TAMPINES MERIDIAN JUNIOR COLLEGE

JC1 PROMOTIONAL EXAMINATION

H2 MATHEMATICS 9758

3 hours

Additional materials: Printed Answer Booklet

List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** the questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 It is given that $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are non-zero real constants. The graph of $y = \mathrm{f}(x)$ passes through the points $(0,3)$ and $(-2,1)$. It is also known that one of the stationary points of $y = \mathrm{f}(x)$ is $(-3,3)$. Find the values of a, b, c and d. [4]

$\mathrm{f}(x) = ax^3 + bx^2 + cx + d$

$\mathrm{f}(0) = 3 \Rightarrow d = 3$

$\therefore \mathrm{f}(x) = ax^3 + bx^2 + cx + 3$

Note that at stationary point, $\mathrm{f}'(x) = 0$.

$\mathrm{f}'(x) = 3ax^2 + 2bx + c$

Since one of the stationary points of $y = \mathrm{f}(x)$ is $(-3,3)$,

$\mathrm{f}'(-3) = 3a(-3)^2 + 2b(-3) + c = 0$

$\therefore 27a - 6b + c = 0$ ------- (1)

Since one of the stationary point of $y = \mathrm{f}(x)$ is $(-3,3)$, it also means that $y = \mathrm{f}(x)$ passes through $(-3,3)$.

$\mathrm{f}(-3) = 3$

$a(-3)^3 + b(-3)^2 + c(-3) + 3 = 3$

$-27a + 9b - 3c = 0$ -------(2)

$\mathrm{f}(-2) = 1$

$a(-2)^3 + b(-2)^2 + c(-2) + 3 = 1$

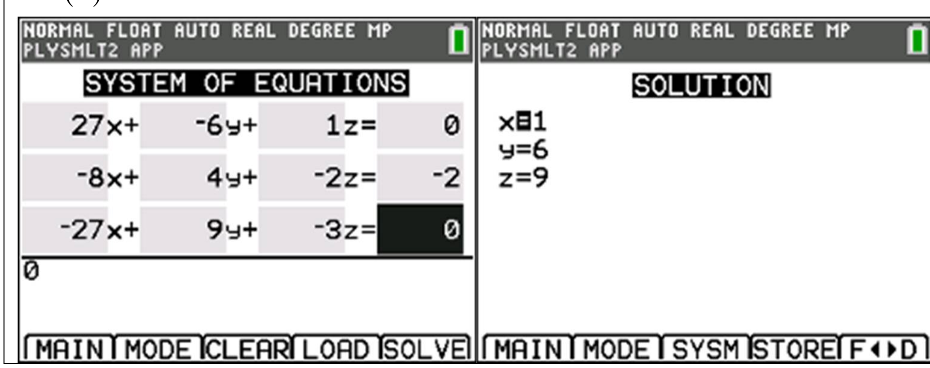
$-8a + 4b - 2c = -2$ -------(3)

Using GC,

$a = 1, b = 6, c = 9$

$d = 3$ (from above)

$\therefore \mathrm{f}(x) = x^3 + 6x^2 + 9x + 3$



2 **(a)** State a sequence of three transformations that will transform the curve with equation $y = \mathrm{f}(x)$ onto the curve with equation $y = 2\mathrm{f}(-x+3)$. [3]

(b) Without using a calculator, solve exactly the inequality

$$\frac{x^2 - 2x + 2}{2x+1} > 1. \qquad [4]$$

(a)

$y = \mathrm{f}(x)$

$\downarrow$ Replace x by $x+3$

$y = \mathrm{f}(x+3)$

$\downarrow$ Replace x by $-x$

$y = \mathrm{f}(-x+3)$

$\downarrow$ Replace y by $\frac{y}{2}$

$\frac{y}{2} = \mathrm{f}(-x+3)$

$\Rightarrow y = 2\mathrm{f}(-x+3)$

The sequence of 3 transformations are as follows:

I. Translation of 3 units in the negative $x-$direction.

II. Reflection in the $y-$axis.

III. Stretch by factor 2 parallel to the $y-$axis.

Descriptions for Transformation of Curves must be technically precise. Do not use your own terms or phrases.

Alternative Method

$y = \mathrm{f}(x)$

$\downarrow$ Replace x by $-x$

$y = \mathrm{f}(-x)$

$\downarrow$ Replace x by $x-3$

$y = \mathrm{f}(-(x-3))$

$\Rightarrow y = \mathrm{f}(-x+3)$

$\downarrow$ Replace y by $\frac{y}{2}$

$\frac{y}{2} = \mathrm{f}(-x+3)$

$\Rightarrow y = 2\mathrm{f}(-x+3)$

The sequence of 3 transformations are as follows:

I. Reflection in the $y-$axis.

II. Translation of 3 units in the positive $x-$direction.

III. Stretch by factor 2 parallel to the $y-$axis.

There are many possible answers. Just use replacement of variables to check that you can indeed go from $y = \mathrm{f}(x)$ to $y = 2\mathrm{f}(-x+3)$.

(b) Without using a calculator, solve exactly the inequality

$$\frac{x^2-2x+2}{2x+1}>1. \qquad [4]$$

(b)

$$\frac{x^2-2x+2}{2x+1}>1, \quad x \neq -\frac{1}{2}$$

$$\frac{x^2-2x+2}{2x+1}-1>0$$

$$\frac{x^2-2x+2-2x-1}{2x+1}>0$$

$$\frac{x^2-4x+1}{2x+1}>0$$

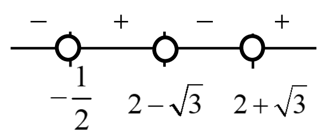
We do not cross multiply when we are not sure how the inequality sign should change. So for this question, we cannot cross multiply by $2x+1$ since it could be negative or positive.

<u>Method 1: Completing the Square</u>

$$\frac{(x-2)^2-3}{2x+1}>0$$

$$\frac{\left(x-2+\sqrt{3}\right)\left(x-2-\sqrt{3}\right)}{2x+1}>0$$

$$\frac{\left(x-\left(2-\sqrt{3}\right)\right)\left(x-\left(2+\sqrt{3}\right)\right)}{2x+1}>0$$



$$x>2+\sqrt{3} \text{ or } -\frac{1}{2}<x<2-\sqrt{3}$$

<u>Method 2: Quadratic Formula</u>

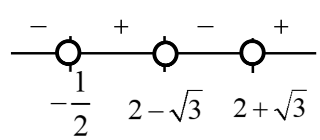
Consider $x^2-4x+1=0$

x^2-4x+1

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$$

$$=\frac{4\pm\sqrt{12}}{2}=\frac{4\pm2\sqrt{3}}{2}=2\pm\sqrt{3}$$

$$\frac{\left(x-\left(2-\sqrt{3}\right)\right)\left(x-\left(2+\sqrt{3}\right)\right)}{2x+1}>0$$



$$x>2+\sqrt{3} \text{ or } -\frac{1}{2}<x<2-\sqrt{3}$$

3 The curve C_1 has equation $y = 2x + 3 + \dfrac{12}{x-2}$, $x \neq 2$.

(a) Sketch C_1, showing clearly the equations of asymptotes, the coordinates of the axial intercepts and the turning points, if any. [4]

(b) By completing the square for both x and y, give a geometrical interpretation of the curve C_2 with equation $x^2 - 4x + y^2 + 6y - 3 = 0$.

