National Junior College
2016 – 2017 H2 Mathematics
Differentiation Applications **Tutorial Solutions**

Basic Mastery Questions

2. $2x^2 - 3xy + y^2 = 5$.

Differentiate implicitly wrt *x*, we get:

$$4x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y - 3x) = 3y - 4x$$

$$\frac{dy}{dx} = \frac{3y-4x}{2y-3x}.$$

At (4, 3), gradient of tangent is

$$\frac{dy}{dx} = \frac{3(3)-4(4)}{2(3)-3(4)} = \frac{7}{6}.$$

Hence equation of tangent is:

$$y - 3 = \frac{7}{6}(x-4)$$

$$y = \frac{7}{6}x - \frac{5}{3}.$$

Gradient of normal, $-\frac{1}{\frac{dy}{dx}} = -\frac{6}{7}$.

Hence equation of normal is:

$$y - 3 = -\frac{6}{7}(x-4)$$

$$y = -\frac{6}{7}x + \frac{45}{7}.$$

3. $x^2 - 8x + y^2 - 4y + 6xy + 4 = 0$.

Differentiate implicitly wrt to *x*, we get:

$$2x - 8 + 2y\frac{dy}{dx} - 4\frac{dy}{dx} + 6y + 6x\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(6x + 2y - 4) = 6x + 2y - 4$$

$$\Rightarrow \frac{dy}{dx} = \frac{8-2x-6y}{6x+2y-4}.$$

For tangent to be parallel to *x*-axis,

$$\frac{dy}{dx}=0$$

$$\frac{8-2x-6y}{6x+2y-4}=0$$

$$8-2x-6y=0$$

$$x=4-3y. \qquad (1)$$

Substituting (1) into
$x^2-8x+y^2-4y+6xy+4=0$,

$$(4-3y)^2-8(4-3y)+y^2-4y$$
$$+6(4-3y)y+4=0$$

$$(4-3y)(4-3y-8+6y)$$
$$+y^2-4y+4=0$$

$$(4-3y)(-4+3y)+(y-2)^2=0$$

$$(y-2)^2-(3y-4)^2=0$$

$$(y-2-3y+4)(y-2+3y-4)=0$$

$$(-2y+2)(2y-3)=0$$

$\Rightarrow y=1$ or $y=\dfrac{3}{2}$

$\Rightarrow x=1$ or $y=\dfrac{1}{2}$ respectively.

Hence coordinates of the points are $(1,1)$ and $\left(-\dfrac{1}{2},\dfrac{3}{2}\right)$.

4. $x=5a\sec\theta$, $y=3a\tan\theta$ where $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$.

$$\frac{dx}{d\theta}=5a\sec\theta\tan\theta, \quad \frac{dy}{d\theta}=3a\sec^2\theta.$$

$$\frac{dy}{dx}=\frac{\left[\frac{dy}{d\theta}\right]}{\left[\frac{dx}{d\theta}\right]}=\frac{3a\sec^2\theta}{5a\sec\theta\tan\theta}=\frac{3}{5\sin\theta}.$$

When the normal is parallel to $y=x$,

$$-\frac{1}{\frac{dy}{dx}}=1$$

$$\frac{dy}{dx}=-1$$

$$\frac{3}{5\sin\theta}=-1$$

$$\theta=\sin^{-1}\left(-\frac{3}{5}\right).$$

Hence,

$$x=5a\sec\left[\sin^{-1}\left(-\frac{3}{5}\right)\right]=\frac{25}{4}a,$$

$$y=3a\tan\left[\sin^{-1}\left(-\frac{3}{5}\right)\right]=-\frac{9}{4}a.$$

The point on the curve where the normal is parallel to the line $y=x$ is $\left(\frac{25}{4}a,-\frac{9}{4}a\right)$.

Practice Questions

5. $x^3+xy+2y^3=k.$ --- (1)

Differentiate implicitly wrt *x*, we get:

$$3x^2+y+x\frac{dy}{dx}+6y^2\frac{dy}{dx}=0$$

$$\left(x+6y^2\right)\frac{dy}{dx}=-\left(3x^2+y\right)$$

$$\frac{dy}{dx}=-\frac{3x^2+y}{x+6y^2}.$$

Since *C* has a tangent which is parallel to the *y*-axis, the normal at the point of contact of the tangent with C is parallel to the *x*-axis, i.e.

