NANYANG JUNIOR COLLEGE

Chapter 4: Vectors(II)

H2 Mathematics

NAME: ______________________

CLASS: ______________________

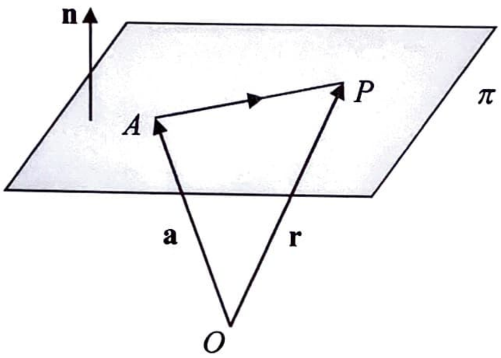
Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

4.5 Equation of a Plane

4.5.1 Equation of a Plane in Scalar Product Form

Let π denotes a plane and **n** is a vector perpendicular to π.
Let A be a given point on π whose position vector is denoted by **a** and let P be any general point on π whose position vector is denoted by **r** as shown in Fig. 4.32 below.

Fig. 4.32



By the Triangular law of vector addition, we have $\overrightarrow{AP} = \mathbf{r} - \mathbf{a}$.
Since $\overrightarrow{AP}$ is perpendicular to **n**, $\overrightarrow{AP} \bullet \mathbf{n} = 0$

$$(\mathbf{r} - \mathbf{a}) \bullet \mathbf{n} = 0$$

$$\mathbf{r} \bullet \mathbf{n} - \mathbf{a} \bullet \mathbf{n} = 0$$

$$\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$$

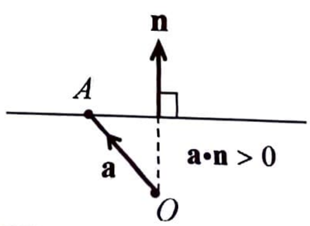
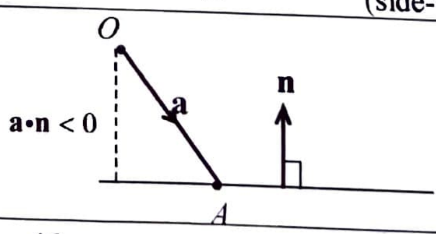
Hence the **scalar product form** of the equation of the plane π is given by

$$\boxed{\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n} = D}$$

where $\mathbf{a} \bullet \mathbf{n} = D$ is some real scalar.

Remarks:

1. The vector **n** is called a **normal vector** of the plane π.
2. If $\mathbf{a} \bullet \mathbf{n} = D > 0$, then the angle between $\overrightarrow{OA}$ and **n** is acute. On the other hand, if $\mathbf{a} \bullet \mathbf{n} = D < 0$, then the angle between $\overrightarrow{OA}$ and **n** is obtuse.
   The sign of D "affects" whether the origin is above or below the plane.

If $\mathbf{a} \bullet \mathbf{n} > 0$, then $\overrightarrow{OA}$ and **n** are in the "same direction"	 (side-view of the plane)
If $\mathbf{a} \bullet \mathbf{n} < 0$, then $\overrightarrow{OA}$ and **n** are in the "opposite direction"	

The concept of vectors is one of the most useful ideas in applied mathematics.

3. In Fig. 4.33 below, F is the foot of the perpendicular from O to the plane π. Let P be any general point on π whose position vector is represented by $\mathbf{r}$. Let θ be the angle between $\mathbf{n}$ and $\mathbf{r}$.

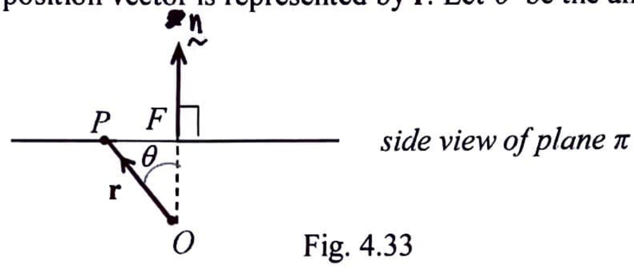


Fig. 4.33

Suppose that $D > 0$ in the equation of the plane $\pi: \mathbf{r} \cdot \mathbf{n} = D$ (i.e. θ is acute)

Dividing throughout by $|\mathbf{n}|$, we obtain $\dfrac{\mathbf{r} \cdot \mathbf{n}}{|\mathbf{n}|} = \dfrac{D}{|\mathbf{n}|} \Rightarrow \mathbf{r} \cdot \hat{\mathbf{n}} = \dfrac{D}{|\mathbf{n}|}$.

From Fig. 4.33 above, the perpendicular distance from O to the plane π is given by

OF = length of projection of $\mathbf{r}$ onto $\mathbf{n}$

$= |\mathbf{r} \cdot \hat{\mathbf{n}}|$ (refer to Section 4.3)

$= \dfrac{|D|}{|\mathbf{n}|}$.

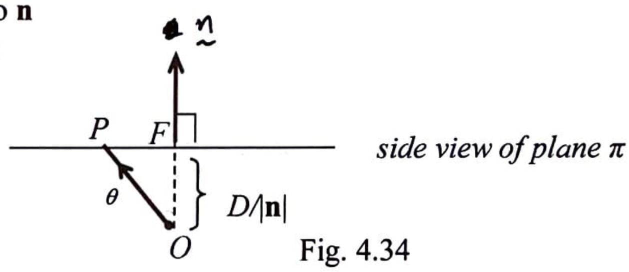


Fig. 4.34

Thus given the equation of a plane π in scalar product form $\mathbf{r} \cdot \mathbf{n} = D$, the perpendicular distance from the origin O to π is given by

$$\boxed{\frac{|D|}{|\mathbf{n}|}}$$

For example, the perpendicular distance from the origin O to the plane π having equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 9$ is $\dfrac{|9|}{\left|\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\right|} = \dfrac{9}{\sqrt{1+4+4}} = \dfrac{9}{\sqrt{9}} = 3$.

Example 4.32

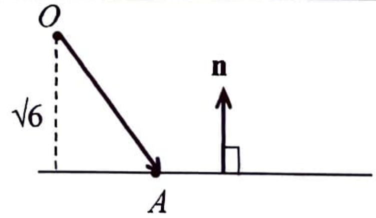
Find the equation of the plane π in scalar product form passing through the point $A(3,4,-1)$ and perpendicular to the vector $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Hence determine the distance of the plane from the origin O.

Solution:

Equation of plane in scalar product form is $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -6 \Rightarrow \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -6$$

Perpendicular distance from O to π is $\dfrac{|-6|}{\left|\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right|} = \dfrac{6}{\sqrt{6}} = \sqrt{6}$ units.



Example 4.33

Find the value(s) of λ such that the distance of the plane $\mathbf{r}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}=\lambda$ from the origin is 4 units.

Solution:

Given $\mathbf{r}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}=\lambda$.

Then perpendicular distance from O to the plane is

$$\frac{|\lambda|}{\left|\begin{pmatrix}1\\2\\2\end{pmatrix}\right|}=4 \qquad \lambda=\pm 12$$

ThinkZone:

Why are there two values of λ?
What is the geometrical interpretation?

4.5.2 Equation of a Plane in Cartesian Form

Letting $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}, \mathbf{n}=\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$ and substituting into the scalar product form $\mathbf{r}\bullet\mathbf{n}=D$ gives

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\bullet\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}=D\Rightarrow n_1x+n_2y+n_3z=D$$

which is the **cartesian equation** of the plane π.

For example, the cartesian equation of the plane $\mathbf{r}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=-6$ is $x-2y+z=-6$.

4.5.3 Equation of a Plane in Vector (Parametric) Form

Consider the plane π which is parallel to vectors **u** and **v** (**u** not parallel to **v**) and which also contains the point A whose position vector is **a**. (See Fig. 4.35)

Let P be any general point on π whose position vector is represented by **r**. Since **u** and **v** are two *non-parallel coplanar vectors* parallel to the plane π and $\overrightarrow{AP}$ is also a vector parallel to π, we can therefore write

$$\overrightarrow{AP}=\lambda\mathbf{u}+\mu\mathbf{v}$$

for some real scalars λ and μ, called **parameters**.

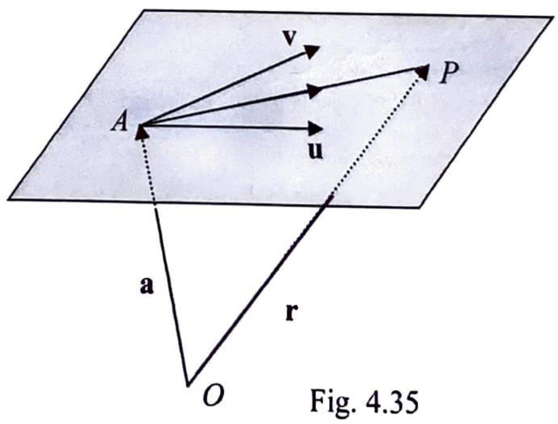


Fig. 4.35

By the Triangular law of vector addition,
$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}\Rightarrow\mathbf{r}=\mathbf{a}+\lambda\mathbf{u}+\mu\mathbf{v}$

Hence the **vector** (or **parametric) equation** of the plane π containing the point A with position vector $\mathbf{a}$ and having two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ both of which are parallel to itself is given by:

$$\boxed{\mathbf{r} = \mathbf{a} + \lambda\mathbf{u} + \mu\mathbf{v}} \quad \text{where } \lambda, \mu \in \mathbb{R}.$$

Remarks:

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are called **direction vectors** of the plane π. Note that the pair of $\mathbf{u}$ and $\mathbf{v}$ are not unique.
2. If we set $\mu = 0$ and allow λ to vary over all real numbers, then the above equation becomes $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u},\ \lambda \in \mathbb{R}$ which is the vector equation of a line through A parallel to $\mathbf{u}$. So this line is contained in the plane π. Therefore a plane can be uniquely determined by a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u},\ \lambda \in \mathbb{R}$ it contains together with a direction vector $\mathbf{v}$ which is *not parallel* to the line.