Hence write down the number of real roots for the equation

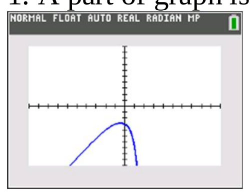
$$(x-2)^2 + \left(2x + 6 + \frac{12}{x-2}\right)^2 = 4^2. \qquad [3]$$

(a) $y = 2x + 3 + \dfrac{12}{x-2}, \quad x \neq 2$

Asymptotes: $y = 2x + 3$ and $x = 2$

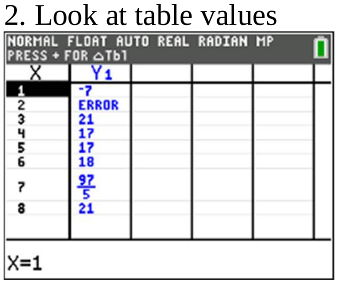
When $x = 0, y = -3$

1. A part of graph is out of standard zoom window

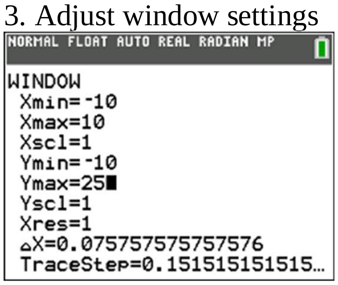


need to know to look for graph for $x > 2$

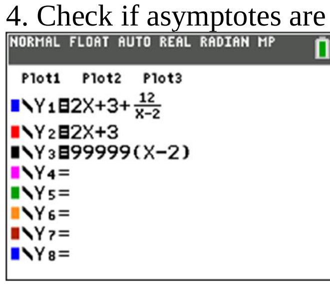
2. Look at table values

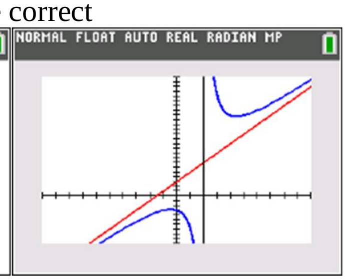


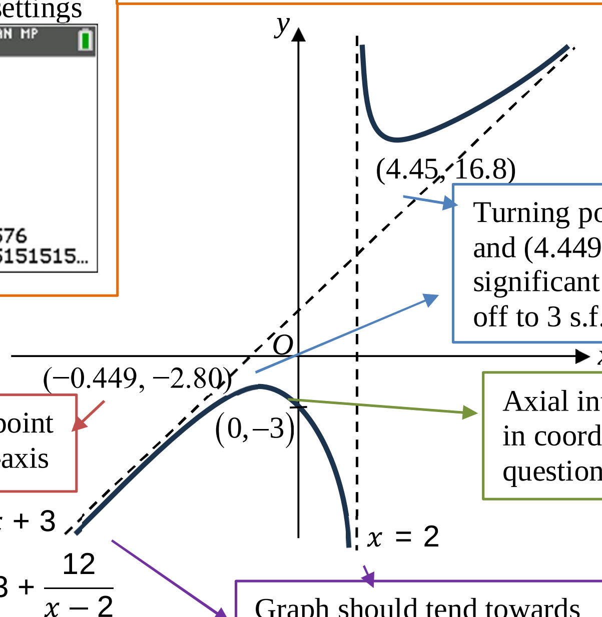
3. Adjust window settings



4. Check if asymptotes are correct







Turning points are (-0.44949,-2.7980) and (4.44948, 16.7980) to 5 significant figures (s.f.), please round off to 3 s.f. (exact not required)

Axial intercept(s) should be in coordinates, based on question requirements.

Note that maximum point is on the left of the y-axis

Graph should tend towards asymptotes (not away/cut) that should be represented by dotted lines

(b) By completing the square for both x and y, give a geometrical interpretation of the curve C_2 with equation $x^2-4x+y^2+6y-3=0$.

Hence write down the number of real roots for the equation

$$(x-2)^2+\left(2x+6+\frac{12}{x-2}\right)^2=4^2. \qquad [3]$$

(b)

$x^2-4x+y^2+6y-3=0$

$(x-2)^2-4+(y+3)^2-9-3=0$

$(x-2)^2+(y+3)^2=4^2$

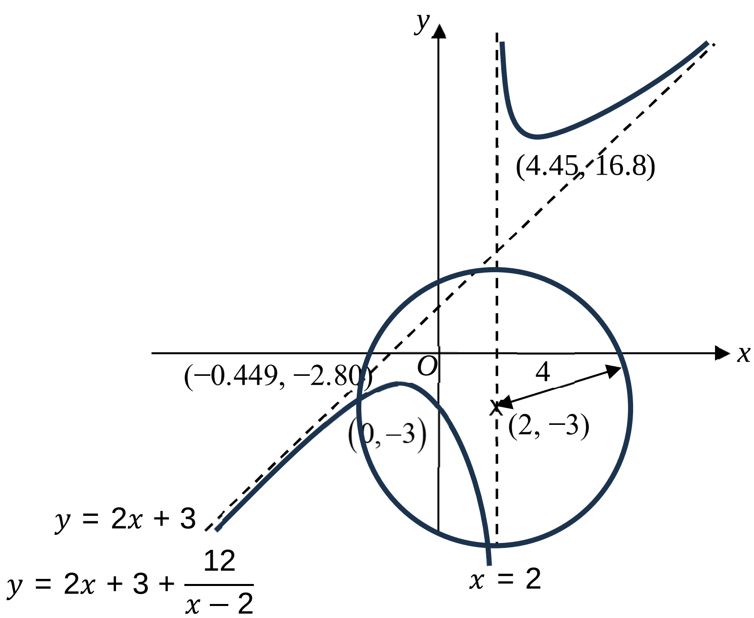
Provide detailed description for geometrical interpretations

C_2 is a **circle** with **centre (2, –3)** and **radius 4 units**.

There are 2 real roots for $(x-2)^2+\left(2x+3+\frac{12}{x-2}+3\right)^2=4^2$.

Read question carefully: "**write down** number of roots", not "find the roots"

"Hence" question, number of real roots for $(x-2)^2+\left(2x+3+\frac{12}{x-2}+3\right)^2=4^2$ is equivalent to the number of intersections between C_1 and C_2.



4 A manufacturer designs a rectangular tank. The base of the tank is a rectangle with adjacent sides of x metres by y metres and **fixed** diagonal length of D metres, where D is a real constant. The area of the base of the tank is denoted by A metres^2.

(a) Express A^2 in terms of x and D. [2]

(b) By differentiation, show that as x varies, the largest area of the base is achieved when $x = \dfrac{D}{\sqrt{2}}$. [4]

The manufacturer designs another type of water tank. Water is poured at a constant rate of 10 metres^3 per minute into the tank. The volume of the water, V metres^3, in the tank at time t minutes, is defined by $V = \dfrac{1}{9}\pi H^3$, where H metres is the height of the water in the tank at time t minutes.

(c) Find the rate of change of the height of the water in the tank in terms of π, when $H = 3$. [3]

(a)	$A = xy$ $D^2 = x^2 + y^2 \Rightarrow y^2 = D^2 - x^2$ $A^2 = x^2 y^2$ $= x^2\left(D^2 - x^2\right)$ $= D^2x^2 - x^4$

(b) By differentiation, show that as x varies, the largest area of the base is achieved when $x=\dfrac{D}{\sqrt{2}}$. [4]

(b)

$A^2 = D^2x^2 - x^4$

It is **not advisable** to use the form $A=\sqrt{D^2x^2-x^4}$ to find $\dfrac{dA}{dx}$ (and $\dfrac{d^2A}{dx^2}$ later on) due to the complicated expression. Using implicit differentiation is much easier for this question.

