k)\le\ln\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)$ $\Rightarrow\frac{1}{n}\left(\ln\left(x_1x_2\ldots x_n\right)\right)\le\ln\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)$ $\Rightarrow\ln\sqrt[n]{x_1x_2\ldots x_n}\le\ln\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)$ From (i), $\mathrm{h}'(x)=\frac{1}{x}>0$ for $x>0$. Therefore h is a strictly increasing function. Since h is strictly increasing, $\ln\sqrt[n]{x_1x_2\ldots x_n}\le\ln\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)$ $\Rightarrow\sqrt[n]{x_1x_2\ldots x_n}\le\frac{x_1+x_2+\ldots+x_n}{n}$ We note that equality holds if and only if $x_1=x_2=\ldots=x_n$.	Generally well done, but due to the number of marks given, students should try to show more working where possible. (ie. Explain that log-function is increasing or non-decreasing)
(iv)	$\frac{x^5+y^5+z^5+2^5+2^5}{5}\ge\sqrt[5]{2^5 2^5 x^5y^5z^5}$ $\Leftrightarrow\frac{x^5+y^5+z^5+64}{5}\ge 4xyz$ $\Leftrightarrow x^5+y^5+z^5-20xyz\ge -64$ and equality holds if and only if $x=y=z=2$. Therefore the minimum value of $x^5+y^5+z^5-20xyz$ is -64, attained when $x=y=z=2$.	Many students do not know where to start despite the hint of (iii). Due to the 5th power, 5 terms should be used and so 2 more k^5 should be added. Some students tried to use AM-GM, but the question wanted (iii) to be used. Note: Jensen's inequality is a very strong result which can show almost EVERY inequatlity. Hence, it cannot be quoted for use in A-levels.

Q8	Solutions	
(i)	Sub $x=y=0$, $\mathrm{f}(0+0)=\mathrm{f}(0)+\mathrm{f}(0)$ $\mathrm{f}(0)=2\mathrm{f}(0)$ $\mathrm{f}(0)=0$	Well done.
(ii)	Sub $y=-x$, $\mathrm{f}(x+(-x))=\mathrm{f}(x)+\mathrm{f}(-x)$ $\mathrm{f}(0)=\mathrm{f}(x)+\mathrm{f}(-x)$ $\mathrm{f}(-x)=-\mathrm{f}(x)$	Well done, but with some poor presentation.
(iii)	Let $P(n)$ be the proposition that $\mathrm{f}(nx)=n\mathrm{f}(x)$. Considering $P(1)$: $\mathrm{f}(1\times x)=1\times\mathrm{f}(x)$ Assume that $P(k)$ is true, consider $P(k+1)$. $\mathrm{f}(x+kx)=\mathrm{f}(x)+\mathrm{f}(kx)$ $=\mathrm{f}(x)+k\mathrm{f}(x)$ $=(k+1)\times\mathrm{f}(x)$ Hence $\mathrm{f}([k+1]x)=[k+1]\mathrm{f}(x)$. By Mathematical Induction, since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$, $P(n)$ is true for all $n\in\mathbb{Z}^+$.	Some students have poor MI presentation.
(iv)	By part **(iii)**, $\mathrm{f}(x)=\mathrm{f}\left(m\dfrac{x}{m}\right)=m\mathrm{f}\left(\dfrac{x}{m}\right)\Leftrightarrow\dfrac{1}{m}\mathrm{f}(x)=\mathrm{f}\left(\dfrac{x}{m}\right)$.	
(v)	When $q\in\mathbb{Q}^+$, $q=\dfrac{n}{m}$. By parts **(iii)** and **(iv)**, $\mathrm{f}\left(\dfrac{n}{m}\right)=\dfrac{n}{m}\mathrm{f}(1)$ for $m,n\in\mathbb{Z}^+$. Hence, $\mathrm{f}(x)=x\mathrm{f}(1)$ for all $\dfrac{n}{m}\in\mathbb{Q}^+$. By part **(ii)**, $\mathrm{f}\left(-\dfrac{n}{m}\right)=-\mathrm{f}\left(\dfrac{n}{m}\right)=-\dfrac{n}{m}\mathrm{f}(1)$. This extends $\mathrm{f}(x)=x\mathrm{f}(1)$ for all $x\in\mathbb{Q}-\{0\}$. By part **(i)**, we have $\mathrm{f}(0)=0=0\cdot\mathrm{f}(1)$. Sub $(1,3)$ into $\mathrm{y=f}(x)$, $\mathrm{f}(1)=3$. Hence $\mathrm{f}(x)=x\mathrm{f}(1)=3x$ for all $x\in\mathbb{Q}$.	Many students neglected the condition that $m,n\in\mathbb{Z}^+$ in (iii) and (iv). These only shows $\dfrac{n}{m}\in\mathbb{Q}^+$. Most did not show $x=0$ and $x\in\mathbb{Q}^-$ cases.