Example 4.34

Write down the vector equation of the plane passing through the point with position vector $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and parallel to the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$.

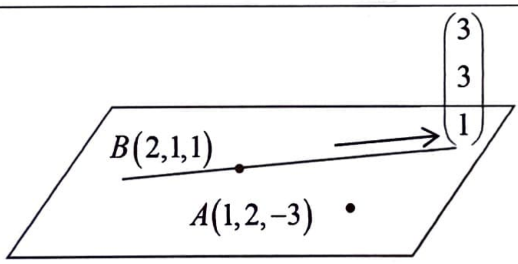
Solution:

Vector equation of the plane is $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix},\ \lambda, \mu \in \mathbb{R}$

Eample 4.35 (Plane containing a point and a line)

Find the vector equation of the plane that passes through the point $(1, 2, -3)$ and contains the line with vector equation $\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \lambda(3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$.

Solution:



Let $A \equiv (1, 2, -3)$ and $B \equiv (2, 1, 1)$.

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

Vector equation of the plane is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix},\ \lambda, \mu \in \mathbb{R}$$

ThinkZone:

Alternatively, another possible vector equation is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix},\ \lambda, \mu \in \mathbb{R}$$

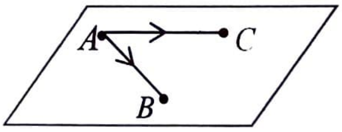
Is the vector equation of a plane unique?

Example 4.36 (Plane containing three points)

Find the vector equation of the plane that passes through the points $A(0,1,1)$, $B(2,1,-3)$ and $C(1,3,2)$.

Solution:

$$\overrightarrow{AB}=\begin{pmatrix}2\\1\\-3\end{pmatrix}-\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}2\\0\\-4\end{pmatrix}$$



$$\overrightarrow{AC}=\begin{pmatrix}1\\3\\2\end{pmatrix}-\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}1\\2\\1\end{pmatrix}$$

Vector equation of the plane is $\mathbf{r}=\begin{pmatrix}0\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-4\end{pmatrix}+\mu\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\lambda,\ \mu\in\mathbb{R}$

ThinkZone:
Can you give another vector equation of the plane?

Example 4.37 (Plane containing two lines)

Find a vector equation of the plane that contains the lines $\mathbf{r}=-3\mathbf{i}+2\mathbf{j}+t(\mathbf{i}-2\mathbf{j}+\mathbf{k})$ and $\mathbf{r}=\mathbf{i}-11\mathbf{j}+4\mathbf{k}+s(2\mathbf{i}-\mathbf{j}+2\mathbf{k})$.

Solution:
$\mathbf{r}=-3\mathbf{i}-2\mathbf{j}+\lambda(\mathbf{i}-2\mathbf{j}+\mathbf{k})+\mu(2\mathbf{i}-\mathbf{j}+2\mathbf{k})$,$\lambda,\mu\in\mathbb{R}$

ThinkZone:

4.5.4 Converting Equation of a Plane from Vector to Scalar Product Form and Vice Versa

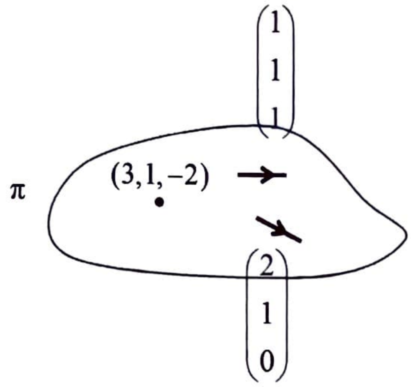
Given a plane π: $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$, $\lambda,\ \mu\in\mathbb{R}$. The normal vector of π, $\mathbf{n}$ is parallel to $\mathbf{b}\times\mathbf{c}$ and the equation of π in scalar product form is therefore $\mathbf{r}\cdot\underbrace{(\mathbf{b}\times\mathbf{c})}_{\mathbf{n}}=\underbrace{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}_{D}=D$.

Vice versa, given a plane : $\mathbf{r}\cdot\mathbf{n}=D$. We need to find 3 non-collinear points A, B and C on π having position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The equation of π in vector form is therefore $\mathbf{r}=\overrightarrow{OA}+\lambda\overrightarrow{AB}+\mu\overrightarrow{AC}\Rightarrow\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})+\mu(\mathbf{c}-\mathbf{a})$, $\lambda,\mu\in\mathbb{R}$.

Example 4.38

Find the equation of the plane passing through the point with position vector $\begin{pmatrix}3\\1\\-2\end{pmatrix}$ and parallel to vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\0\end{pmatrix}$, giving your answer in scalar product form.

Solution:



A normal to the plane is given by

$$\mathbf{n}=\begin{pmatrix}1\\1\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\2\\-1\end{pmatrix}.$$

Equation of the plane is

$$\mathbf{r}\bullet\begin{pmatrix}-1\\2\\-1\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\-1\end{pmatrix}=1$$

Thus, we have $\pi:\mathbf{r}\bullet\begin{pmatrix}-1\\2\\-1\end{pmatrix}=1$

ThinkZone:

Is $\pi:\mathbf{r}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=-1$ another possible answer?

Example 4.39

The equation of a plane Π in scalar product form is $\mathbf{r}\bullet\begin{pmatrix}1\\2\\-1\end{pmatrix}=3$.

$\begin{pmatrix}a\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}=3$
$a+2=3$
$a=1$

Since $\begin{pmatrix}3\\0\\0\end{pmatrix}$ lies on the plane, one direction vector b_1 $=\begin{pmatrix}3\\0\\0\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\0\end{pmatrix}=\frac{1}{\sqrt5}\begin{pmatrix}2\\-1\\0\end{pmatrix}$

Since $\begin{pmatrix}0\\0\\-3\end{pmatrix}$ lies on the plane $b_2=\begin{pmatrix}0\\0\\-3\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\-1\\-3\end{pmatrix}=\frac{1}{\sqrt{11}}\begin{pmatrix}-1\\-1\\-3\end{pmatrix}$

Convert this equation into the vector form $\mathbf{r}=\begin{pmatrix}a\\1\\0\end{pmatrix}+\lambda\mathbf{b}_1+\mu\mathbf{b}_2,\ \lambda,\mu\in\mathbb{R}$

where a is an integer to be determined and $\mathbf{b}_1$ and $\mathbf{b}_2$ are mutually perpendicular unit vectors.

Solution:

Since $(a,1,0)$ lies in Π, $\begin{pmatrix}a\\1\\0\end{pmatrix}\bullet\begin{pmatrix}1\\2\\-1\end{pmatrix}=3\Rightarrow a+2=3\Rightarrow a=1$.

So $A(1,1,0)$ is on Π.

Take another point $B(3,0,0)$ on Π.

ThinkZone

How do you know B lies on Π? Can you state another point on Π?

Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

Then $\overrightarrow{AB}=\begin{pmatrix}3\\0\\0\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$ is a direction vector of Π.

Let $\mathbf{b}_1$ is a unit vector of $\overrightarrow{AB}$.

Consider $\overrightarrow{AB}\times\mathbf{n}$.

$$\overrightarrow{AB}\times\mathbf{n}=\begin{pmatrix}2\\-1\\0\end{pmatrix}\times\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}1\\2\\5\end{pmatrix}$$

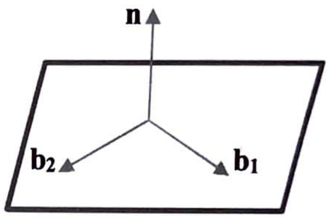
$\overrightarrow{AB}\times\mathbf{n}$ is parallel to Π (and hence a direction vector of Π) since $\overrightarrow{AB}\perp\mathbf{n}$ (Why?).

Let $\mathbf{b}_2$ be a unit vector of $\overrightarrow{AB}\times\mathbf{n}$.

Thus $\mathbf{b}_1=\frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\\0\end{pmatrix}$ and $\mathbf{b}_2=\frac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\5\end{pmatrix}$.

Hence required plane is $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{pmatrix}+\mu\begin{pmatrix}1/\sqrt{30}\\2/\sqrt{30}\\5/\sqrt{30}\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$

Draw a diagram to see this



4.5.5 The Special Planes – *xy*-, *xz*-, and *yz*-Planes

$\mathbf{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$

$z=0$

Plane	Equation of Plane		
	Cartesian Form	Scalar Product Form	Vector Form
xy-plane	$z=0$	$\mathbf{r}\cdot\mathbf{k}=0$	$\mathbf{r}=\lambda\mathbf{i}+\mu\mathbf{j}$, $\lambda,\mu\in\mathbb{R}$
yz-plane	$x=0$	$\mathbf{r}\cdot\mathbf{i}=0$	$\mathbf{r}=\lambda\mathbf{j}+\mu\mathbf{k}$, $\lambda,\mu\in\mathbb{R}$
xz-plane	$y=0$	$\mathbf{r}\cdot\mathbf{j}=0$	$\mathbf{r}=\lambda\mathbf{i}+\mu\mathbf{k}$, $\lambda,\mu\in\mathbb{R}$

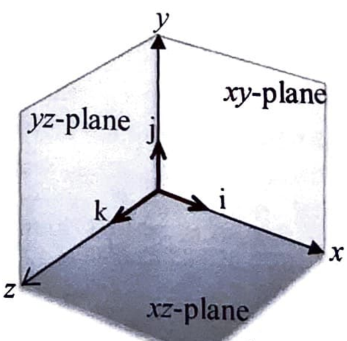


Fig. 4.36

4.5.6 Questions Involving a Plane and a Point

4.5.6.1 Determine if a given point lies on a plane

Example 4.40

Given that π: $\mathbf{r}\bullet\begin{pmatrix}-1\\2\\-1\end{pmatrix}=1$. Determine if the following points lie on the plane.