Differentiate wrt x:

$$2A\frac{dA}{dx}=2D^2x-4x^3$$

$$A\frac{dA}{dx}=D^2x-2x^3$$

When area is the largest, $\dfrac{dA}{dx}=0$

As a "show" question, you need to work towards $x=\dfrac{D}{\sqrt{2}}$ by solving for $\dfrac{dA}{dx}=0$. **Do not merely verify the result** by substituting $x=\dfrac{D}{\sqrt{2}}$ onto $\dfrac{dA}{dx}$.

$$D^2x-2x^3=0$$

$$x\left(D^2-2x^2\right)=0$$

$$x=0 \text{ or } x=\frac{D}{\sqrt{2}} \text{ or } x=-\frac{D}{\sqrt{2}}$$

As a "show" question, you need to state the reason for rejecting $x=0$ and $x=-\dfrac{D}{\sqrt{2}}$.

Since $x>0$, $x=\dfrac{D}{\sqrt{2}}$. (shown)

Differentiate wrt x,

$$\left(\frac{dA}{dx}\right)^2+A\frac{d^2A}{dx^2}=D^2-6x^2$$

When $x=\dfrac{D}{\sqrt{2}}$, $\dfrac{dA}{dx}=0$,

$$\frac{d^2A}{dx^2}=\frac{D^2-6\left(\frac{D}{\sqrt{2}}\right)^2}{A}$$

$$=-\frac{2D^2}{A}$$

To use the second derivate test, you need to substitute $x=\dfrac{D}{\sqrt{2}}$ into $\dfrac{d^2A}{dx^2}$ and simplify it sufficiently before concluding that $\dfrac{d^2A}{dx^2}<0$.

Since $A>0$, $D>0$, $\dfrac{d^2A}{dx^2}=-\dfrac{2D^2}{A}<0$

Hence, the cross-sectional area of the tank is the largest when $x=\dfrac{D}{\sqrt{2}}$.

The manufacturer designs another type of water tank. Water is poured at a constant rate of 10 metres^3 per minute into the tank. The volume of the water, V metres^3, in the tank at time t minutes, is defined by $V = \frac{1}{9}\pi H^3$, where H metres is the height of the water in the tank at time t minutes.

(c) Find the rate of change of the height of the water in the tank in terms of π, when $H = 3$. [3]

(c)	$V = \frac{1}{9}\pi H^3$ $\frac{\mathrm{d}V}{\mathrm{d}H} = \frac{1}{3}\pi H^2$ When $H = 3$, $\frac{\mathrm{d}V}{\mathrm{d}H} = 3\pi$ $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}H} \times \frac{\mathrm{d}H}{\mathrm{d}t}$ $\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{10}{3\pi}$ metres per minute Use chain rule to connect the given rate of change in the question, $\frac{\mathrm{d}V}{\mathrm{d}t} = 10$, and the rate of change we want to find, $\frac{\mathrm{d}H}{\mathrm{d}t}$. Be careful, $\frac{10}{3\pi} \neq \frac{10}{3}\pi$.

5 **(a)** By using the standard series from List of Formulae (MF27), find the series expansion of $e^{x^2}\left(\sqrt{1+2x}\right)$, up to and including the term in x^2. [3]

(b) Given that $y=\sin^{-1}x$, show that $\dfrac{d^2y}{dx^2}=x\left(\dfrac{dy}{dx}\right)^3$.

Hence, by further differentiation, find the Maclaurin series for $\sin^{-1}x$ in ascending powers of x up to and including the term in x^3. [5]

(a)	$e^{x^2}\left(\sqrt{1+2x}\right)=e^{x^2}(1+2x)^{\frac{1}{2}}$ $=\left(1+x^2+\cdots\right)\left(1+\frac{1}{2}(2x)+\dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(2x)^2+\cdots\right)$ $=\left(1+x^2+\cdots\right)\left(1+x-\frac{1}{2}x^2+\cdots\right)$ $=1+x+\frac{1}{2}x^2+\cdots\quad\left(\text{up to } x^2\right)$

Use MF 27 Standard Series for e^x and replace x by x^2

Use MF 27 Standard Series for $(1+x)^n$ and replace n by $\frac{1}{2}$ and x by $2x$

(b) Given that $y = \sin^{-1} x$, show that $\dfrac{d^2y}{dx^2} = x\left(\dfrac{dy}{dx}\right)^3$.

Hence, by further differentiation, find the Maclaurin series for $\sin^{-1} x$ in ascending powers of x up to and including the term in x^3. [5]

(b)	$y = \sin^{-1} x$ $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-\frac{1}{2}}$ $\dfrac{d^2y}{dx^2} = \left(-\dfrac{1}{2}\right)\left(1-x^2\right)^{-\frac{3}{2}}(-2x)$ $= x\left(1-x^2\right)^{-\frac{3}{2}}$ $= x\left(\dfrac{dy}{dx}\right)^3$ (Shown) $\dfrac{d^3y}{dx^3} = \left(\dfrac{dy}{dx}\right)^3 + 3x\left(\dfrac{dy}{dx}\right)^2\left(\dfrac{d^2y}{dx^2}\right)$ Remember to apply product rule here. Also, it is highly recommended to further differentiate from the “Shown” result, rather than uinsg your own expression or the previous un-simplified expression. when $x = 0$, $y = 0,\ \dfrac{dy}{dx} = 1,\ \dfrac{d^2y}{dx^2} = 0,\ \dfrac{d^3y}{dx^3} = 1.$ $\sin^{-1} x = x + \dfrac{1}{3!}x^3 + \cdots$ $= x + \dfrac{1}{6}x^3 + \cdots$ (up to x^3)

6 **(a)** The points A, B and C have position vectors $\begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -9 \\ 8 \\ -27 \end{pmatrix}$, with respect to the origin O.

(i) Show that the points A, B and C are collinear. [2]

(ii) Find the exact area of the triangle OAB. [3]

(b) The vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are defined such that $|\mathbf{u}| = 1$, $|\mathbf{v}| = 2$ and $|\mathbf{w}| = 3$.

(i) Suppose that $\mathbf{u} + \mathbf{v} = \mathbf{w}$. With the aid of a diagram, or otherwise, explain why $\mathbf{u}$ is a unit vector that is parallel to $\mathbf{w}$. [2]

(ii) Given instead that each of the three vectors is perpendicular to the sum of the other two vectors, find the exact value of $|\mathbf{u} + \mathbf{v} + \mathbf{w}|$. [4]

(a)(i)

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -9 \\ 8 \\ -27 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} -10 \\ 5 \\ -20 \end{pmatrix}$$

$$= -\frac{5}{2}\begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix}$$

$$= -\frac{5}{2}\overrightarrow{AB}$$

Express the scalar multiple of parallel vectors carefully.