<table>
<tr><th>Q9</th><th>Solutions</th><th>Comments</th></tr>
<tr><td>(i)</td><td>

$2I_{n+1} - I_n = \int_0^1 \frac{2t^n}{(t+1)^{n+1}} - \frac{t^{n-1}}{(t+1)^n} \ \mathrm{d}t$

$= \int_0^1 \frac{2t^n - t^{n-1}(t+1)}{(t+1)^{n+1}} \ \mathrm{d}t = \int_0^1 \frac{t^n - t^{n-1}}{(t+1)^{n+1}} \ \mathrm{d}t$

Since $0 \le t \le 1$, $t^n - t^{n-1} = t^{n-1}(t-1) \le 0$, therefore,

$\mathrm{f}(t) = \frac{t^n - t^{n-1}}{(t+1)^{n+1}} \le 0$. Hence $2I_{n+1} - I_n \le 0$, and $I_{n+1} \le \frac{1}{2} I_n$.

<u>Alternative solution</u>

$I_{n+1} = \int_0^1 \frac{t^n}{(t+1)^{n+1}} \ \mathrm{d}t = \int_0^1 \frac{t}{t+1} \cdot \frac{t^{n-1}}{(t+1)^n} \ \mathrm{d}t$

Let $\mathrm{f}(t) = \frac{t}{t+1} = 1 - \frac{1}{t+1}$. For $0 \le t \le 1$, $\mathrm{f}'(t) = \frac{1}{(t+1)^2} > 0$.

Therefore, f is strictly increasing for $0 \le t \le 1$ and

$\frac{t}{t+1} = \mathrm{f}(t) \le \mathrm{f}(1) = \frac{1}{2}$ for $0 \le t \le 1$. Hence,

$I_{n+1} = \int_0^1 \frac{t}{t+1} \cdot \frac{t^{n-1}}{(t+1)^n} \ \mathrm{d}t \le \frac{1}{2} \int_0^1 \frac{t^{n-1}}{(t+1)^n} \ \mathrm{d}t = \frac{1}{2} I_n$.

</td><td>Many students skipped this question (possibly due to lack of time). Instead of skipping the question completely, students should attempt the easier parts like (ii) [use integration by parts], and (iii) [use MOD].</td></tr>
<tr><td>(ii)</td><td>

Using integration by parts,

$I_{n+1} = \int_0^1 \frac{t^n}{(t+1)^{n+1}} \ \mathrm{d}t = \int_0^1 t^n (t+1)^{-n-1} \ \mathrm{d}t$

$= \left[t^n \frac{(t+1)^{-n}}{-n} \right]_0^1 - \int_0^1 n t^{n-1} \frac{(t+1)^{-n}}{-n} \ \mathrm{d}t$

$= -\frac{1}{n2^n} + \int_0^1 \frac{t^{n-1}}{(t+1)^n} \ \mathrm{d}t = -\frac{1}{n2^n} + I_n$

From (i), $-\frac{1}{n2^n} + I_n = I_{n+1} \le \frac{1}{2} I_n$. Hence

$\frac{1}{2} I_n \le \frac{1}{n2^n}$

$I_n \le \frac{1}{n2^{n-1}}$

</td><td></td></tr>
<tr><td>(iii)</td><td>

From $I_{n+1} = -\frac{1}{n2^n} + I_n$, we have $\frac{1}{n2^n} = I_n - I_{n+1}$. Therefore,