(a) $(0,1,1)$, (b) $(-1,2,-1)$

Solution:

Since $\begin{pmatrix}0\\1\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\-1\end{pmatrix}=2-1=1$, the point $(0,1,1)$ lies on the plane Since $\begin{pmatrix}-1\\2\\-1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\-1\end{pmatrix}=6\neq 1$, the point $(-1,2,-1)$ does not lies on the plane	

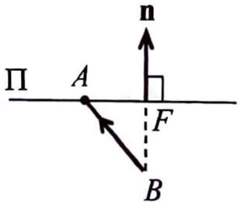
4.5.6.2 Find the

- **perpendicular (shortest) distance from a point to a plane**
- **foot of perpendicular from a point to a plane**
- **reflection of a point in a plane**

Given a plane Π : $\mathbf{r}\bullet\mathbf{n}=D$ and a point B (not on Π) with position vector $\mathbf{b}$. We wish to compute the perpendicular distance from B to Π. We present two ways to do this.

The first way involved the idea of using projection. Take an arbitrary point A with position vector $\mathbf{a}$ on Π and connect B to A to form the vector $\overrightarrow{BA}$.

Fig. 4.37



The perpendicular distance from B to Π, BF, is equal to the length of projection of $\overrightarrow{BA}$ onto $\mathbf{n}$

$$=\left|\overrightarrow{BA}\bullet\hat{\mathbf{n}}\right|$$

$$=\left|(\mathbf{a}-\mathbf{b})\bullet\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$=\frac{|(\mathbf{a}-\mathbf{b})\bullet\mathbf{n}|}{|\mathbf{n}|}$$

$$=\frac{|\mathbf{a}\bullet\mathbf{n}-\mathbf{b}\bullet\mathbf{n}|}{|\mathbf{n}|}$$

$$=\frac{|D-\mathbf{b}\bullet\mathbf{n}|}{|\mathbf{n}|}$$ since $\mathbf{a}\bullet\mathbf{n}=D$ (Why?)

Thus , the perpendicular distance from B (with position vector $\mathbf{b}$) to Π having equation $\mathbf{r}\bullet\mathbf{n}=D$ is

$$\boxed{\frac{|D-\mathbf{b}\bullet\mathbf{n}|}{|\mathbf{n}|}}$$

Another way to view this formula is from the perspective of distance between two planes. The idea is to create a plane Π^* containing the point B and parallel to Π. The required distance is then the difference between the perpendicular distances from the origin O to the planes Π and Π^*.

The equation of Π^* is given by $\mathbf{r}\bullet\mathbf{n}=\mathbf{b}\bullet\mathbf{n}=D^*$ (Note that we use the same normal vector $\mathbf{n}$ as Π).

From our discussion under remark 3, section 4.5.1, we have

perpendicular distance from B to Π = $=\left|\frac{D}{|\mathbf{n}|}-\frac{D^*}{|\mathbf{n}|}\right|=\frac{|D-D^*|}{|\mathbf{n}|}=\frac{|D-\mathbf{b}\bullet\mathbf{n}|}{|\mathbf{n}|}$ as before.

Therefore, another expression for finding the perpendicular distance from B to Π having equation $\mathbf{r}\bullet\mathbf{n}=D$ is

$$\boxed{\frac{|D-D^*|}{|\mathbf{n}|}}$$ where $D^*=\mathbf{b}\bullet\mathbf{n}$

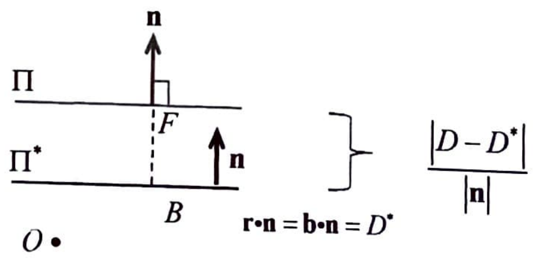


Fig. 4.38

To find the position vector of F, the foot of the perpendicular from B to Π, we find the vector equation of the line BF and then intersect it with the equation of the plane, π, to obtain the position vector of F.
Equation of line BF : $\mathbf{r}=\mathbf{b}+\lambda\mathbf{n},\lambda\in\mathbb{R}$

Note: We use $\mathbf{n}$ as the direction vector for the line BF since BF is perpendicular to the plane and therefore parallel to $\mathbf{n}$.

Substitute $\mathbf{r}=\mathbf{b}+\lambda\mathbf{n}$ into $\mathbf{r}\bullet\mathbf{n}=D$ gives

$$(\mathbf{b}+\lambda\mathbf{n})\bullet\mathbf{n}=D\Rightarrow\mathbf{b}\bullet\mathbf{n}+\lambda\mathbf{n}\bullet\mathbf{n}=D\Rightarrow\lambda=\frac{D-\mathbf{b}\bullet\mathbf{n}}{|\mathbf{n}|^2}$$ since $\mathbf{n}\bullet\mathbf{n}=|\mathbf{n}|^2$.

Putting λ back into $\mathbf{r}=\mathbf{b}+\lambda\mathbf{n}$ gives the position vector of F.

Example 4.41

The plane Π has equation $\mathbf{r}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=4$, and point A has position vector $\begin{pmatrix}0\\2\\5\end{pmatrix}$. Find

(i) the distance from A to Π,
(ii) the position vector of the foot of the perpendicular from A to Π,
(iii) the position vector of the point of reflection A' of A in Π.

Solution:

	Think Zone:
(i) $D=4$, $\mathbf{b}=\begin{pmatrix}0\\2\\5\end{pmatrix}$ and $\mathbf{n}=\begin{pmatrix}-2\\1\\1\end{pmatrix}$. Substituting into the formula $\dfrac{\lvert D-\mathbf{b}\cdot\mathbf{n}\rvert}{\lvert\mathbf{n}\rvert}$ gives required distance $=\dfrac{\left\lvert 4-\begin{pmatrix}0\\2\\5\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\rvert}=\dfrac{\lvert-3\rvert}{\sqrt{6}}=\dfrac{3}{\sqrt{6}}=\dfrac{3\sqrt{6}}{6}=\dfrac{\sqrt{6}}{2}$. Alternatively, consider $\mathbf{r}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}0\\2\\5\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=7$ Applying $\dfrac{\lvert D-D^*\rvert}{\lvert\mathbf{n}\rvert}=\dfrac{\lvert 4-7\rvert}{\left\lvert\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\rvert}=\dfrac{3}{\sqrt{6}}=\dfrac{\sqrt{6}}{2}$, same answer as above. (ii) Let F be the foot of perpendicular from A to Π. Line AF has equation $\mathbf{r}=\begin{pmatrix}0\\2\\5\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$. Since F lies on line AF, $\overrightarrow{OF}=\begin{pmatrix}0\\2\\5\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. ----- (*) Since F also lies in plane Π, $\overrightarrow{OF}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=4$.	Alternative method: take $C(0,4,0)$ on the plane. Then required distance = length of projection of $\overrightarrow{AC}$ onto $\mathbf{n}$ $=\lvert\overrightarrow{AC}\cdot\hat{\mathbf{n}}\rvert$ $=\left\lvert\begin{pmatrix}0\\2\\-5\end{pmatrix}\cdot\dfrac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\rvert$ $=\dfrac{1}{\sqrt{6}}\lvert-3\rvert$ $=\dfrac{3}{\sqrt{6}}=\dfrac{\sqrt{6}}{2}$

$$\left(\begin{pmatrix}0\\2\\5\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\1\end{pmatrix}\right)\bullet\begin{pmatrix}-2\\1\\1\end{pmatrix}=4$$

$$7+6\lambda=4 \Rightarrow \lambda=-\frac{1}{2}$$

Hence $\overrightarrow{OF}=\begin{pmatrix}0\\2\\5\end{pmatrix}+\left(-\frac{1}{2}\right)\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}1\\\frac{3}{2}\\\frac{9}{2}\end{pmatrix}$ from (*)

(iii) By Ratio Theorem, $\overrightarrow{OF}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OA'}\right)$

$$\Rightarrow \overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}=2\begin{pmatrix}1\\\frac{3}{2}\\\frac{9}{2}\end{pmatrix}-\begin{pmatrix}0\\2\\5\end{pmatrix}=\begin{pmatrix}2\\1\\4\end{pmatrix}.$$

Remark: The method of finding foot of perpendicular from a point to a plane is the same as finding the point of intersection between line and a plane. (See **Example 4.43**)

Self-Review 4.13 (2008/Prelim/MI/I/11 modified)

Two planes π_1 and π_2 have equations $\mathbf{r}\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}=9$ and $\mathbf{r}\bullet\begin{pmatrix}4\\3\\-1\end{pmatrix}=8$ respectively. The point P has coordinates $(1,-1,3)$.

(i) Show that P lies on plane π_1.

(ii) Find the shortest distance from P to plane π_2. [$10/\sqrt{26}$]

4.5.7 Problems Involving a Line and a Plane

4.5.7.1 Determine if a line is parallel or perpendicular to a plane

Given a line l with direction vector $\mathbf{d}$ and a plane Π with normal vector $\mathbf{n}$.

