Secondary 4 Additional Mathematics: Differentiation

1. Rules of Differentiation

Rule	Example
(a) $\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$, where n is a real number.	$\frac{\mathrm{d}}{\mathrm{d}x}[x^4] = 4x^{4-1}$ $= 4x^3$
(b) In particular, $\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0$, where c is a constant.	$\frac{\mathrm{d}}{\mathrm{d}x}(3) = 0$
(c) $\frac{\mathrm{d}}{\mathrm{d}x}[kx^n] = k\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = knx^{n-1}$, where k is a real number.	$\frac{\mathrm{d}}{\mathrm{d}x}[3x^5] = 5\frac{\mathrm{d}}{\mathrm{d}x}[x^5]$ $= 5(5x^{5-1})$ $= 25x^4$

Examples

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3 + 3x^2 - 2x + 7) = \frac{\mathrm{d}}{\mathrm{d}x}(x^3) + 3\frac{\mathrm{d}}{\mathrm{d}x}(x^2) - 2\frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}x}(7)$

$= 3x^{3-1} + 3(2x^{2-1}) - 2(1x^{1-1}) + 0$

$= 3x^2 + 6x - 2$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^3 - \frac{3}{x^2}\right) = 2\frac{\mathrm{d}}{\mathrm{d}x}(x^3) - 3\frac{\mathrm{d}}{\mathrm{d}x}(x^{-2})$

$= 2(3x^{3-1}) - 3(-2x^{-3})$

$= 6x^2 + 6x^{-3}$

$= 6x^2 + \frac{6}{x^3}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}[(x-2)(x+1)] = \frac{\mathrm{d}}{\mathrm{d}x}(x^2 - x - 2)$ → *Expand before differentiation

$= 2x^{2-1} - 1x^{1-1} - 0$

$= x - 1$

2. **Chain Rule:**

If y is a function of u and u is a function of x, that is $y = \mathrm{f}(u)$, where $u = \mathrm{g}(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

3. **Derivatives of some functions:**

$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$	$\frac{\mathrm{d}}{\mathrm{d}x}(k\sin x) = k\cos x$
$\frac{\mathrm{d}}{\mathrm{d}x}(ke^x) = ke^x$	$\frac{\mathrm{d}}{\mathrm{d}x}(k\cos x) = -k\sin x$
$\frac{\mathrm{d}}{\mathrm{d}x}(k\ln x) = \frac{k}{x}$	$\frac{\mathrm{d}}{\mathrm{d}x}(k\tan x) = k\sec^2 x$

Examples of chain rule and derivatives of some functions:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(3x+2)^{\frac{1}{2}} = \frac{1}{2}(3x+2)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(3x+2)$ →

$= \frac{1}{2}(3x+2)^{-\frac{1}{2}}(3)$

$= \frac{3}{2}(3x+2)^{-\frac{1}{2}}$

$= \frac{3}{2\sqrt{3x+2}}$

> Differentiate from the outermost function to the innermost function, that is,
> - differentiate the square root function, then
> - differentiate $3x + 2$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)^3 = 3(\ln x)^2\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)$ →

$= 3(\ln x)^2\left(\frac{1}{x}\right)$

$= \frac{3(\ln x)^2}{x}$

> Differentiate from the outermost function to the innermost function, that is,
> - differentiate the cube function, then
> - differentiate ln$x$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x^2+2x)) = \cos(x^2+2x)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+2x)$

$= (2x+2)\cos(x^2+2x)$

$= 2(x+1)\cos(x^2+2x)$

4. **Product Rule:**

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

5. **Quotient Rule:**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

Examples of Product rule and Chain rule:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}\cos x\right) = e^{x^2}(-\sin x) + 2xe^{x^2}(\cos x)$

$= e^{x^2}(2x\cos x - \sin x)$ → Always factorize common multiple!

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-1}{\sqrt{1-2x}}\right) = \frac{\sqrt{1-2x}(1)-(x-1)\left(\frac{1}{2}\right)(1-2x)^{-\frac{1}{2}}(-2)}{1-2x}$ → Quotient rule

$= \frac{\sqrt{1-2x}+(x-1)(1-2x)^{-\frac{1}{2}}}{1-2x}$

$= \frac{(1-2x)^{-\frac{1}{2}}[(1-2x)^{1}+(x-1)]}{1-2x}$ → Factorize common factor $(1-2x)^{-\frac{1}{2}}$

$= \frac{1-2x+x-1}{(1-2x)^{\frac{3}{2}}}$ → Bring down $(1-2x)^{-\frac{1}{2}}$

$= \frac{-x}{(1-2x)^{\frac{3}{2}}}$ → Simplify

6. **Problems on applications of differentiation include the following:**

(a) Gradients, tangents and normal.