</td><td></td></tr>
</table>

	$\sum_{r=1}^{n}\frac{1}{r2^r}=\sum_{r=1}^{n}\left(I_r-I_{r+1}\right)$ $=I_1-I_2$ $+I_2-I_3$ $+\ldots$ $+I_n-I_{n+1}$ $=I_1-I_{n+1}$ Note that $I_1=\int_0^1\frac{t^{1-1}}{(t+1)^1}\,\mathrm{d}t=\int_0^1\frac{1}{t+1}\,\mathrm{d}t=\left[\ln\lvert t+1\rvert\right]_0^1=\ln 2.$ Therefore, $\sum_{r=1}^{n}\frac{1}{r2^r}=\ln 2-I_{n+1}$ and we obtain $\ln 2=\sum_{r=1}^{n}\frac{1}{r2^r}+I_{n+1}.$	
(iv)	Since $I_{n+1}\geq 0$, from (iii), we have $\ln 2\geq\sum_{r=1}^{n}\frac{1}{r2^r}.$ On the other hand, $\ln 2=\sum_{r=1}^{n}\frac{1}{r2^r}+I_{n+1}\leq\sum_{r=1}^{n}\frac{1}{r2^r}+\frac{1}{(n+1)2^n}$ from (ii) and (iii). Together, $\sum_{r=1}^{n}\frac{1}{r2^r}\leq\ln 2\leq\sum_{r=1}^{n}\frac{1}{r2^r}+\frac{1}{(n+1)2^n}$ Choosing $n=4$, we have $0.68854\leq\ln 2\leq 0.69479$ $\approx 0.69\leq\ln 2\leq 0.69$ (corrected to 2 d.p. within the inequalities). Hence $\ln 2\approx 0.69.$	This part is poorly done. The essence is to obtain a lower bound and an upper bound for ln2 using (ii) and (iii) and then choose a suitable value for *n* to approximate ln2.

<table>
<tr><td>Q10</td><td>Solutions</td><td>Comments</td></tr>
<tr><td>(i)</td><td>

$15=3\times 5$

$\mathrm{f}(15)=15\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)=8$

$180=2^2\times 3^2\times 5$

$\mathrm{f}(180)=180\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)=48$

</td><td>The whole question is generally well attempted.</td></tr>
<tr><td>(ii)</td><td>

For any positive integer *N*, let $N=p_1^{k_1}p_2^{k_2}\dots p_l^{k_l}$ be its unique prime factorisation, $k_i\in\mathbb{Z}^+$.

Thus

$\mathrm{f}(N)=N\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_l}\right)$

$=\left(p_1^{k_1}p_2^{k_2}\dots p_l^{k_l}\right)\left(\frac{p_1-1}{p_1}\right)\left(\frac{p_2-1}{p_2}\right)\dots\left(\frac{p_l-1}{p_l}\right)$

$=\left(p_1^{k_1-1}p_2^{k_2-1}\dots p_l^{k_l-1}\right)(p_1-1)(p_2-1)\dots(p_l-1)$

where k_i-1 is an integer greater than or equals to zero, and (p_i-1) is an integer. Hence $\mathrm{f}(N)$ is an integer.

</td><td></td></tr>
<tr><td>(iii)(a)</td><td>

$\mathrm{f}(15)\,\mathrm{f}(12)$

$=8\times 12\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)$

$=8\times 4$

$=32$

$\neq 48$

$=\mathrm{f}(180)$

Hence statement is not true.

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<tr><td>(iii)(b)</td><td>

(IF) Let *a* and *b* be coprime to each other.