(a) The line and the plane are *parallel* if and only if $\mathbf{n}\perp\mathbf{d}$, i.e. $\mathbf{n}\bullet\mathbf{d}=0$;

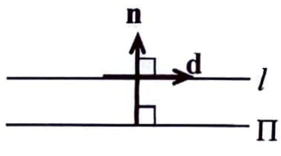


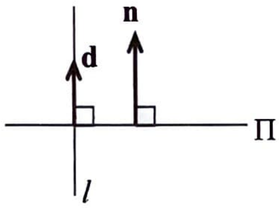
Fig. 4.39

E.g. The line l_1 with equation $\mathbf{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ is parallel to the plane $\mathbf{r}\bullet\begin{pmatrix}2\\-1\\3\end{pmatrix}=4$ since

$$\mathbf{d}\bullet\mathbf{n}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\3\end{pmatrix}=0.$$

(b) The line and the plane are *perpendicular* if $\mathbf{n}//\mathbf{d}$, i.e. $\mathbf{n}=\lambda\mathbf{d}$ for some $\lambda\in\mathbb{R}$.

Fig. 4.40



E.g The line l_2 with equation $\mathbf{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\-3\end{pmatrix}$ is perpendicular to the plane $\mathbf{r}\bullet\begin{pmatrix}2\\-1\\3\end{pmatrix}=4$ since

$$\mathbf{d}=\begin{pmatrix}-2\\1\\-3\end{pmatrix}=-\begin{pmatrix}2\\-1\\3\end{pmatrix}=-\mathbf{n},\text{ i.e., }\mathbf{n}//\mathbf{d}.$$

4.5.7.2 Determine if a line lies on a plane

Example 4.42

Show that the line $\mathbf{r}=2\mathbf{i}+2\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}+3\mathbf{j}+2\mathbf{k}),\ \lambda\in\mathbb{R}$ lies in the plane $\mathbf{r}\bullet(4\mathbf{i}-2\mathbf{j}+\mathbf{k})=5$.

Solution:

Substitute $\mathbf{r}=\begin{pmatrix}2\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\2\end{pmatrix}$ into the equation of the plane gives

$$\begin{aligned}\text{LHS}&=\left[\begin{pmatrix}2\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\2\end{pmatrix}\right]\bullet\begin{pmatrix}4\\-2\\1\end{pmatrix}\\&=\begin{pmatrix}2\\2\\1\end{pmatrix}\bullet\begin{pmatrix}4\\-2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\2\end{pmatrix}\bullet\begin{pmatrix}4\\-2\\1\end{pmatrix}\\&=(8-4+1)+\lambda(4-6+2)\\&=5\\&=\text{RHS.}\end{aligned}$$

Thus, the line lies in the plane.

Alternative Solution:

ThinkZone:

Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

Let $\mathbf{a}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ be the position vector of a point on the line, $\mathbf{d}=\begin{pmatrix}1\\3\\2\end{pmatrix}$ the direction vector of the line, and $\mathbf{n}=\begin{pmatrix}4\\-2\\1\end{pmatrix}$ the normal vector of the plane. Step 1: First show that the line is parallel to the plane. $\mathbf{d}\bullet\mathbf{n}=\begin{pmatrix}1\\3\\2\end{pmatrix}\bullet\begin{pmatrix}4\\-2\\1\end{pmatrix}=4-6+2=0$ Hence the line is parallel to the plane. Step 2: Show that there is a point on the line which also lies on the plane. $\mathbf{a}\bullet\mathbf{n}=\begin{pmatrix}2\\2\\1\end{pmatrix}\bullet\begin{pmatrix}4\\-2\\1\end{pmatrix}=8+(-4)+1=5$ The point $(2,2,1)$ on the line also lies in the plane. Steps 1 and 2 together imply that *all* points on the line lie in the plane. Hence the line lies in the plane.	

4.5.7.3 Determine the point of intersection of a line and a plane

If a line is not parallel to a plane, then they will intersect at *a point*. To find the point of intersection, we substitute the equation of line into the equation of the plane and then solve for the parameter.

Example 4.43

Find the point of intersection of the line l: $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$, $\lambda\in\mathbb{R}$ and the plane Π: $\mathbf{r}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}=6$.

Solution:

Let X be the point of intersection of the line l and the plane Π. Since X lies on l, $\overrightarrow{OX}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. Since X lies on Π, $\overrightarrow{OX}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}=6$, Thus $\left(\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}\right)\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}=6\Rightarrow 11+5\lambda=6\Rightarrow\lambda=-1$.	

Hence $\overrightarrow{OX} = \begin{pmatrix}1\\2\\3\end{pmatrix} + (-1)\begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\3\\0\end{pmatrix}$. $\therefore$ the point of intersection is (0, 3, 0).	

Example 4.44

Find the position vector of the point of intersection of the line $\mathbf{r} = \begin{pmatrix}2\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$, $\lambda \in \mathbb{R}$ and the plane $\mathbf{r} = \begin{pmatrix}1\\-1\\-1\end{pmatrix} + s\begin{pmatrix}4\\3\\0\end{pmatrix} + t\begin{pmatrix}1\\1\\2\end{pmatrix}$, $s, t \in \mathbb{R}$.

Solution:

Let the point of intersection be X.

Then $\overrightarrow{OX} = \begin{pmatrix}2\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ for some $\lambda \in \mathbb{R}$ ----- (*)

and $\overrightarrow{OX} = \begin{pmatrix}1\\-1\\-1\end{pmatrix} + s\begin{pmatrix}4\\3\\0\end{pmatrix} + t\begin{pmatrix}1\\1\\2\end{pmatrix}$ for some $s,\ t \in \mathbb{R}$.

Therefore $\begin{pmatrix}2\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\end{pmatrix} + s\begin{pmatrix}4\\3\\0\end{pmatrix} + t\begin{pmatrix}1\\1\\2\end{pmatrix}$

$$\begin{matrix} 2+\lambda = 1+4s+t & & \lambda - 4s - t = -1 \\ \Rightarrow\ 2-\lambda = -1+3s+t & \Rightarrow & -\lambda - 3s - t = -3 \\ 3\lambda = -1+2t & & 3\lambda \quad -2t = -1 \end{matrix}$$

Solving the above system of linear equations, we get $\lambda = 1,\ s = 0,\ t = 2$

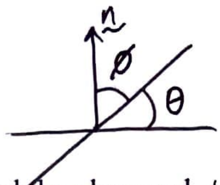
Sub $\lambda = 1$ in (*): $\overrightarrow{OX} = \begin{pmatrix}2\\2\\0\end{pmatrix} + \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}3\\1\\3\end{pmatrix}$

ThinkZone:
Can you make use of $s=0,\ t=2$ to get $\overrightarrow{OX}$?

Self-Review 4.14

Find the point of intersection of the line $x-2 = 2y+1 = 3-z$ and the plane $x+2y+z=3$.[$(1,-1,4)$]

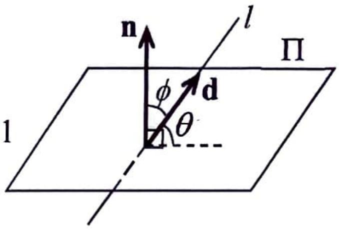
4.5.7.4 Find the acute angle between a line and a plane



Let θ be the acute angle between the line and the plane and ϕ be the *acute* angle between the line l and the normal **n**. Since ϕ is acute, $\cos\phi > 0$. So

$$|\mathbf{n}\bullet\mathbf{d}| = |\mathbf{n}||\mathbf{d}|\cos\phi$$
$$= |\mathbf{n}||\mathbf{d}|\cos\left(\frac{\pi}{2}-\theta\right)$$
$$= |\mathbf{n}||\mathbf{d}|\sin\theta$$

Fig. 4.41



The acute angle θ between a line with direction vector **d** and a plane with normal vector **n** is given by $|\mathbf{n}\bullet\mathbf{d}| = |\mathbf{n}||\mathbf{d}|\sin\theta$, i.e.

$$\sin\theta = \frac{|\mathbf{n}\bullet\mathbf{d}|}{|\mathbf{n}||\mathbf{d}|}$$

Example 4.45

Find the acute angle between the plane $\mathbf{r}\bullet(3\mathbf{i}-5\mathbf{k})=5$ and the line $\mathbf{r}=2\mathbf{i}-12\mathbf{j}+11\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\lambda\in\mathbb{R}$.

Solution:

Let θ be the acute angle between the plane and the line. $\left\lvert\begin{pmatrix}3\\0\\-5\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rvert = \left\lvert\begin{pmatrix}3\\0\\-5\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rvert\sin\theta \Rightarrow \sin\theta = \frac{2}{\sqrt{34}\sqrt{3}} \Rightarrow \theta = 11.4^\circ$ (1 d.p.) Thus, the acute angle between the plane and the line is 11.4°.	Alternatively, find angle ϕ between **n** and **d**. If ϕ is acute, then $\theta = 90^\circ - \phi$. If ϕ is obtuse, then $\theta = \phi - 90^\circ$.

Self-Review 4.15

Find the sine of the acute angle between the line and plane whose equations are $\frac{x-2}{2}=\frac{y+1}{6}=\frac{z+3}{3}$, $2x-y-2z=4$ respectively. [8/21]

4.5.7.5 Find the image of a line reflected in a plane

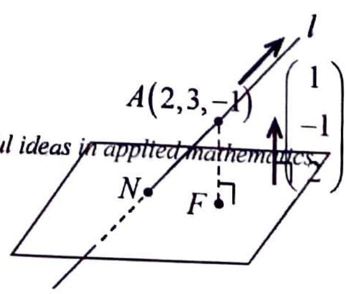
Example 4.46

The equation of line l is $x-2=\frac{3-y}{2}=\frac{z+1}{-3}$.

(i) Find the coordinates of the point of intersection N of l and the plane Π: $\mathbf{r}\bullet\begin{pmatrix}1\\-1\\2\end{pmatrix}=15$.

(ii) Find equation of the line of reflection l' of the line l in the plane Π.