- $\frac{\mathrm{d}y}{\mathrm{d}x}$ refers to the gradient of the tangent at any point of the curve.
- The gradient of the normal is = $-\frac{1}{m_{tangent}}$ as tangent ⊥ normal

(b) Stationary points, increasing and decreasing functions.

- At stationary points or inflection points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.
- For increasing functions, , $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ (positive gradient)
- For decreasing functions, , $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ (negative gradient)

(c) Connected rates of change.

- Example: $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}l} \times \frac{\mathrm{d}l}{\mathrm{d}t}$ (from chain rule)

(d) Maxima and minima.

- We can use the *first derivative test* or the *second derivative test* to determine the nature of the stationary point.
- Second derivative test:
  - If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, then the stationary point is a maximum point.
  - If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, then the stationary point is a minimum point.

Examples of First and Second Derivative tests

Find the stationary point of the curve $y = 4x^2 - 16x$ and determine the nature of the stationary points.

(a) Use the first derivative test.

$y = 4x^2 - 16x$

$\frac{dy}{dx} = 8x - 16$

At stationary point, $\frac{dy}{dx} = 0$

$8x - 16 = 0$

$x = 2$

At $x = 2,\ y = 4(2)^2 - 16(2) = -16$

At $(2, -16)$

***x* value**	1.9	2	2.1
$\mathbf{\frac{dy}{dx}}$	$\frac{dy}{dx} = 8(1.9) - 16$ $= -0.8$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 8(2.1) - 16$ $= 0.8$
Shape			

It is a minimum point.

(b) Use the second derivative test.

$y = 4x^2 - 16x$

$\frac{dy}{dx} = 8x - 16$

$\frac{d^2y}{dx^2} = 8 > 0$ for all values of x

Thus since $\frac{d^2y}{dx^2} > 0$, the stationary point is a minimum point.

7. Differentiation of Composite Trigonometric Functions

Function	Example
(a) $\frac{\mathrm{d}}{\mathrm{d}x}[k\sin f(x)] = k[\cos f(x)][f'(x)]$	$\frac{\mathrm{d}}{\mathrm{d}x}[2\sin(3x+1)] = (3)(2)\cos(3x+1)$ $= 6\cos(3x+1)$
(b) $\frac{\mathrm{d}}{\mathrm{d}x}[k\cos f(x)] = -k[\sin f(x)][f'(x)]$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\cos(x^2+1)\right]$ $= \left(\frac{1}{2}\right)[-\sin(x^2+1)](2x)$ $= -x\sin(x^2+1)$
(c) $\frac{\mathrm{d}}{\mathrm{d}x}[k\tan f(x)] = k[\sec^2 f(x)][f'(x)]$	$\frac{\mathrm{d}}{\mathrm{d}x}[3\tan(x^3+3x)]$ $= 3(3x^2+3)\sec^2(x^3+3x)$
(d) $\frac{\mathrm{d}}{\mathrm{d}x}[k\sin^{\mathrm{n}} f(x)] = kn[\sin^{n-1} f(x)][\cos f(x)][f'(x)]$	
(e) $\frac{\mathrm{d}}{\mathrm{d}x}[k\cos^{\mathrm{n}} f(x)] = -kn[\cos^{n-1} f(x)][\sin f(x)][f'(x)]$	
(f) $\frac{\mathrm{d}}{\mathrm{d}x}[k\tan^{\mathrm{n}} f(x)] = kn[\tan^{\mathrm{n}-1} f(x)][\sec^2 f(x)][f'(x)]$	

Examples of Composite Trigonometric Function

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\cos^4(x^3+2x)\right] = \frac{1}{2}(4)[\cos^3(x^3+2x)][-\sin(x^3+2x)](3x^2+2)$