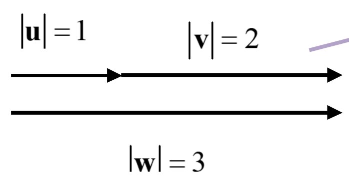
Since $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$ with a common point A, the points A, B and C are collinear. (Shown)

(ii) Find the exact area of the triangle *OAB*. [3]

(ii)	Area of triangle *OAB* $=\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}1\\3\\-7\end{pmatrix}\times\begin{pmatrix}5\\1\\1\end{pmatrix}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}10\\-36\\-14\end{pmatrix}\right\|$ $=\sqrt{398}$

(b) The vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are defined such that $\|\mathbf{u}\|=1$, $\|\mathbf{v}\|=2$ and $\|\mathbf{w}\|=3$.

(i) Suppose that $\mathbf{u}+\mathbf{v}=\mathbf{w}$. With the aid of a diagram, or otherwise, explain why $\mathbf{u}$ is a unit vector that is parallel to $\mathbf{w}$. [2]

(b)(i)	Since $\mathbf{u}+\mathbf{v}=\mathbf{w}$, and $\|\mathbf{u}\|=1$, $\|\mathbf{v}\|=2$ and $\|\mathbf{w}\|=3$, $\mathbf{u}$ and $\mathbf{v}$ must both be parallel to $\mathbf{w}$, as seen in the diagram.  Draw diagram with clear indication of the fact that $\|\mathbf{u}\|=1$ and $\|\mathbf{v}\|=2$ add up to $\|\mathbf{w}\|=3$. For $\mathbf{u}+\mathbf{v}=\mathbf{w}$, due to their lengths, $\mathbf{u}$ and $\mathbf{v}$ must both be parallel to $\mathbf{w}$. Since $\|\mathbf{u}\|=1$, $\mathbf{u}$ is a unit vector parallel to $\mathbf{w}$. Explain clearly why $\mathbf{u}$ is a unit vector.

(ii) Given instead that each of the three vectors is perpendicular to the sum of the other two vectors, find the exact value of $|\mathbf{u}+\mathbf{v}+\mathbf{w}|$. [4]

(ii)	Since each of the three vectors is perpendicular to the sum of the other two vectors, $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=0$ $\mathbf{v}\cdot(\mathbf{u}+\mathbf{w})=0$ $\mathbf{w}\cdot(\mathbf{v}+\mathbf{u})=0$ The dot products of the 2 perpendicular vectors equals 0. $\lvert\mathbf{u}+\mathbf{v}+\mathbf{w}\rvert^2=(\mathbf{u}+\mathbf{v}+\mathbf{w})\cdot(\mathbf{u}+\mathbf{v}+\mathbf{w})$ $=\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})+\mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot(\mathbf{u}+\mathbf{w})+\mathbf{w}\cdot\mathbf{w}+\mathbf{w}\cdot(\mathbf{v}+\mathbf{u})$ $=\lvert\mathbf{u}\rvert^2+\lvert\mathbf{v}\rvert^2+\lvert\mathbf{w}\rvert^2$ $=1^2+2^2+3^2$ $=14$ Note: $\lvert\mathbf{a}\rvert^2=\mathbf{a}\bullet\mathbf{a}$ $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=0$ $\mathbf{v}\cdot(\mathbf{u}+\mathbf{w})=0$ $\mathbf{w}\cdot(\mathbf{v}+\mathbf{u})=0$ Since $\lvert\mathbf{u}+\mathbf{v}+\mathbf{w}\rvert>0$, $\lvert\mathbf{u}+\mathbf{v}+\mathbf{w}\rvert=\sqrt{14}$

7 **(a)** Find $\int \frac{1}{\sqrt{4-x^2}}\,dx$. [1]

(b) Find $\int \cos^2 x\,dx$. [2]

(c) Using integration by parts, find $\int \tan^{-1} x\,dx$. [3]

(d) Use the substitution $u=\sqrt{x-2}$ to find the exact value of $\int_2^4 \frac{x+1}{x\sqrt{x-2}}\,dx$. [4]

(a)	$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{2^2-x^2}}\,dx$ $= \sin^{-1}\left(\frac{x}{2}\right)+C$
(b)	$\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx$ $= \frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+C$ Remember to apply Golden Rule!
(c)	$\int \tan^{-1} x\,dx$ $\int \tan^{-1} x\,dx = \int (1)\left(\tan^{-1} x\right)dx$ Keep, Differentiate "$\tan^{-1} x$". Integrate "1". $= x\tan^{-1} x - \int x\left(\frac{1}{1+x^2}\right)dx$ $= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$ $= x\tan^{-1} x - \frac{1}{2}\ln\left|1+x^2\right|+C$ $= x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right)+C,\quad$ since $1+x^2>0$

(d) Use the substitution $u=\sqrt{x-2}$ to find the exact value of $\int_2^4 \frac{x+1}{x\sqrt{x-2}}\,\mathrm{d}x$. [4]

Remember to change the limits too!

(d)	$\int_2^4 \frac{x+1}{x\sqrt{x-2}}\,\mathrm{d}x$ $=\int_0^{\sqrt{2}} \frac{(u^2+2)+1}{(u^2+2)u}2u\,\mathrm{d}u$ $=2\int_0^{\sqrt{2}} \frac{(u^2+2)+1}{(u^2+2)}\,\mathrm{d}u$ $=2\int_0^{\sqrt{2}} 1+\frac{1}{\left(u^2+\sqrt{2}^2\right)}\,\mathrm{d}u$ $=2\left[u+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\right]_0^{\sqrt{2}}$ $=2\left[\sqrt{2}+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)-0\right]$ $=2\left[\sqrt{2}+\frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)\right]$ $=\sqrt{2}\left(2+\frac{\pi}{4}\right)$ Ensure that all substitutions are completed in 1 step to reduce careless mistakes. It is recommended to express the integral in standard form first before applying MF27 integration formula. This usually reduces the chance of making careless mistakes.	$u=\sqrt{x-2}$ $u^2=x-2$ Differentiating with respect to x, $2u\frac{\mathrm{d}u}{\mathrm{d}x}=1$ When $x=4$, $u=\sqrt{4-2}=\sqrt{2}$, When $x=2$, $u=\sqrt{2-2}=0$,

8 A curve C has parametric equations

$$x = 2t - \sin 2t, \qquad y = 1 - \cos 2t, \qquad \text{where } -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

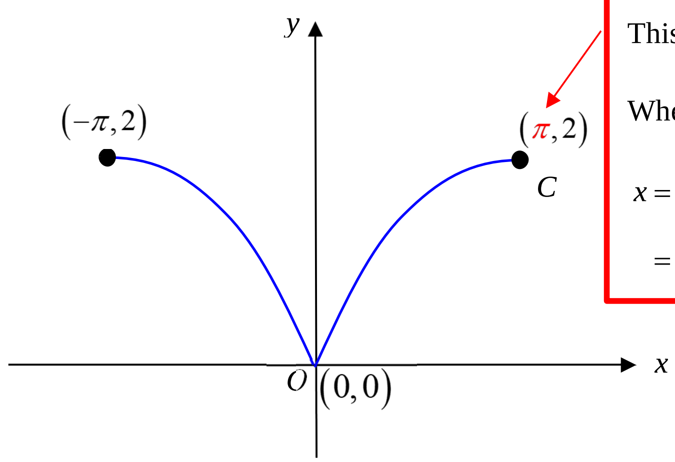
(a) Sketch the curve C. [2]

(b) The point P on C has coordinates $(2p - \sin 2p, 1 - \cos 2p)$. Show that the equation of the tangent at P is $y = x\cot p - 2p\cot p + 2$. [4]

The tangent to C at point Q, where $t = \frac{\pi}{4}$, meets the y-axis at the point L. The tangent to C at point R, where $t = -\frac{\pi}{4}$, also meets the y-axis at the point L.