Solution:



(i) $\frac{x-2}{1}=\frac{y-3}{-2}=\frac{z-(-1)}{-3} \Rightarrow \mathbf{r}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-3\end{pmatrix},\ \lambda\in\mathbb{R}$

Substitute into Π: $\mathbf{r}\bullet\begin{pmatrix}1\\-1\\2\end{pmatrix}=15$

$$\left(\begin{pmatrix}2\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-3\end{pmatrix}\right)\bullet\begin{pmatrix}1\\-1\\2\end{pmatrix}=15$$

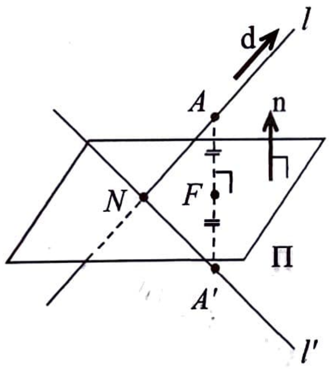
$-3-3\lambda=15$

$\lambda=-6$

Therefore $\overrightarrow{ON}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+(-6)\begin{pmatrix}1\\-2\\-3\end{pmatrix}=\begin{pmatrix}-4\\15\\17\end{pmatrix}$. So coordinates of $N\equiv(-4,15,17)$.

(ii) To find the equation of l' (reflection of l in plane l), we

- Take an arbitrary point A on l and find F, the foot of the perpendicular from A to the plane l
- Find A' , which is the reflected image of A in the plane l , using ratio theorem
- Find N, which is the point of intersection of l and l
- The line l' is formed using point A' and N



Let $A\equiv(2,3,-1)$ and F be the foot of the perpendicular from A to plane Π.

F lies on the line $AF \Rightarrow \overrightarrow{OF}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$ for some $\mu\in\mathbb{R}$.

F also lies in plane $\Pi \Rightarrow \overrightarrow{OF}\bullet\begin{pmatrix}1\\-1\\2\end{pmatrix}=15$.

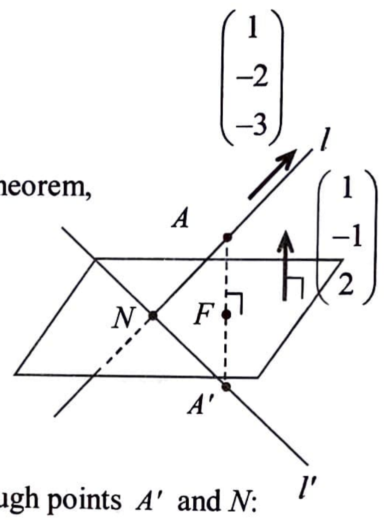
$$\left(\begin{pmatrix}2\\3\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\2\end{pmatrix}\right)\bullet\begin{pmatrix}1\\-1\\2\end{pmatrix}=15$$

$-3+6\mu=15$

$\mu=3$

$$\therefore \quad \overrightarrow{OF}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+3\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}5\\0\\5\end{pmatrix}$$

$$\overrightarrow{OF}=\frac{|AF|\overrightarrow{OA'}+|AE|\overrightarrow{OA}}{|AF|+|AE|}$$



Let A' be the reflection of A in the plane Π. By Ratio Theorem,

$\overrightarrow{OF}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OA'}\right)$.

$\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}$

$$\overrightarrow{OA'}=2\begin{pmatrix}5\\0\\5\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}8\\-3\\11\end{pmatrix}$$

The reflection l' of the line l in the plane Π passes through points A' and N:

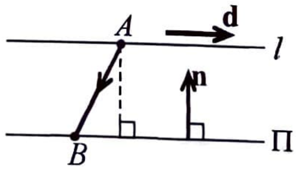
$$\overrightarrow{A'N}=\overrightarrow{ON}-\overrightarrow{OA'}=\begin{pmatrix}-4\\15\\17\end{pmatrix}-\begin{pmatrix}8\\-3\\11\end{pmatrix}=\begin{pmatrix}-12\\18\\6\end{pmatrix}$$

Therefore equation of l' is $\mathbf{r}=\begin{pmatrix}8\\-3\\11\end{pmatrix}+s\begin{pmatrix}-12\\18\\6\end{pmatrix},\ s\in\mathbb{R}$

4.5.7.6 Determine the distance between a parallel line and a plane

To find the distance between a parallel line and a plane, the method is similar to finding the distance from a point A on the line to the plane.

Fig. 4.42



Example 4.47

The concept of vectors is one of the most useful ideas in applied mathematics.

Show that the line l whose vector equation is $\mathbf{r}=2\mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda(\mathbf{i}-\mathbf{j}+4\mathbf{k})$ is parallel to the plane Π whose equation is $\mathbf{r}\cdot(\mathbf{i}+5\mathbf{j}+\mathbf{k})=5$ and find the distance between them.

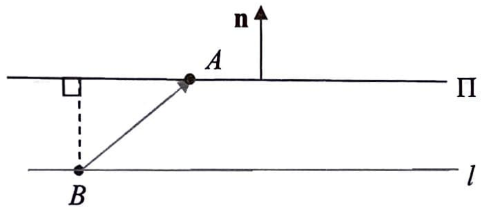
Solution:

Let the direction vector of the line be **d** and the normal of the plane be **n**.

$$\mathbf{n}\cdot\mathbf{d}=\begin{pmatrix}1\\5\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\4\end{pmatrix}=1-5+4=0.$$

Since the normal of the plane Π is perpendicular to the direction vector of the line l, line l is parallel to the plane Π.

Let B be $(2,-2,3)$. Since B lies on l, the perpendicular distance between l and Π is the perpendicular distance between the point B and Π. Thus,



Distance between l and Π

$$=\frac{|D-\mathbf{b}\cdot\mathbf{n}|}{|\mathbf{n}|}=\frac{\left|5-\begin{pmatrix}2\\-2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\5\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\5\\1\end{pmatrix}\right|}=\frac{|5-(-5)|}{\sqrt{27}}=\frac{10}{\sqrt{27}}.$$

Think Zone

Alternatively, take $A(0,1,0)$ on Π.

Required distance

$$=\left|\overrightarrow{AB}\cdot\mathbf{n}\right|$$

$$=\left|\begin{pmatrix}2\\-3\\3\end{pmatrix}\cdot\frac{1}{\sqrt{27}}\begin{pmatrix}1\\5\\1\end{pmatrix}\right|=\frac{10}{\sqrt{27}}.$$

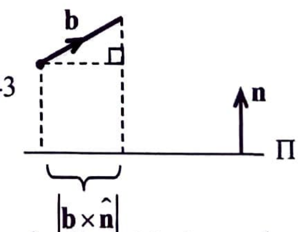
Self-Review 4.16

Show that the line $x+1=y=\dfrac{z-3}{2}$ is parallel to the plane $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})=3$ and find the distance between them. $[7\sqrt{3}/3]$

4.5.7.7 Find the length of projection of a vector onto a plane

Length of projection of **b** onto plane Π $=\left|\mathbf{b}\times\hat{\mathbf{n}}\right|$

Fig. 4.43



Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

Example 4.48

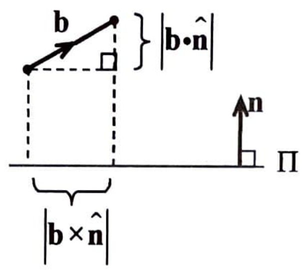
The plane Π has equation $2x + y - 2z = 8$. Find the length of projection of the vector $\mathbf{j} + 4\mathbf{k}$ onto the plane.

Solution:

length of projection of the vector $\begin{pmatrix}0\\1\\4\end{pmatrix}$ onto plane Π

$$= \left| \begin{pmatrix}0\\1\\4\end{pmatrix} \times \frac{1}{3}\begin{pmatrix}2\\1\\-2\end{pmatrix} \right|$$

$$= \frac{1}{3}\left| \begin{pmatrix}-6\\8\\-2\end{pmatrix} \right| = \frac{2}{3}\sqrt{26}.$$



Think Zone:

What does $|\mathbf{b} \cdot \hat{\mathbf{n}}|$ represent in this case?

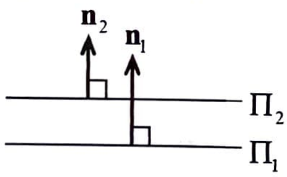
4.5.8 Problems Involving Two Planes

4.5.8.1 Determine if two planes are parallel or perpendicular

Given two planes Π_1, Π_2 with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively,

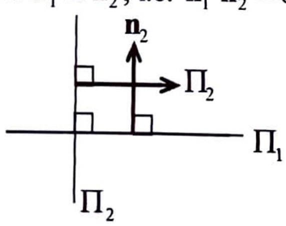
(a) the planes are *parallel* if $\mathbf{n}_1 // \mathbf{n}_2$, i.e. $\mathbf{n}_2 = \lambda \mathbf{n}_1$ for some $\lambda \in \mathbb{R}$

Fig. 4.44



(b) the planes are *perpendicular* if $\mathbf{n}_1 \perp \mathbf{n}_2$, i.e. $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.

Fig. 4.45



E.g. Let $\mathbf{n}_i$ be the normal vector to the plane $\Pi_i, i = 1,2,3,4$

Planes $\Pi_1 : \mathbf{r} \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} = 0$ and $\Pi_2 : \mathbf{r} \cdot \begin{pmatrix}3\\3\\6\end{pmatrix} = -2$ are parallel since $\mathbf{n}_2 = \begin{pmatrix}3\\3\\6\end{pmatrix} = 3\begin{pmatrix}1\\1\\2\end{pmatrix} = 3\mathbf{n}_1$.

