CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC2 Preliminary Examination

CANDIDATE NAME	
CLASS	
INDEX NUMBER	

MATHEMATICS 9758/01

Paper 1

29 Aug 2023

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your class, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use a HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
Marks											
Total	4	6	8	9	10	11	13	15	12	12	**100**

This document consists of 3 printed pages, including this cover page.

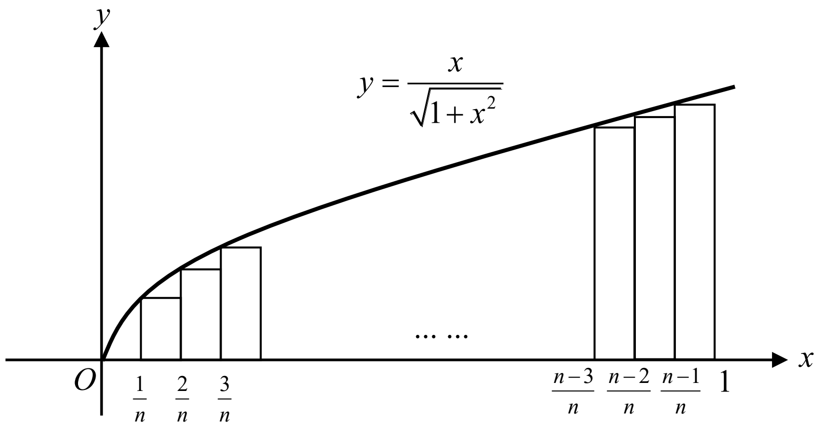
1 The complex numbers z and w satisfy the following equations.

$$2z+1=|w|$$

$$2w-z=4+24\text{i}$$

Find z and w, giving your answers in the form $a+\text{i}b$ where a and b are real numbers. [4]

2 **(a)** The diagram below shows a sketch of the curve $y=\dfrac{x}{\sqrt{1+x^2}}$ for $x \geq 0$. Rectangles, each of width $\dfrac{1}{n}$, where $n \in \mathbb{Z}^+$, are drawn under the curve for $0 \leq x \leq 1$.



Show that the total area of all the rectangles, A, can be written as $\dfrac{1}{n}\displaystyle\sum_{r=1}^{n-1}\dfrac{r}{\sqrt{n^2+r^2}}$. [3]

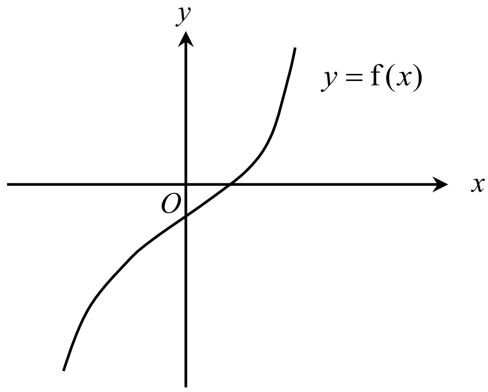
(b) Find the exact value of $\lim\limits_{n\to\infty} A$. [3]

3 The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively. The points P and Q are fixed and R varies.

(a) Given that $\mathbf{p}$ is non-zero and $(\mathbf{r}-\mathbf{q})\times\mathbf{p}=\mathbf{0}$, find a linear relationship between $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. Describe geometrically the set of all possible positions of the point R. [4]

(b) Given that $\mathbf{r}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{p}=\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$, $\mathbf{q}=\begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix}$ and $(\mathbf{r}-\mathbf{q})\cdot\mathbf{p}=0$, find a relationship between x, y and z. Describe geometrically the set of all possible positions of the point R. [4]

4 It is given that $f(x)$ is a cubic polynomial with real coefficients. The diagram shows the curve with equation $y = f(x)$.



(a) What can be said about all the roots of the equation $f(x) = 0$? [2]

Suppose $f(x) = 2x^3 - 7x^2 + 16x + c$, where c is a real number.

(b) Show that $c = -15$, if the equation $f(x) = 0$ has a root $x = 1 - 2\text{i}$. [1]

(c) Without using a calculator, determine the other roots of the equation $f(x) = 0$. [3]

(d) Hence, find the roots of the equation $-15w^3 + 16w^2 - 7w + 2 = 0$. [3]

5 The first four terms of a sequence of numbers are -4, -2, 12 and 38. The sum of the first n terms of this sequence is denoted by S_n.