(c) Explain why $\angle QLR$ is a right angle. Hence draw the triangle QLR on the same diagram as curve C in part **(a)**. [4]

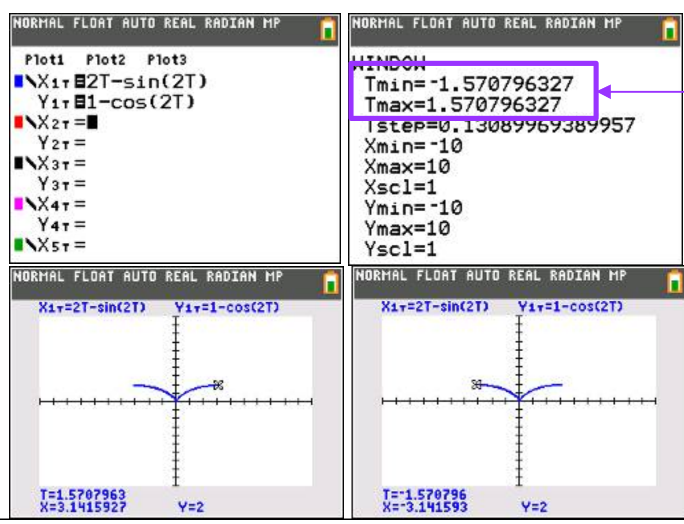
(a)



This is the x-coordinate when $t = \frac{\pi}{2}$.

When $t = \frac{\pi}{2}$,

$$x = 2\left(\frac{\pi}{2}\right) - \sin 2\left(\frac{\pi}{2}\right)$$
$$= \pi$$



Set Tmin and Tmax according to the range of values of t, i.e $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

(b) The point P on C has coordinates $(2p - \sin 2p, 1 - \cos 2p)$. Show that the equation of the tangent at P is $y = x\cot p - 2p\cot p + 2$. [4]

(b)	$x = 2t - \sin 2t$ $\quad\quad$ $y = 1 - \cos 2t$ $\frac{dx}{dt} = 2 - 2\cos 2t$ $\quad\quad$ $\frac{dy}{dt} = 2\sin 2t$ $\frac{dy}{dx} = \frac{2\sin 2t}{2 - 2\cos 2t}$ $= \frac{\sin 2t}{1 - \cos 2t}$ $= \frac{2\sin t\cos t}{1 - (1 - 2\sin^2 t)}$ $= \frac{2\sin t\cos t}{2\sin^2 t}$ $= \cot t$ At point P, $t = p \Rightarrow \frac{dy}{dx} = \cot p$ Equation of tangent at point P is $y - (1 - \cos 2p) = \cot p(x - (2p - \sin 2p))$ $y = x\cot p - 2p\cot p + \cot p\sin 2p + 1 - \cos 2p$ $y = x\cot p - 2p\cot p + \frac{\cos p}{\sin p}(2\sin p\cos p) + 1 - (1 - 2\sin^2 p)$ $y = x\cot p - 2p\cot p + 2\cos^2 p + 2\sin^2 p$ $y = x\cot p - 2p\cot p + 2$ (shown) Use of double angle formula is expected in order to simplify to the required form. Do not skip steps as this is a showing question.

The tangent to C at point Q, where $t = \frac{\pi}{4}$, meets the y-axis at the point L. The tangent to C at point R, where $t = -\frac{\pi}{4}$, also meets the y-axis at the point L.

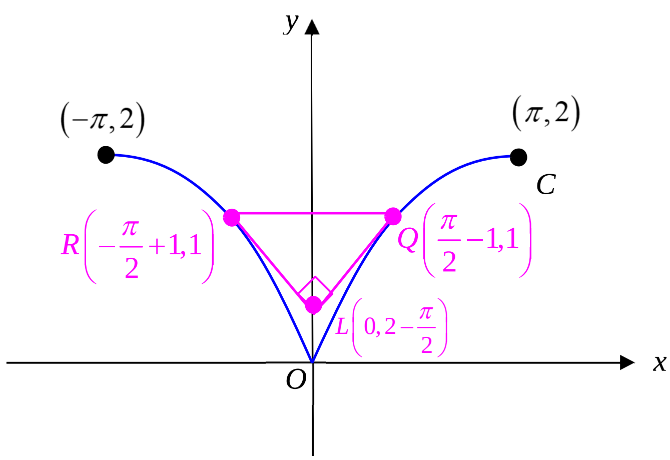
(c) Explain why $\angle QLR$ is a right angle. Hence draw the triangle QLR on the same diagram as curve C in part **(a)**. [4]

(c)

At $t = \frac{\pi}{4}$, the gradient of the tangent is $\cot\left(\frac{\pi}{4}\right) = 1$.

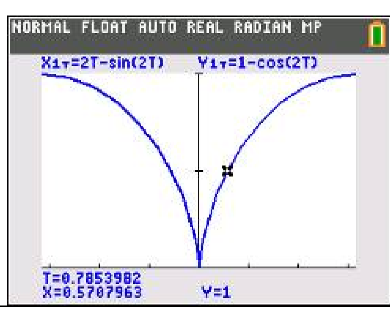
At $t = -\frac{\pi}{4}$, the gradient of the tangent is $\cot\left(-\frac{\pi}{4}\right) = -1$.

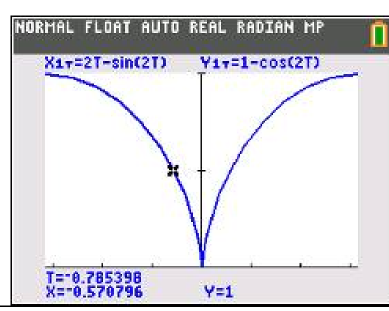
Since the gradient of tangent at point $Q = 1$ and the gradient of tangent at point $R = -1$, the product of their gradients $= -1$. Thus the 2 tangents are perpendicular to one another. Hence angle QLR is a right angle.



The triangle MUST be drawn out clearly, so draw it big and clear. Things to look out:

- The y-coordinate of R and Q are same, so the line joining R and Q should be horizontal.
- The y-coordinate of R and Q is 1, so point R and Q should be below the end point of curve C.
- L is not at the origin.
- Angle QLR is right angle.





9 The plane p has equation $\mathbf{r}=\begin{pmatrix}5\\0\\5\end{pmatrix}+s\begin{pmatrix}1\\0\\-1\end{pmatrix}+t\begin{pmatrix}m\\1\\0\end{pmatrix}$, where m is a real constant and s and t are real parameters. The line ℓ_1 passes through the point $A(1,-1,2)$ and is parallel to the vector $3\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

(a) In the case where ℓ_1 and p do not intersect, find the value of m. [3]

For the rest of the question, the plane p has equation $x-y+z=10$.

(b) Find the position vector of the foot of the perpendicular from A to p. [3]

The line ℓ_2 passes through A and has the equation $\dfrac{x-1}{3}=\dfrac{y+1}{-1}=\dfrac{z-2}{2}$.