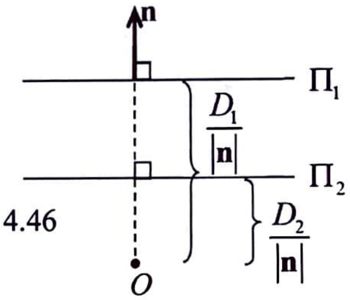
Planes $\Pi_3:\ \mathbf{r}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}=3$ and $\Pi_4:\ \mathbf{r}\bullet\begin{pmatrix}2\\-1\\0\end{pmatrix}=4$ are perpendicular since $\mathbf{n_3}\bullet\mathbf{n_4}\begin{pmatrix}1\\2\\2\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\0\end{pmatrix}=0$.

4.5.8.2 Determine the distance between two parallel planes and find the reflection of a plane in another parallel plane

Given two parallel planes with equations in scalar product form
$\Pi_1:\ \mathbf{r}\bullet\mathbf{n}=D_1$ and $\Pi_2:\ \mathbf{r}\bullet\mathbf{n}=D_2$.

Then perpendicular distance from O to Π_1 and Π_2 are $\frac{D_1}{|\mathbf{n}|}$ and $\frac{D_2}{|\mathbf{n}|}$ respectively.

Fig. 4.46



The distance between Π_1 and Π_2 is given by

$$\left|\frac{D_1}{|\mathbf{n}|}-\frac{D_2}{|\mathbf{n}|}\right|=\frac{|D_1-D_2|}{|\mathbf{n}|}$$

Note: The normal vector **n** must be the *same* in both equations.

Example 4.49

The equations of three parallel planes are as follows:

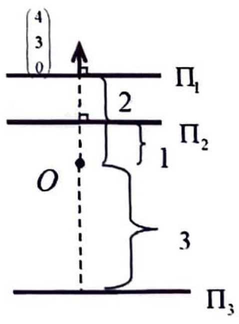
$$\Pi_1:\ \mathbf{r}\bullet\begin{pmatrix}4\\3\\0\end{pmatrix}=10,\quad \Pi_2:\ \mathbf{r}\bullet\begin{pmatrix}4\\3\\0\end{pmatrix}=5 \text{ and } \Pi_3:\ \mathbf{r}\bullet\begin{pmatrix}-4\\-3\\0\end{pmatrix}=15$$

Find the perpendicular distance between
(a) Π_1 and Π_2, (b) Π_1 and Π_3.

Solution:

$$\Pi_1:\ \mathbf{r}\bullet\begin{pmatrix}4\\3\\0\end{pmatrix}=10,\quad \Pi_2:\ \mathbf{r}\bullet\begin{pmatrix}4\\3\\0\end{pmatrix}=5 \text{ and } \Pi_3:\ \mathbf{r}\bullet\begin{pmatrix}-4\\-3\\0\end{pmatrix}=15$$

Re-write $\Pi_3:\ \mathbf{r}\bullet\begin{pmatrix}4\\3\\0\end{pmatrix}=-15$



(a) Perpendicular distance between Π_1 and Π_2 = $\frac{|D_1-D_2|}{|\mathbf{n}|}=\frac{|10-5|}{5}=1$

Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

Alternatively, distance between Π_1 and $\Pi_2 = \left|\frac{D_1}{|\mathbf{n}|} - \frac{D_2}{|\mathbf{n}|}\right| = \left|\frac{10}{5} - \frac{5}{5}\right| = |2-1| = 1$

(b) Perpendicular distance between Π_1 and Π_3 = $\frac{|D_1 - D_3|}{|\mathbf{n}|} = \frac{|10-(-15)|}{5} = 5$.

Alternatively, distance between Π_1 and $\Pi_3 = \left|\frac{D_1}{|\mathbf{n}|} - \frac{D_3}{|\mathbf{n}|}\right| = \left|\frac{10}{5} - \frac{(-15)}{5}\right| = |2-(-3)| = 5$

Example 4.50

Show that the planes
$\pi_1: \mathbf{r} = 6\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{j} + \mathbf{k})$ and $\pi_2: \mathbf{r} = 2\mathbf{i} + \mathbf{k} + s(\mathbf{i} - \mathbf{j}) + t(3\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ are parallel.
Find
(a) the distance between the planes;
(b) an equation of the mirror image of π_1 in π_2.

Solution:

	ThinkZone

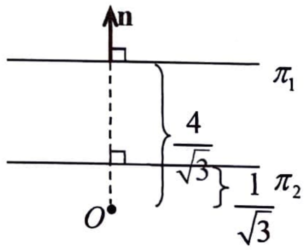
$$\mathbf{n}_1 = \begin{pmatrix}4\\-1\\3\end{pmatrix} \times \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}-4\\-4\\4\end{pmatrix} = -4\begin{pmatrix}1\\1\\-1\end{pmatrix} \text{ and } \mathbf{n}_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix} \times \begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}-4\\-4\\4\end{pmatrix} = -4\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

Since the normal vectors to both planes are parallel (in fact equal), the planes are parallel.

Taking the common normal vector to be $\begin{pmatrix}1\\1\\-1\end{pmatrix}$

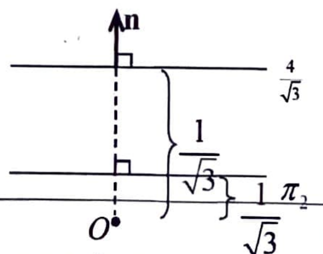
$$\pi_1: \mathbf{r} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}6\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} = 4$$

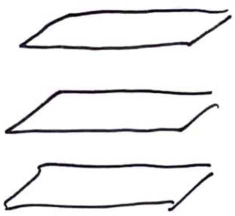
$$\pi_2: \mathbf{r} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} = 1$$



(a) Distance between π_1 and π_2 is $\left|\frac{D_1}{|\mathbf{n}|} - \frac{D_2}{|\mathbf{n}|}\right| = \left|\frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| = \frac{3}{\sqrt{3}} = \sqrt{3}$ units.

(b) Let π_1' be the image of π_1 reflected in π_2. $\pi_1': \mathbf{r} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} = D_1'$



$\pi_1: \mathbf{r}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=4$ $\pi_2: \mathbf{r}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=1$ $\pi_1': \mathbf{r}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=D_1'$  Equation of π_1': $\frac{D_1'+4}{2}=1$, $D_1'=-2$ Hence the mirror image of π_1 in π_2 is π_1': $\mathbf{r}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=-2$	Why do we have a negative sign on the RHS of the equation?

4.5.8.3 Find the line of intersection of two non-parallel planes

Two non-parallel planes must intersect along a straight line.

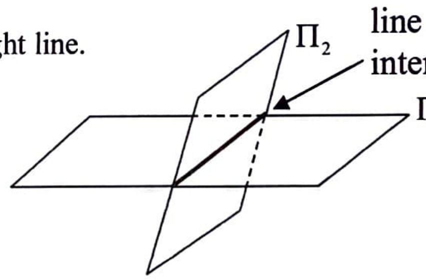


Fig. 4.47

Method 1 (GC): Use GC to solve the Cartesian equations of the planes simultaneously

Example 4.51

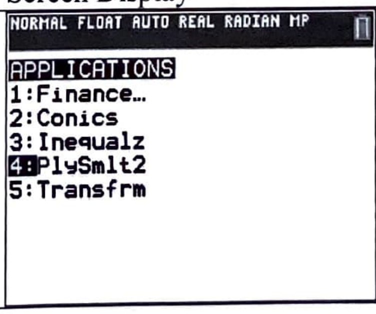
Find a vector equation of the line of intersection of the planes π_1 : $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})=6$ and π_2 : $\mathbf{r}\cdot(2\mathbf{i}-3\mathbf{j}+2\mathbf{k})=2$.

Solution:

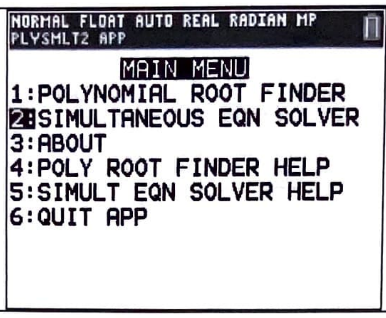
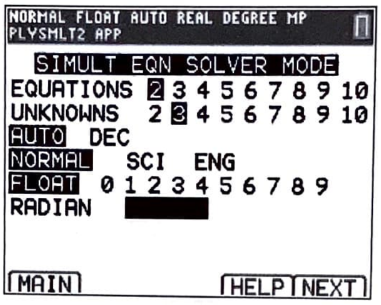
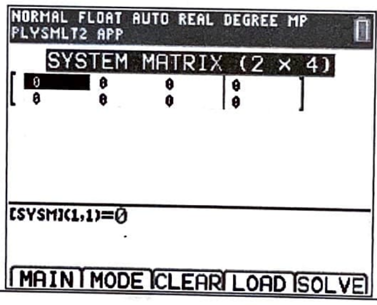
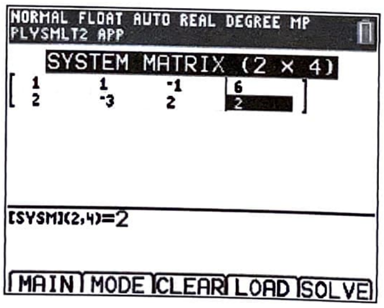
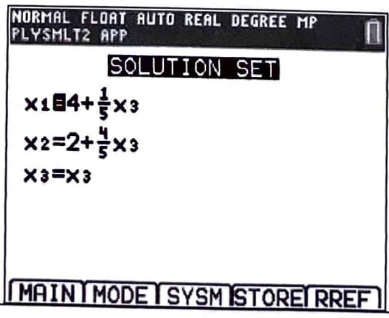
Express the equations of π_1 and π_2 in Cartesian form:

π_1 : $x+y-z=6$; π_2 : $2x-3y+2z=2$

Solve the above system of linear equations using GC:

Keystrokes	Screen Display
Press 	

Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

press 2	
select 2 **EQUATIONS** 3 **UNKNOWNS** and press GRAPH to select **NEXT**	
	
Press 1 ENTER 1 ENTER (−) 1 ENTER 6 ENTER And so on. Press GRAPH to select **SOLVE**	
	

The concept of vectors is one of the most useful ideas in applied mathematics.