(a) Explain why S_n cannot be a quadratic polynomial in n. [2]

It is given that S_n is a cubic polynomial.

(b) Find S_n in terms of n. [3]

(c) Show that the nth term of the sequence, u_n is $6n^2 - 16n + 6$. [2]

(d) Hence find $\sum_{n=10}^{2m} \left(u_n - u_{n-1}\right)$ in terms of m. [3]

6 A curve C has equation $y=\dfrac{ax^2+bx+c}{x-d}$, where a, b, c and d are constants. It is given that two of its asymptotes are $y=x+2$ and $x=1$.

(a) State the value of d, and show that $a=b=1$. [2]

(b) Using differentiation, find the range of values of c such that the graph of C contains two stationary points. [4]

Use $c=14$ for the rest of the question.

(c) Sketch C, showing clearly the equations of asymptotes and the coordinates of the turning points. [3]

(d) State the maximum number of roots to the equation

$$k^2(x-5)^2+\left(\frac{ax^2+bx+c}{x-d}-3\right)^2=k^2\text{, where } k>0.$$

Deduce the range of values of k for the maximum number of roots to occur. [2]

7 It is given that

$$\mathrm{f}: x \mapsto \begin{cases} (x-2)^2 & , \quad 0<x\le 3, \\ 3x-8 & , \quad 3<x\le 4, \end{cases}$$

(a) **(i)** Sketch the graph of $y=\mathrm{f}(x)$, labelling the coordinates of any turning points and end-points. Explain why f^{-1} does not exist. [3]

(ii) If the domain of f is restricted to $(0, k]$, state the largest value of k such that f^{-1} exists. Hence, for this value of k, find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(iii) The function g is such that

$$\mathrm{g}: x \mapsto \mathrm{e}^x+3\ , \quad x\le 0.$$

Find the function fg, giving your answer in similar form. [3]

(b) It is given further that $\mathrm{f}(x)=\mathrm{f}(x+4)$.

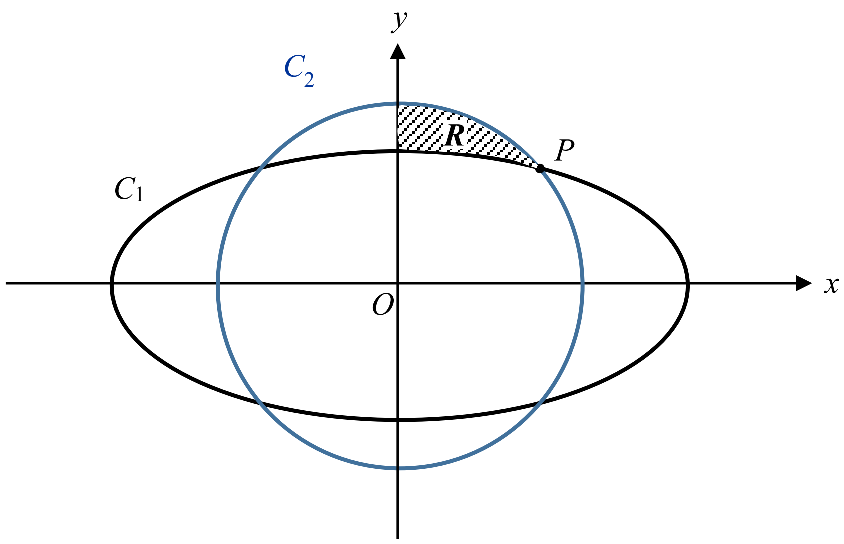
(i) Evaluate $\mathrm{f}(25)$ and $\mathrm{f}(-8)$. [2]

(ii) Sketch the graph of $y=\mathrm{f}\left(\dfrac{1}{2}x-1\right)$ for $-8<x\le 10$. [2]

8 **(a)** Use the substitution $x=\sqrt{15}\sin\theta$ to show that

$$\int\sqrt{15-x^2}\,\mathrm{d}x=\frac{1}{2}x\sqrt{15-x^2}+\frac{15}{2}\sin^{-1}\left(\frac{x}{\sqrt{15}}\right)+C.$$ [5]

The diagram below shows a sketch of the curves C_1 and C_2.