(c) Find the coordinates of the point of intersection between ℓ_2 and p. [3]

(d) By considering parts **(b)** and **(c)**, find an equation of the line of reflection of ℓ_2 in p. [3]

(a)	$\mathbf{n}=\begin{pmatrix}1\\0\\-1\end{pmatrix}\times\begin{pmatrix}m\\1\\0\end{pmatrix}=\begin{pmatrix}1\\-m\\1\end{pmatrix}$ Since ℓ_1 and p do not intersect at all, they are parallel to each other which means that the direction vector of ℓ_1 is perpendicular to the normal vector of p. $\begin{pmatrix}3\\1\\-2\end{pmatrix}\bullet\begin{pmatrix}1\\-m\\1\end{pmatrix}=0$ $3-m-2=0$ $m=1$ Notice that there is no need to find the equations of ℓ_1 and p, since we only need to compare the respective direction and normal vectors.

For the rest of the question, the plane p has equation $x - y + z = 10$.

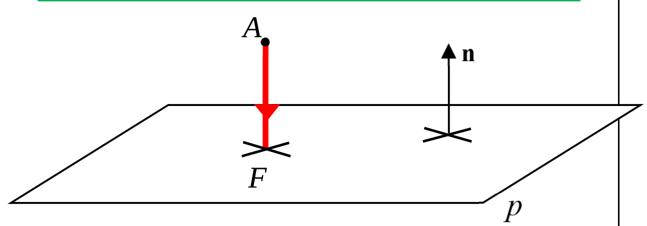
(b) Find the position vector of the foot of the perpendicular from A to p. [3]

(b) Let the foot of perpendicular of A to p be F.

$$\ell_{AF} : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \beta \in \mathbb{R} \text{ ------ (1)}$$

$$p : \mathbf{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 10 \text{ ------ (2)}$$

Observe that F lies on both ℓ_{AF} and p.



Since F lies on ℓ_{AF}, $\overrightarrow{OF} = \begin{pmatrix} 1+\beta \\ -1-\beta \\ 2+\beta \end{pmatrix}$ for some $\beta \in \mathbb{R}$.

$$\begin{pmatrix} 1+\beta \\ -1-\beta \\ 2+\beta \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 10$$

Subt (1) into (2) to find $\overrightarrow{OF}$

$$1+\beta+1+\beta+2+\beta = 10$$
$$3\beta = 6$$
$$\beta = 2$$

$$\overrightarrow{OF} = \begin{pmatrix} 1+2 \\ -1-2 \\ 2+2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}$$

The line ℓ_2 passes through A and has the equation $\dfrac{x-1}{3}=\dfrac{y+1}{-1}=\dfrac{z-2}{2}$.

(c) Find the coordinates of the point of intersection between ℓ_2 and p. [3]

(c)

$$\frac{x-1}{3}=\frac{y+1}{-1}=\frac{z-2}{2}=\gamma$$

$$\ell_2:\mathbf{r}=\begin{pmatrix}1\\-1\\2\end{pmatrix}+\gamma\begin{pmatrix}3\\-1\\2\end{pmatrix},\gamma\in\mathbb{R}$$

$$p:\mathbf{r}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=10$$

Let the point of intersection between ℓ_2 and p be B.

$$\begin{pmatrix}1+3\gamma\\-1-\gamma\\2+2\gamma\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=10$$

$$1+3\gamma+1+\gamma+2+2\gamma=10$$

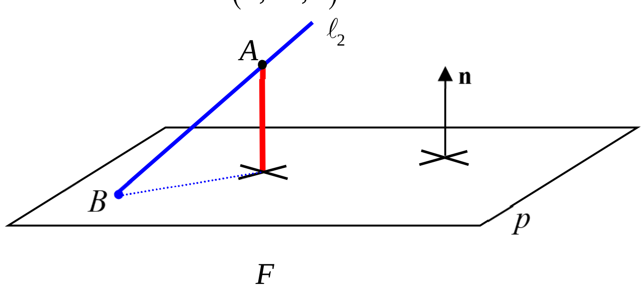
$$6\gamma=6$$

$$\gamma=1$$

$$\overrightarrow{OB}=\begin{pmatrix}1+3(1)\\-1-1(1)\\2+2(1)\end{pmatrix}=\begin{pmatrix}4\\-2\\4\end{pmatrix}$$

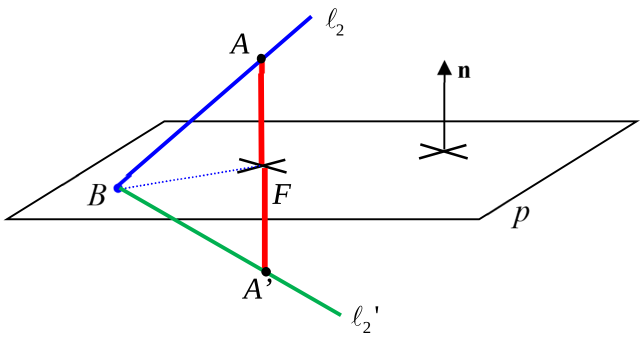
Note:
Question asks for coordinates

$\therefore$ Coordinates of $B=(4,-2,4)$



(d) By considering parts **(b)** and **(c)**, find an equation of the line of reflection of ℓ_2 in p. [3]

(d)



Using Ratio Theorem,

$$\overrightarrow{OF} = \frac{\overrightarrow{OA} + \overrightarrow{OA}'}{2}$$

$$\overrightarrow{OA}' = 2\overrightarrow{OF} - \overrightarrow{OA}$$

$$\overrightarrow{OA}' = 2\begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 6 \end{pmatrix}$$

$$\overrightarrow{BA}' = \begin{pmatrix} 5 \\ -5 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

Find a direction vector of ℓ_2'

$$\ell_2' : \mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} + \phi \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \phi \in \mathbb{R}$$

10 The function f is defined by

$$\mathrm{f} : x \mapsto \begin{cases} x\sqrt{8-x^2}, & 0 < x \leq 2, \\ 8-2x, & 2 < x \leq 4. \end{cases}$$

and $\mathrm{f}(x) = \mathrm{f}(x+4)$ for all real values of x.

(a) Find the values of $\mathrm{f}(-1)$ and $\mathrm{f}(6)$. [2]

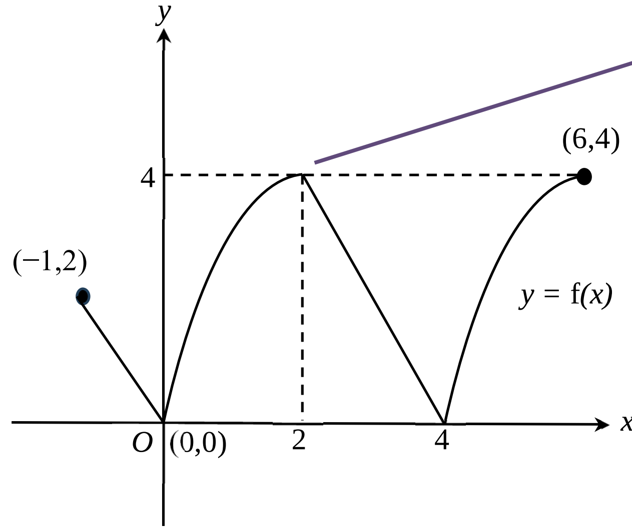
(b) Sketch the graph of $y = \mathrm{f}(x)$ for $-1 \leq x \leq 6$. [4]

(c) Hence find the area under the curve $y = \mathrm{f}(x)$ for $-1 \leq x \leq 6$, giving your answer correct to 3 significant figures. [3]