Let $x = x_1, y = x_2, z = x_3$ and letting $x_3 = \lambda$, we obtain $x = 4 + \frac{1}{5}\lambda,\ y = 2 + \frac{4}{5}\lambda,\ z = \lambda$

Hence $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix}$ and writing $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

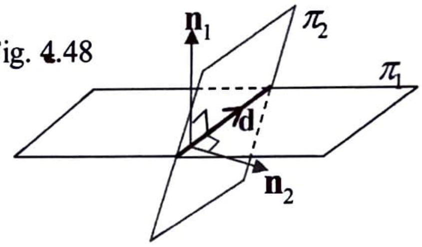
equation of the line of intersection of π_1 and π_2 is $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}$

Method 2 (Non-GC): Find a direction vector of the line of intersection and a point on the line

Let $\pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = D_1$ and $\pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = D_2$ be two non-parallel planes and l: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d},\ \lambda \in \mathbb{R}$ the line of intersection of π_1 and π_2.

Since l is parallel to both planes, its direction vector $\mathbf{d}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Hence the direction vector of the line of intersection of π_1 and π_2, $\mathbf{d}$, is given by

Fig. 4.48



$$\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$$

Finally, a point P on l is a common point of π_1 and π_2 and this can be found by letting $\mathbf{r} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and substituting into the equations of π_1 and π_2 to give *two equations in three unknowns*. This means the system of linear equations has **one degree of freedom** and we can let any of the variables x, y or z be any number we like, usually the number 0. The other two variables can then be found by solving the two simultaneous equations. The point P is now determined and the equation of l can now be written down.

Example 4.52 (Independent Learning)

Without the use of GC, find the vector equation of the line of intersection of the pair of planes

π_1: $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ π_2: $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$

Solution:

$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ \mathbf{n}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$	**ThinkZone:**

Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

To find a point common to π_1 and π_2, let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

π_1: $\mathbf{r}\cdot\mathbf{n}_1=\mathbf{a}\cdot\mathbf{n}_1$ $\qquad$ π_2: $\mathbf{r}\cdot\mathbf{n}_2=\mathbf{a}\cdot\mathbf{n}_2$

$$\mathbf{r}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=1 \qquad \mathbf{r}\cdot\begin{pmatrix}-2\\1\\-1\end{pmatrix}=\begin{pmatrix}2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\-1\end{pmatrix}=-4+1=-3$$

$$x+y=1.........(1) \qquad -2x+y-z=-3\Rightarrow 2x-y+z=3.........(2)$$

Put $x=0$ into equations (1) and (2), we have

(1): $\quad y=1$

(2): $\quad -y+z=3\Rightarrow -1+z=3\Rightarrow z=4$

$\therefore$ a point common to π_1 and π_2 is $(0,1,4)$

Hence the vector equation of the line of intersection of the planes is

$$\mathbf{r}=\begin{pmatrix}0\\1\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\3\end{pmatrix},\quad \lambda\in\mathbb{R}$$

Can we set $y = 0$ or $z = 0$ instead?

Sometimes putting $x = 0$ may not work. Try finding the line of intersection of planes given by $\mathbf{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=2$ & $\mathbf{r}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}=3$.

Example 4.53

The two planes $\pi_1:\mathbf{r}\cdot(2\mathbf{i}+\mathbf{j}+\alpha\mathbf{k})=6$ where $\alpha\in\mathbb{R}$ and $\pi_2:\mathbf{r}\cdot(4\mathbf{i}-\mathbf{j}+3\mathbf{k})=2$ intersect along a line l.

(i) Find a vector equation of l in terms of α.

(ii) Given that l is parallel to the vector $2\mathbf{i}-\mathbf{j}-3\mathbf{k}$. Hence or otherwise, find the value of α.

Solution:

(i) Let $\mathbf{n}_i$ be the normal to the plane $\pi_i, i=1,2$

$$\mathbf{n}_1\times\mathbf{n}_2=\begin{pmatrix}2\\1\\\alpha\end{pmatrix}\times\begin{pmatrix}4\\-1\\3\end{pmatrix}=\begin{pmatrix}3+\alpha\\-(6-4\alpha)\\-2-4\end{pmatrix}=\begin{pmatrix}3+\alpha\\-6+4\alpha\\-6\end{pmatrix}$$

To find a point common to π_1 and π_2:

π_1: $2x+y+\alpha z=6$

π_2: $4x-y+3z=2$

ThinkZone:

Set $z = 0$. Then $\begin{matrix} 2x+y=6 \\ 4x-y=2 \end{matrix}$

$$\Rightarrow 6x = 8 \quad \therefore x = \frac{4}{3}, y = \frac{10}{3}.$$

Equation of the line of intersection is

$$\mathbf{r} = \begin{pmatrix} \frac{4}{3} \\ \frac{10}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3+\alpha \\ -6+4\alpha \\ -6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Can we set $y = 0$ or $x = 0$ instead?

(b) "Hence" Method:

$$\begin{pmatrix} 3+\alpha \\ -6+4\alpha \\ -6 \end{pmatrix} = k \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

$$\Rightarrow 3+\alpha = 2k$$
$$-6+4\alpha = -k$$
$$-6 = -3k \Rightarrow k = 2$$
$$\therefore \alpha = 1$$

"Otherwise" Method:

$$\mathbf{d} \perp \mathbf{n}_1 \quad \Rightarrow \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ \alpha \end{pmatrix} = 0 \Rightarrow 4-1-3\alpha = 0 \Rightarrow \alpha = 1.$$

4.5.8.4 Find the angle between two non-parallel planes

Let π_1 and π_2 be two planes with normals $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively.
The acute angle θ between two planes is given by $|\mathbf{n}_1 \bullet \mathbf{n}_2| = |\mathbf{n}_1||\mathbf{n}_2|\cos\theta$ where $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vectors of the planes.
Note: The modulus is applied to the LHS to ensure that $\cos\theta > 0$ so that θ is acute.
The acute angle θ between two planes is therefore

$$\cos\theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$$

Fig. 4.49

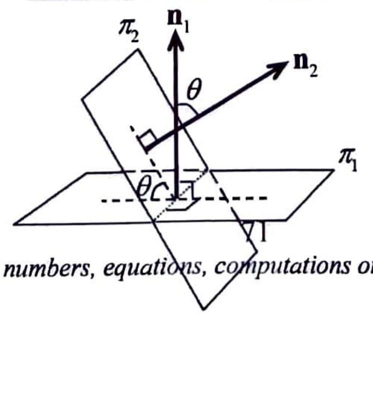
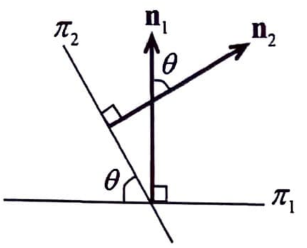


Fig. 4.50
(side-view)



ThinkZone:
Can you observe that the angle between 2 planes is also the same as the angle between the normal of the 2 planes?

Example 4.54

Find the acute angle between the planes $\mathbf{r}\bullet(\mathbf{i}+\mathbf{j}+\mathbf{k})=3$ and $\mathbf{r}\bullet(2\mathbf{i}-2\mathbf{j}-\mathbf{k})=1$.

Solution:

$$\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\bullet\begin{pmatrix}2\\-2\\-1\end{pmatrix}\right| = \left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|\left|\begin{pmatrix}2\\-2\\-1\end{pmatrix}\right|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{1`}{3\sqrt{3}}.$$

Thus $\theta = 78.9°$ (1 d.p.)

Self-Review 4.17:

A tetrahedron has vertices at the points $A(2,-1,0)$, $B(3,0,1)$, $C(1,-1,2)$, $D(-1,3,0)$. Find the cosine of the angle between the faces ABC and ABD.

$[\frac{4}{\sqrt{259}}]$

4.6 Miscellaneous Examples

Example 4.55 (2014/ACJC Prelim/I/8)

The planes p_1 and p_2 have equations $\mathbf{r}\bullet\begin{pmatrix}-1\\-2\\2\end{pmatrix}=-1$ and $\mathbf{r}\bullet\begin{pmatrix}-7\\4\\4\end{pmatrix}=1$ respectively.

(i) Find the acute angle between p_1 and p_2. [2]

(ii) The point A $(2, \alpha, 3)$ is equidistant from the planes p_1 and p_2. Calculate the two possible values of α. [5]

(iii) Find the position vector of the foot of perpendicular from B $(0, 1, 2)$ to the plane p_1. Hence find the cartesian equation of the plane p_3 such that p_3 is parallel to p_1 and point B is equidistant from planes p_1 and p_3. [4]

(iv) Find a vector equation of the line of intersection between p_1 and p_2. [1]

Solution:

The concept of vectors is one of the most useful ideas in applied mathematics.