The curve C_1 has parametric equations

$$x=6\cos\theta, \qquad y=2\sqrt{2}\sin\theta$$

for $0\le\theta\le 2\pi$.

The curve C_2 has equation

$$x^2+y^2=15.$$

Given that P is a point of intersection between C_1 and C_2,

(b) determine the exact coordinates of P. [4]

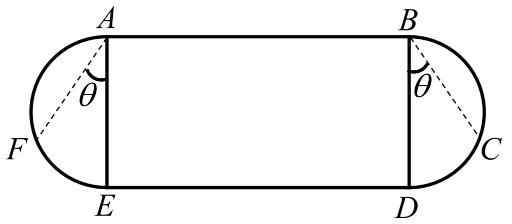
The region R is bounded by curves C_1 and C_2 and the y-axis in the first quadrant.

(c) Show that the area of R is given by

$$m\sin^{-1}\left(n\sqrt{15}\right)-\sqrt{2}\pi$$

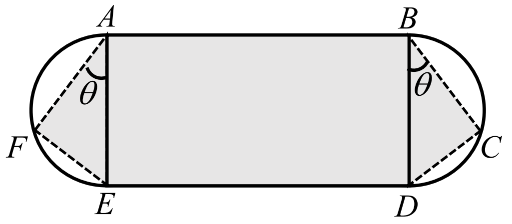
where m and n are constants to be determined. [6]

9 The following diagram shows a plot of land formed by two semicircles joined to a rectangle $ABDE$.



Point C lies on the arc BD, with $\angle CBD = \theta$, where $0 < \theta < \frac{\pi}{2}$, and point F lies on the arc AE, with $\angle FAE = \theta$.

As part of a training regime, Nigel runs along the perimeter of the shaded portion $ABCDEF$ as shown in the diagram.



It is given that AE is fixed at $2r$ metres, and AB is twice the length of AE.

(a) Show that the perimeter, P metres, of $ABCDEF$ is
$$P = 4r\left(2 + \cos\theta + \sin\theta\right).$$ [2]

(b) Find the exact value of θ which maximises P and hence find the exact maximum distance that Nigel can run in one round of $ABCDEF$, giving your answer in terms of r. [5]

(c) Nigel plans to run one round with maximum distance at a constant speed of 6 metres per second within 3 minutes. Find the maximum value of r, giving your answer in the form $a + b\sqrt{2}$, where a and b are constants to be determined. [2]

(d) To clearly mark out the shape $ABCDEF$, the management wishes to plant grass within the shape $ABCDEF$. It costs \$0.15 to plant 1 m^2 of grass and the management has a budget of \$10000. Using the value of θ found in part **(b)** and the value of r found in part **(c)**, determine, with justifications, if the management is able to afford to cover the entire shape $ABCDEF$ with grass. [3]

10 A marketing manager of a company wishes to advertise a new product. He has tasked his team to create an engaging video and upload it on InstaFame social platform. He hopes the video would go viral on the internet so that the product will sell well.

According to some analysts, a video is considered to have gone viral when it gets at least a total of 5 million views at the end of the seventh day after its initial posting.

The video is uploaded at the start of a particular day and the number of daily views at the end of the first day is 1196. On each subsequent day, the number of daily views at the end of the day will be three times that of the previous day.

(a) Find the number of daily views at the end of the third day. [2]

(b) Determine if the video will go viral. [2]

The marketing manager also looks at the number of comments being posted on the InstaFame social platform. On each subsequent day, the number of daily comments at the end of the day will be 780 more than that of the previous day. It is given that the number of daily comments posted at the end of the first day is 576.

(c) Find the least number of days for the total number of comments to exceed 100 000. [3]

When the total number of comments reaches 100 000, a software in InstaFame social platform will be activated to remove the oldest comments **at the start of the following day**, helping to save storage space. Upon the activation of the software, w comments will be removed at the start of each day and the number of comments at the end of the day is 3% more than the number of comments at the start of the day. The software will remain activated even when the number of comments drops below 100 000 at any one time.

(d) By taking Day 1 as the day which the software starts removing comments, show that the number of comments at the end of Day n is

$$(1.03)^n M - \frac{103w}{3}\left[(1.03)^n - 1\right]$$

where M is a constant to be determined. [3]

(e) Hence find the range of values of w such that all comments are removed by the end of Day 31. Give your answer correct to the nearest integer. [2]

– END –