(a)	$\mathrm{f}(-1) = \mathrm{f}(3)$ $= -2(3) + 8$ $= 2$ $\mathrm{f}(6) = \mathrm{f}(2)$ $= 2\sqrt{8-2^2}$ $= 4$

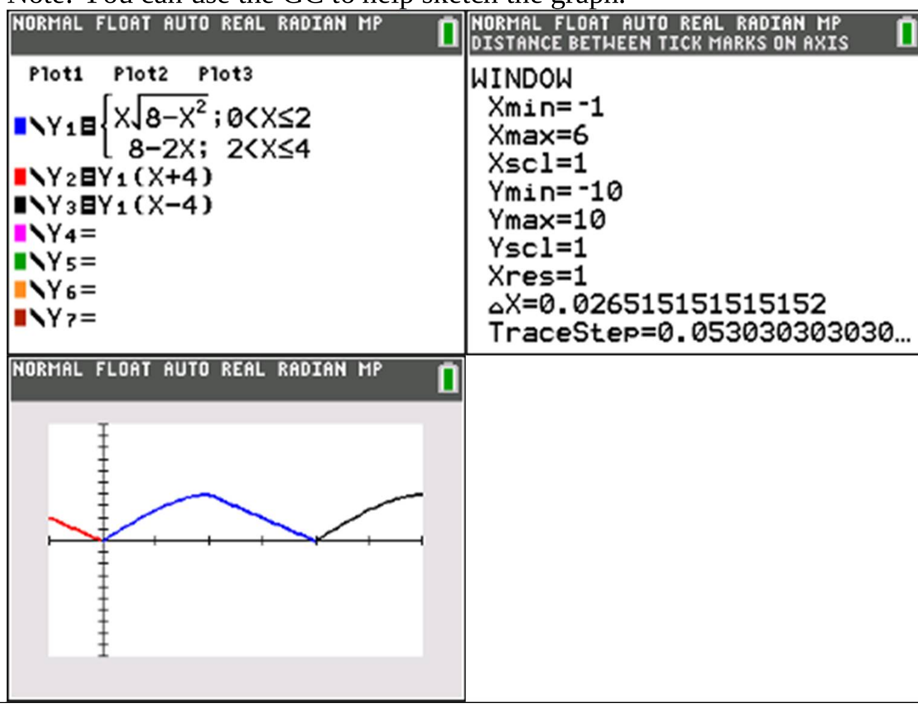
(b) Sketch the graph of $y = \mathrm{f}(x)$ for $-1 \le x \le 6$. [4]

(b)



Please be reminded to provide coordinates for end points.
For kinks, you may label using coordinates or dotted lines to the axis (and then indicating the relevant values).

Note: You can use the GC to help sketch the graph.



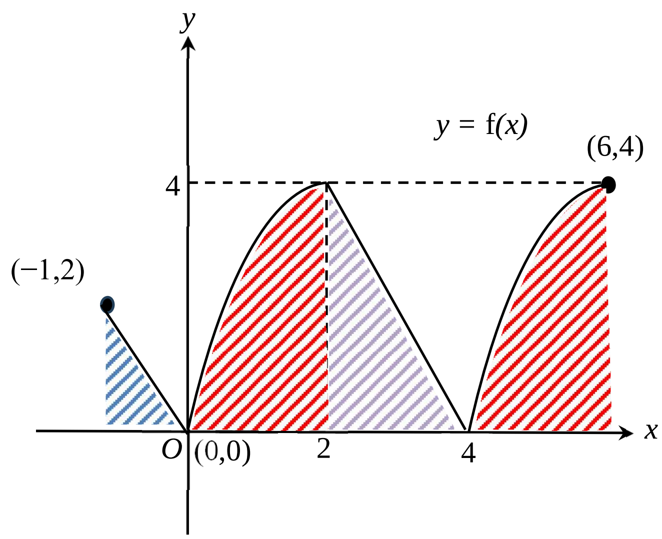
(c) Hence find the area under the curve $y = \mathrm{f}(x)$ for $-1 \le x \le 6$, giving your answer correct to 3 significant figures. [3]

(c)	**Method 1 (Using similar areas, then use GC)**

Required Area $= \int_{-1}^{6} \mathrm{f}(x)\,\mathrm{d}x$

$$= \frac{1}{2}(1)(2) + 2\int_{0}^{2} x\sqrt{8-x^2}\,\mathrm{d}x + \frac{1}{2}(2)(4)$$

$$= 14.8 \text{ units}^2 \text{ (3s.f)}$$



Required Area $= \int_{-1}^{6} \mathrm{f}(x)\,\mathrm{d}x$

$$= \int_{-1}^{0} \mathrm{f}(x)\,\mathrm{d}x + \int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x + \int_{2}^{4} \mathrm{f}(x)\,\mathrm{d}x + \int_{4}^{6} \mathrm{f}(x)\,\mathrm{d}x$$

Note that we cannot just substitute in expressions for $\mathrm{f}(x)$ for the integrals from -1 to 0 and 4 to 6 as f is only defined for $0 < x \le 4$.

The most common way to address this is to spot similar shapes and match to regions in $0 < x \le 4$. So for example, $\int_{4}^{6} \mathrm{f}(x)\,\mathrm{d}x$ is exactly the same as $\int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x$. For $\int_{-1}^{0} \mathrm{f}(x)\,\mathrm{d}x$, we see that this is just the area of a triangle so we can just calculate the area directly.

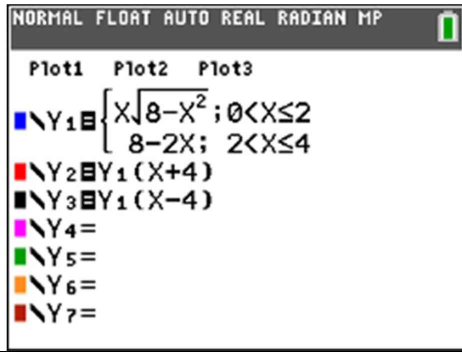
Hence required Area $= \frac{1}{2}(1)(2) + 2\int_{0}^{2} x\sqrt{8-x^2}\,\mathrm{d}x + \frac{1}{2}(2)(4)$.

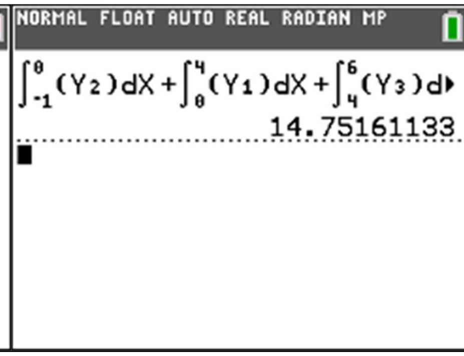
Method 2 (Using Piecewise Function in GC)

Required Area $= \int_{-1}^{6} \mathrm{f}\left(x\right) \mathrm{d}x$

$= 14.8 \text{ units}^2$ (3s.f)

> If you used the GC to sketch the graph in part **(b)**, you can use the GC to find the area very quickly here. However, do note that if the answer is incorrect, you will not obtain any working marks due to the lack of working.