(i)	Acute angle between p_1 and p_2, $\theta = \cos^{-1}\left\{\dfrac{\left|\begin{pmatrix}-7\\4\\4\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}\right|}{\sqrt{7^2+4^2+4^2}\sqrt{1^2+2^2+2^2}}\right\} = 74.97° = 75.0°$
(ii)	Equation of plane containing A and parallel to p_1. $p_1^*: \mathbf{r}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix} = \begin{pmatrix}2\\\alpha\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix} = 4-2\alpha$ Equation of plane containing A and parallel to p_2. $p_2^*: \mathbf{r}\cdot\begin{pmatrix}-7\\4\\4\end{pmatrix} = \begin{pmatrix}2\\\alpha\\3\end{pmatrix}\cdot\begin{pmatrix}-7\\4\\4\end{pmatrix} = -2+4\alpha$ Distance between p_1 and $p_1{}^* = \dfrac{\lvert -1-(4-2\alpha)\rvert}{\left\lvert\begin{pmatrix}-1\\-2\\2\end{pmatrix}\right\rvert} = \dfrac{1}{\sqrt{1^2+2^2+2^2}}\lvert 2\alpha-5\rvert = \dfrac{\lvert 2\alpha-5\rvert}{3}$ Distance between p_2 and $p_2{}^* = \dfrac{\lvert 1-(-2+4\alpha)\rvert}{\left\lvert\begin{pmatrix}-7\\4\\4\end{pmatrix}\right\rvert} = \dfrac{\lvert 3-4\alpha\rvert}{\sqrt{7^2+4^2+4^2}} = \dfrac{1}{9}\lvert 3-4\alpha\rvert$ $\therefore \left\lvert\dfrac{2\alpha-5}{3}\right\rvert = \left\lvert\dfrac{3-4\alpha}{9}\right\rvert \Rightarrow 3\lvert 2\alpha-5\rvert = \lvert 3-4\alpha\rvert$ Squaring, $3^2(-5+2\alpha)^2 = (3-4\alpha)^2 \Rightarrow [3(-5+2\alpha)+3-4\alpha][3(-5+2\alpha)-(3-4\alpha)] = 0$ which simplifies to $(\alpha-6)(5\alpha-9)=0 \Rightarrow \alpha = 6$ or $\dfrac{9}{5}$.
(iii)	Let M be the position vector of the foot of perpendicular from B to p_1. Equation of line segment BM $r = \begin{pmatrix}0\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}-1\\-2\\2\end{pmatrix}$ When line segment BM intersects p_1.

$$\begin{pmatrix} -\lambda \\ 1-2\lambda \\ 2+2\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = -1$$

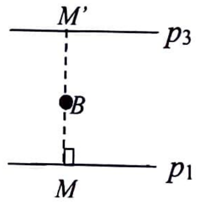
$$\lambda - 2 + 4\lambda + 4 + 4\lambda = -1$$

$$\lambda = -\frac{1}{3}$$

$$\therefore \overrightarrow{OM} = \frac{1}{3}\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

If B is equidistant from p_1 and p_3, then B is the midpoint of MM'

Let point M' be a point in plane p_3 such that M' is the foot of ⊥ from B to p_3



$$\overrightarrow{OB} = \frac{1}{2}(\overrightarrow{OM} + \overrightarrow{OM'})$$

$$\overrightarrow{OM'} = 2\overrightarrow{OB} - \overrightarrow{OM}$$

$$\overrightarrow{OM'} = 2\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 8 \end{pmatrix}$$

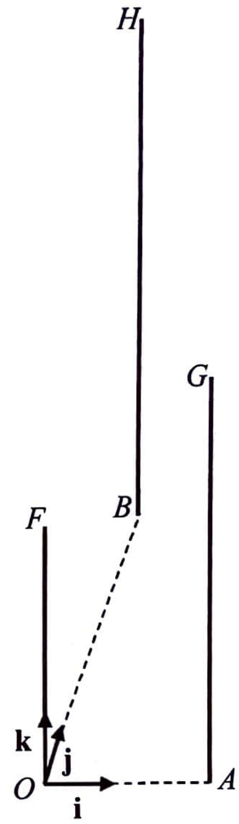
Equation of plane p_3 is

$$\mathbf{r} \cdot \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = 5$$

(iv)

By GC, equation of line of intersection is $\mathbf{r} = \begin{pmatrix} 1/9 \\ 4/9 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix}, \lambda \in \mathbb{R}$

Example 4.56 (2017/NYJC JC1 Block Test/I/8)(Application Question)



The diagram (not drawn to scale) shows three vertical flagpoles, *OF*, *AG*, *BH*, with bases *O*, *A*, *B* respectively on an open and flat field, where $OA = 4$ metres and $OB = 8$ metres. The flagpoles have heights 10 metres, 14 metres and 18 metres respectively. The point *O* is taken as the origin, with unit vectors **i** along *OA*, **j** along *OB* and **k** along *OF*.

The chairman of a neighbourhood committee wants to hold a songbird competition in the open field. He needs to erect a triangular flat shade sail using the flagpoles as supports and with fixing points at *F*, *G* and *H*.

(i) Show that the cartesian equation of this shade sail is $x + y - z = -10$. **[3]**

To strengthen the structure, he has to erect another vertical pole *MN* where the base *M* is at position vector $2\mathbf{i} + 3\mathbf{j}$.

(ii) Calculate the height of the pole if *N* is just touching the shade sail. **[4]**

In order to hang some bird cages, he needs to tie a taut rope from the midpoint of *FH* to a point *P* on *GH*.

(iii) Find the coordinates of *P* if the length of the rope used is to be minimum. **[4]**

(iv) To help the residents find the location of the competition easily, the chairman ties a red helium balloon at *H* using another rope such that the balloon is vertically above *H*. Assuming that there is no wind, find the angle between this rope and the shade sail. **[2]**

Solution:

(i) $\overrightarrow{OF} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}, \overrightarrow{OG} = \begin{pmatrix} 4 \\ 0 \\ 14 \end{pmatrix}, \overrightarrow{OH} = \begin{pmatrix} 0 \\ 8 \\ 18 \end{pmatrix}$ $\overrightarrow{FG} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}, \overrightarrow{HG} = \begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix}$	

$\mathbf{n}=\begin{pmatrix}4\\0\\4\end{pmatrix}\times\begin{pmatrix}4\\-8\\-4\end{pmatrix}=\begin{pmatrix}32\\32\\-32\end{pmatrix}=32\begin{pmatrix}1\\1\\-1\end{pmatrix}$ $\mathbf{r}\bullet\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\10\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=-10$ $x+y-z=-10$	
(ii) Let $\overrightarrow{ON}=\begin{pmatrix}2\\3\\z\end{pmatrix}$ Since N lies on the shade sail, $\begin{pmatrix}2\\3\\z\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=-10,$ $2+3-z=-10$ $z=15$ Height of the pole MN = 15 m	Why the coordinates of N is $(2,3,z)$, where z is to be determined?
(iii) Let X be the midpoint of FH $\overrightarrow{OX}=\frac{1}{2}\left(\overrightarrow{OF}+\overrightarrow{OH}\right)=\begin{pmatrix}0\\4\\14\end{pmatrix}$ Equation of line GH is $\mathbf{r}=\begin{pmatrix}4\\0\\14\end{pmatrix}+\mu\begin{pmatrix}4\\-8\\-4\end{pmatrix},\ \mu\in\mathbb{R}$ Since point P lies on line GH, $\overrightarrow{OP}=\begin{pmatrix}4\\0\\14\end{pmatrix}+\mu\begin{pmatrix}4\\-8\\-4\end{pmatrix}$, for some $\mu\in\mathbb{R}$ $\overrightarrow{XP}=\begin{pmatrix}4\\0\\14\end{pmatrix}+\mu\begin{pmatrix}4\\-8\\-4\end{pmatrix}-\begin{pmatrix}0\\4\\14\end{pmatrix}=\begin{pmatrix}4\\-4\\0\end{pmatrix}+\mu\begin{pmatrix}4\\-8\\-4\end{pmatrix}$ $\overrightarrow{XP}\perp\overrightarrow{GH}$	

The concept of vectors is one of the most useful ideas in applied mathematics.

$\left[\begin{pmatrix}4\\-4\\0\end{pmatrix}+\mu\begin{pmatrix}4\\-8\\-4\end{pmatrix}\right]\cdot\begin{pmatrix}4\\-8\\-4\end{pmatrix}=0$ $48+96\mu=0\Rightarrow\mu=-\frac{1}{2}$ $\overrightarrow{OP}=\begin{pmatrix}4\\0\\14\end{pmatrix}-\frac{1}{2}\begin{pmatrix}4\\-8\\-4\end{pmatrix}=\begin{pmatrix}2\\4\\16\end{pmatrix}$ Coordinates of P is (2, 4, 16) OR Equation of line GH is $\mathbf{r}=\begin{pmatrix}0\\8\\18\end{pmatrix}+\mu\begin{pmatrix}4\\-8\\-4\end{pmatrix},\ \mu\in\mathbb{R}$ $\overrightarrow{OP}=\begin{pmatrix}0\\8\\18\end{pmatrix}+\mu\begin{pmatrix}4\\-8\\-4\end{pmatrix}$, for some $\mu\in\mathbb{R}$ $\overrightarrow{XP}=\begin{pmatrix}0\\8\\18\end{pmatrix}+\mu\begin{pmatrix}4\\-8\\-4\end{pmatrix}-\begin{pmatrix}0\\4\\14\end{pmatrix}=\begin{pmatrix}0\\4\\4\end{pmatrix}+\mu\begin{pmatrix}4\\-8\\-4\end{pmatrix}$ $\left[\begin{pmatrix}0\\4\\4\end{pmatrix}+\mu\begin{pmatrix}4\\-8\\-4\end{pmatrix}\right]\cdot\begin{pmatrix}4\\-8\\-4\end{pmatrix}=0\Rightarrow\mu=\frac{1}{2}$	
(iv) Let the angle between this rope and the shade sail be θ. $\sin\theta=\dfrac{\left\|\begin{pmatrix}1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}{\sqrt{3}}=\dfrac{1}{\sqrt{3}}$ Since θ is an obtuse angle, $\theta=180^o-35.3^o=144.7^o$	