$= -2(3x^2+2)\cos^3(x^3+2x)\sin(x^3+2x)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left[3\tan^2\left(\frac{1}{2}x^2+2\right)\right] = 3(2)\left[\tan\left(\frac{1}{2}x^2+2\right)\right]\left[\sec^2\left(\frac{1}{2}x^2+2\right)\right]\left[\frac{1}{2}(2x)\right]$

$= 6x\tan\left(\frac{1}{2}x^2+2\right)\sec^2\left(\frac{1}{2}x^2+2\right)$

8. Differentiation of Composite Exponential and Logarithmic Functions

Function	Example
(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left[ke^{f(x)}\right] = kf'(x)\,e^{f(x)}$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[2e^{\frac{1}{3}x+2}\right] = 2\left(\frac{1}{3}\right)e^{\frac{1}{3}x+2}$ $= \frac{2}{3}e^{\frac{1}{3}x+2}$
(b) $\frac{\mathrm{d}}{\mathrm{d}x}[\ln f(x)] = \frac{f'(x)}{f(x)}$	$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(3x^2+x)] = \frac{6x+1}{3x^2+1}$

Differentiation – Practice Questions 1a

1. Differentiate the following with respect to x:

   (a) $2x^2 + 5x + 7$

Answer: (a) ____________________

   (b) $\frac{7}{2}x^4 + \frac{5}{3x^4} + x$

Answer: (b) ____________________

   (c) $4x^2 + \frac{2}{3\sqrt{x}} + 7$

Answer: (c) ____________________

(d) $5ax^2 + 3bx^3 + 5$

Answer: (d) ____________________

(e) $\frac{3x^2-4x^3}{x^3}$

Answer: (e) ____________________

(f) $\frac{3\sqrt{x}-4}{\sqrt{x}}$

Answer: (f) ____________________

(g) $\sqrt{x} + \frac{1}{\sqrt{x}}$

Answer: (g) ____________________

2. Find the gradient of the curve $y = \frac{5x-4}{x^2}$ at the point where the curve crosses the x-axis.

Answer: *gradient* = ____________________

3. The gradient of the curve at $y = \frac{a}{x^2} + \frac{b}{x}$ at the point $(-1, 5)$ is 4.

Find the values of a and b.

Answer: a = ____________________

b = ____________________

4. Differentiate the following with respect to x:

(a) $(2x+5)^7$

Answer: (a) ____________________

(b) $3(x+4)^5$

Answer: (b) ____________________

(c) $\frac{2}{3}\left(\frac{x}{6}-1\right)^4$

Answer: (c) ____________________

(d) $\frac{1}{3x+2}$

Answer: (d) ____________________

(e) $\frac{12}{2+3x^2}$

Answer: (e) ____________________

(f) $\sqrt{5x^2+6}$

Answer: (f) ____________________

5. Find the gradient of the curve $y = (3x^2 - 5x + 3)^3$ at the point where $x = 1$.

Answer: *gradient* = ____________________

6. Find the equation of the tangent to the curve $y = 2 + \frac{12}{(3x-4)^2}$ at the point (2, 5).

Answer: y = ____________________

7. The curve $y = \frac{a}{2+bx}$ passes through the point (1, 1) and the gradient at that point is $\frac{3}{5}$.

Calculate the values of a and b.

Answer: a = ____________________

b = ____________________

Differentiation – Practice Questions 1b

1. Differentiate the following with respect to *x*:

   (a) $(3x + 2)(2 - x^2)$

   Answer: (a) ____________________

   (b) $(x + 1)^3(x + 3)^5$

   Answer: (b) ________________________________

(c) $(x+5)^3(x-4)^6$

Answer: (c) ________________________________

(d) $(3x-1)\sqrt{2x^2+3}$

Answer: (d) ________________________________

(e) $(x^3 + x^2)(x - 2)^7$

Answer: (e) ______________________________

2. Given that $y = (x + 3)^4(x - 5)^7$, find $\frac{dy}{dx}$ and the values of x for which $\frac{dy}{dx} = 0$.

Answer: x = _________ , _________ , _________

3. Given that $y = (3x + 2)(2 - x)^{-1}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the values of x for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 8$.