$$-\frac{1}{\frac{dy}{dx}}=0$$

$$\frac{x+6y^2}{3x^2+y}=0$$

$$x=-6y^2.$$

Substitute $x=-6y^2$ into (1), we get:

$$\left(-6y^2\right)^3+\left(-6y^2\right)y+2y^3=k$$

$$-216y^6-6y^3+2y^3=k$$

$$216y^6+4y^3+k=0 \text{ (shown)}.$$

Hence

$$y^3=\frac{-4\pm\sqrt{(4)^2-4(216)k}}{2(216)}.$$

For real values of y^3,

$$(4)^2-4(216)k\geq 0$$

$$k\leq\frac{1}{54} \quad \text{(shown)}.$$

When $x=-6$,

$$-6y^2=-6$$

$$y^2=1$$

$$y=1 \text{ or } -1.$$

Hence from (1), when $y=1$,

$$k=(-6)^3+(-6)(1)+2(1)^3=-220,$$

And when $y=-1$,

$$k=(-6)^3+(-6)(-1)+2(-1)^3=-212.$$

6. $x = t^2,\ y = t^3.$

$\frac{dx}{dt} = 2t,\ \ \frac{dy}{dt} = 3t^2.$

$$\frac{dy}{dx} = \frac{\left[\frac{dy}{dt}\right]}{\left[\frac{dx}{dt}\right]} = \frac{3t^2}{2t} = \frac{3t}{2}.$$

Then equation of tangent is

$$y - t^3 = \frac{3t}{2}\left(x - t^2\right)$$
$$2y - 2t^3 = 3tx - 3t^3$$
$$2y - 3tx + t^3 = 0 \text{ (proven). ----(1)}$$

(i) For tangents that pass through (X, Y), $2Y - 3tX + t^3 = 0.$

Since the equation of the tangent passing through (X, Y) is a cubic equation in *t*, there are at most only 3 real roots for *t*, hence there cannot be more than 3 tangents passing through (X, Y).

[Note that each value of *t* results in one tangent equation when substituted into (1).]

(ii) Equation of tangent at $t = 2$:

$$2y - 3(2)x + (2)^3 = 0$$
$$2y - 6x + 8 = 0$$
$$y - 3x + 4 = 0.$$

Since the tangent meets the curve again at $t = u$, substitute $x = u^2$ and $y = u^3$ into $y - 3x + 4 = 0$,

$$u^3 - 3u^2 + 4 = 0$$
$$(u+1)\left(u^2 - 4u + 4\right) = 0$$
$$(u+1)(u-2)^2 = 0$$

Hence $u = -1$.

7(i) $x = 2t - 1,\ \ y = \frac{1}{2t+1}.$

$\frac{dx}{dt} = 2,\ \ \frac{dy}{dt} = -\frac{2}{(2t+1)^2}.$

$$\frac{dy}{dx} = \frac{\left[\frac{dy}{dt}\right]}{\left[\frac{dx}{dt}\right]} = -\frac{1}{(2t+1)^2} < 0.$$

Therefore curve shows a decreasing function.

(ii) At $t = -1,\ x = -3$ and $y = -1$.

At $t = \frac{5}{8},\ x = \frac{1}{4}$ and $y = \frac{4}{9}$.

Hence,

$$\text{gradient of chord} = \frac{\frac{4}{9} - (-1)}{\frac{1}{4} - (-3)} = \frac{4}{9}.$$

For tangent to be parallel to chord,

$$\frac{dy}{dx} = \frac{4}{9}$$
$$-\frac{1}{(2t+1)^2} = \frac{4}{9} \Rightarrow (2t+1)^2 = -\frac{9}{4}$$

Since $(2t+1)^2 > 0$, t is undefined, and hence there is no tangent to the curve parallel to the chord.

(iii) Gradient of normal to the curve is given by:

$$-\frac{1}{\frac{dy}{dx}} = (2t+1)^2$$

At $t = \frac{1}{2},\ x = 0,\ y = \frac{1}{2},\ -\frac{1}{\frac{dy}{dx}} = 4.$

Hence equation of normal to the curve at $t = \frac{1}{2}$ is:

$$y - \frac{1}{2} = 4(x - 0)$$
$$2y = 8x + 1.$$

Since the normal meets the curve again,

$$2\left(\frac{1}{2t+1}\right)=8(2t-1)+1$$

$$2=8\left(4t^2-1\right)+(2t+1)$$

$$32t^2+2t-9=0$$

$$\left(t-\frac{1}{2}\right)\left(t+\frac{9}{16}\right)=0$$

$$t=\frac{1}{2} \text{ or } t=-\frac{9}{16}.$$

At $t=-\frac{9}{16}$, $x=-\frac{17}{8}$ and $y=-8$.

Hence the normal to the curve at $t=\frac{1}{2}$ meets the curve again at $\left(-\frac{17}{8},-8\right)$.

(iv) $4x+\frac{1}{4}=0$

$\Rightarrow x=-\frac{1}{8}$.

Coordinates of P are $\left(-\frac{1}{8},0\right)$.

Therefore area of triangle OPQ is

$$\left(\frac{1}{8}\times\frac{1}{2}\right)\times\frac{1}{2}=\frac{1}{32}\ .$$