11 A company is launching a new limited-edition watch, L-Watch. The profit, p thousand dollars, will be a function of the selling price, s thousand dollars, of each L-Watch. One of the business analysts in the company predicts that the profit function can be represented by

$$p : s \mapsto -\frac{1}{2}(s-4)^2 + 3, \; s > 0.$$

(a) **(i)** Explain why the inverse function p^{-1} does not exist. [1]

(ii) The function p^{-1} exists if the domain of p is restricted to $0 < s \leq k$, where $k \in \mathbb{R}$. State the largest value of k. [1]

(iii) Using the domain defined in part **(a)(ii)**, find $p^{-1}(x)$ and state the domain of p^{-1}. [4]

For the rest of the question, the domain of p is as originally defined.

Based on past market trends, the company adjusts the selling price of the L-watch periodically. The business analysts proposes that the selling price, s thousand dollars, of the L-Watch at time t months after the launch of the L-Watch can be modelled by the function

$$s : t \mapsto \frac{2t+5}{t+1}, \; t \geq 0.$$

(b) Show that the composite function ps exists. [2]

(c) Find $ps(2)$ and give an interpretation of $ps(2)$ in context. [3]

(d) Find the range of ps. [2]

11 A company is launching a new limited-edition watch, L-Watch. The profit, p thousand dollars, will be a function of the selling price, s thousand dollars, of each L-Watch. One of the business analysts in the company predicts that the profit function can be represented by

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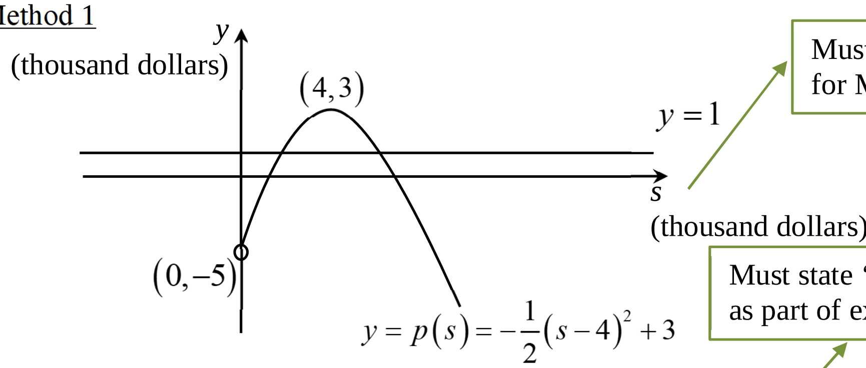
(a) **(i)** Explain why the inverse function p^{-1} does not exist. [1]

(a)(i)

Method 1



Must draw graph for Method 1.

Must state ‘***p* is not 1-1**’ as part of explanation.

Since horizontal line $y=1$ cuts the graph of p more than once, ***p* is not 1-1** and p^{-1} does not exist.

Give a specific example of a horizontal line that cuts the graph more than once.

Method 2

Since $p(2) = p(6) = 1$, ***p* is not 1-1** and p^{-1} does not exist.

(ii) The function p^{-1} exists if the domain of p is restricted to $0 < s \le k$, where $k \in \mathbb{R}$. State the largest value of k. [1]

(a)(ii)	Largest value of $k = 4$

(iii) Using the domain defined in part **(a)(ii)**, find $p^{-1}(x)$ and state the domain of p^{-1}. [4]

(a)(iii)

$$y = -\frac{1}{2}(s-4)^2 + 3$$

$$(s-4)^2 = 6 - 2y$$

$$s = 4 \pm \sqrt{6-2y}$$

$$= 4 - \sqrt{6-2y} \quad (\text{since } s \le 4)$$

$$p^{-1}(x) = 4 - \sqrt{6-2x}$$

$$D_{p^{-1}} = R_p = (-5, 3]$$

Answer to the question. Ensure same variable throughout.

Must take note that $(s-4)^2 = 6-2y$, $s - 4 = \pm\sqrt{6-2y}$

Explain which expression is chosen based on the domain of p.

If question asks for range or domain, the answer must be given in set notation (such as interval notation).

For the rest of the question, the domain of p is as originally defined.

Based on past market trends, the company adjusts the selling price of the L-watch periodically. The business analysts proposes that the selling price, s thousand dollars, of the L-Watch at time t months after the launch of the L-Watch can be modelled by the function

$$s : t \mapsto \frac{2t+5}{t+1},\ t \geq 0.$$

(b) Show that the composite function ps exists. [2]

(b)

Exclude 2 as $s = 2$ is a horizontal asymptote.

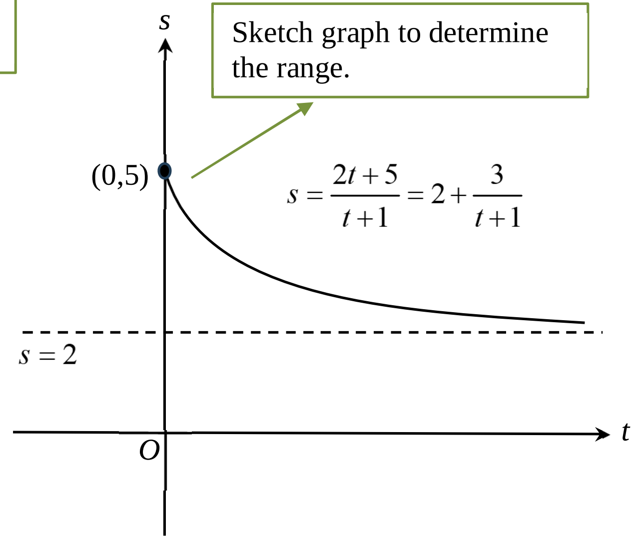
$R_s = (2,5]$

$D_p = (0,\infty)$

Since $R_s \subseteq D_p$, ps exists.

Need to remember the correct condition to prove that composite function exists.
Since $R_s \subseteq D_p$, ps exists.

Sketch graph to determine the range.



(c) Find $ps(2)$ and give an interpretation of $ps(2)$ in context. [3]

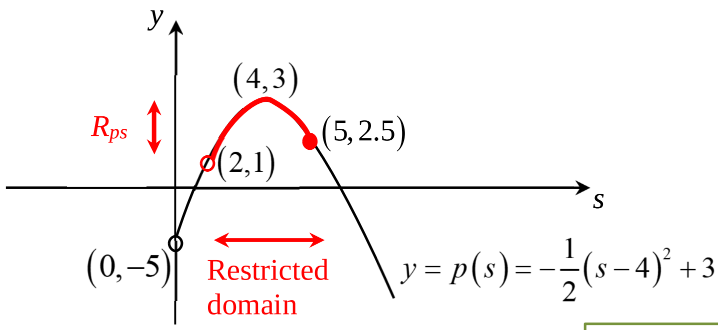
(c)

$$ps(2) = p\left(\frac{2(2)+5}{(2)+1}\right)$$
$$= p(3)$$
$$= -\frac{1}{2}(3-4)^2 + 3$$
$$= 2.5$$

Since question asks for "in context", you have to refer to the question and answer in context as best as you can.

$ps(2) = 2.5$ is the **profit**, in **thousands of dollars**, for **each** L-watch **2 months after** launch.

(d) Find the range of *ps*. [2]

(d)	 $D_s = [0, \infty) \xrightarrow{s} \underbrace{(2,5]}_{\text{Restricted Domain of } p} \xrightarrow{p} (1,3]$ $R_{ps} = (1,3]$ Highly encouraged to use mapping method. Finding the composite function first then sketching the graph to observe the range is tedious and difficult. Students who did so are generally unsuccessful. Need to include the maximum point which lies within the restricted domain.