Let $a=p_1^{a_1}p_2^{a_2}\dots p_m^{a_m}$ and $b=q_1^{b_1}q_2^{b_2}\dots q_n^{b_n}$ be the prime factorizations of *a* and *b* respectively. Since *a* and *b* are coprime to each other, $p_i\neq q_j$ for all $1\le i\le m$, $1\le j\le n$. We also note that $p_1^{a_1}p_2^{a_2}\dots p_m^{a_m}q_1^{b_1}q_2^{b_2}\dots q_n^{b_n}$ is a prime factorization of *ab*. Since prime factorizations are unique, it must be the only one. Hence $p_1,\dots,p_m,q_1,q_2,\dots q_n$ are the prime factors of *ab*. Thus,

</td><td>This is the part which most students lost marks because they only showed one direction.</td></tr>
</table>

$$\mathrm{f}(ab) = ab\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_m}\right)\left(1-\frac{1}{q_1}\right)\left(1-\frac{1}{q_2}\right)\ldots\left(1-\frac{1}{q_n}\right)$$

$$= a\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_m}\right)b\left(1-\frac{1}{q_1}\right)\left(1-\frac{1}{q_2}\right)\ldots\left(1-\frac{1}{q_n}\right)$$

$$= \mathrm{f}(a)\mathrm{f}(b)$$

(ONLY IF) Let $\mathrm{f}(ab) = \mathrm{f}(a)\mathrm{f}(b)$.

Suppose *a* and *b* are not coprime to each other. Thus, let r_1, r_2, $\ldots$, r_c be all the prime factors common to both *a* and *b*.

Hence we will have the prime factorization of *a* and *b* as

$$a = p_1^{a_1} p_2^{a_2} \ldots p_m^{a_m} \ldots r_1^{a_{m+1}} r_2^{a_{m+2}} \ldots r_c^{a_{m+c}},$$

$$b = q_1^{b_1} q_2^{b_2} \ldots q_n^{b_n} r_1^{b_{n+1}} r_2^{b_{n+2}} \ldots r_c^{b_{n+c}},$$

where $p_i \neq q_j$ for all $1 \le i \le m$, $1 \le j \le n$. By recycling the proof in the (IF) part, we know that p_1, p_2, $\ldots$, p_m, q_1, q_2, $\ldots$, q_n, r_1, $\ldots$, r_c are all the prime factors of *ab*. So

$$\mathrm{f}(ab) = \mathrm{f}(a)\mathrm{f}(b)$$

$$\Rightarrow ab\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_m}\right)\left(1-\frac{1}{q_1}\right)\left(1-\frac{1}{q_2}\right)\ldots\left(1-\frac{1}{q_n}\right)\times$$
$$\left(1-\frac{1}{r_1}\right)\left(1-\frac{1}{r_2}\right)\ldots\left(1-\frac{1}{r_c}\right)$$

$$= a\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_m}\right)\left(1-\frac{1}{r_1}\right)\left(1-\frac{1}{r_2}\right)\ldots\left(1-\frac{1}{r_c}\right)\times$$
$$b\left(1-\frac{1}{q_1}\right)\left(1-\frac{1}{q_2}\right)\ldots\left(1-\frac{1}{q_n}\right)\left(1-\frac{1}{r_1}\right)\left(1-\frac{1}{r_2}\right)\ldots\left(1-\frac{1}{r_c}\right)$$

$$\Rightarrow 1 = \left(1-\frac{1}{r_1}\right)\left(1-\frac{1}{r_2}\right)\ldots\left(1-\frac{1}{r_c}\right)$$

a contradiction, since $\left(1-\frac{1}{r_i}\right)$ is a proper fraction which means

$$\left(1-\frac{1}{r_1}\right)\left(1-\frac{1}{r_2}\right)\ldots\left(1-\frac{1}{r_c}\right) < 1.$$

Hence *a* and *b* must be coprime to each other.

Some of the students who attempted to prove this direction were not clear in their explanations.

(iv)	$f(p^k) = p^k\left(1-\frac{1}{p}\right) = p^{k-1}(p-1)$ Note that 1 is coprime to any positive integer. Since *p* is prime, the integers less than or equal to p^k that are not coprime to p^k are of the form mp, where *m* is a positive integer such that $1 \le m \le p^{k-1}$. Hence there are p^{k-1} integers that are not coprime to p^k. Hence the number of positive integers that are less than or equal to p^k and are coprime to p^k $= p^k - p^{k-1} = p^{k-1}(p-1) = f(p^k)$ (shown)	Some students were unable to explain why number of positive integers that are less than or equal to p^k and are coprime to p^k is $p^k - p^{k-1}$.