Answer: x = _________ , _________

4. Differentiate the following with respect to *x*:

(a) $\frac{3x+2}{1-4x}$

Answer: (a) ____________________

(b) $\frac{x^2}{2x-1}$

Answer: (b) ____________________

(c) $\frac{\sqrt{x}}{3+x}$

Answer: (c) ____________________

(d) $\frac{2x-7}{\sqrt{x+1}}$

Answer: (d) ____________________

8. A curve has the equation $y = \frac{x+2}{\sqrt{3x+1}}$.

(a) Find the gradient of the tangent to the curve at $x = 1$.

(b) Find the equation of the same tangent.

Answer: (a) *gradient* = ____________________

(b) y = ____________________

9. The line $2x + 9y = 3$ meets the curve $xy + y + 2 = 0$ at the points P and Q.

   (a) Find the coordinates of points P and Q.

   (b) Find the gradient of the tangent to the curve at point P and point Q.

Answer: (a) P = (_________ , _________) , Q = (_________ , _________)

(b) *gradient* = ____________________ and ____________________

Applications of Differentiation – Practice Questions 2

1. Find the equation of the tangent and of the normal to the curve $y = \sqrt{1-2x}$ at $y = 3$.

Answer: *Equation of tangent – y =* ____________________

Equation of normal – y = ____________________

2. Find the equation of the normal to the curve $y = \frac{3x+1}{1-x}$, for $x \neq 1$, at the point where the curve crosses the x-axis.

Answer: *Equation of normal – y =* ____________________

3. (a) Find the equation of the tangent to the curve $x^4y = 1$ at $x = 1$.

(b) This tangent meets the axes at *A* and *B*. Find the midpoint of *AB*.

Answer: (a) *Equation of tangent* – y = ____________________

(b) Midpoint of *AB* = (_________ , _________)

4. Find the point where tangent to the curve $y = 4x^3 - 3x$ at the point $\left(-\frac{1}{2}, 1\right)$ meets the curve again.

Answer: (_________ , _________)

5. A curve is given by $y = \frac{3(x^2-1)}{2x-1}$.

(a) Find $\frac{dy}{dx}$ and the values of x for which $\frac{dy}{dx} = 2$.

(b) Find the equation and that of the normal to the curve at the point where $x = 1$.

Answer: (a) x = __________ or __________

(b) *Equation of tangent* – y = ____________________

Equation of normal – y = ____________________

6. Given that the curve $y = \frac{2x^3}{x-1}$, $x \neq 1$, find

(a) the gradient of the curve when $x = -3$,

(b) find the coordinates of the points where the tangent to the curve is horizontal.

Answer: (a) *gradient* = ____________________

(b) (_________ , _________) & (_________ , _________)

7. Find the equation of the tangent to the curve $(y-2)^2 = x$, which is parallel to the line $x - 2y = 4$.

Answer: *Equation of tangent* – y = ____________________

Further Differentiation – Practice Questions 3

1. For each of the following functions, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and determine if y is increasing or decreasing.

   (a) $y = 4x - 1$

   Answer: (a) $\frac{\mathrm{d}y}{\mathrm{d}x} =$ ____________________

   y is ____________________

   (b) $y = -9 - x$

   Answer: (b) $\frac{\mathrm{d}y}{\mathrm{d}x} =$ ____________________

   y is ____________________

   (c) $y = -(x^3 + 1)$

   Answer: (c) $\frac{\mathrm{d}y}{\mathrm{d}x} =$ ____________________

   y is ____________________

(d) $y = x^2 + 2x$ for $x > 0$

Answer: (d) $\frac{\mathrm{d}y}{\mathrm{d}x} =$ ____________________

y is ____________________

2. A function is defined by the equation $y = \frac{x}{x^2+1}$. Show that y is decreasing for $x > 1$ or $x < -1$.

3. The percentage of a drug being absorbed into the bloodstream is x hours after the drug is taken is given by

$$f(x) = \frac{5x}{4x^2 + 16}$$

Find the time interval in which the absorption level is increasing.

4. Find the stationary points of each of the following curves and determine the nature of the stationary points. *Use the first derivative test.

(a) $y = x^2 + 4x$

Answer: (a) (__________ , __________)

Nature of stationary point ____________________

(b) $y = -x^2 + 6x$

Answer: (b) (__________ , __________)

Nature of stationary point ____________________

(c) $y = 3x^4 - 4x^3 + 5$

Answer: (c) (__________ , __________) & (__________ , __________)

Nature of stationary point ________________________ & ________________________

(d) $y = \frac{(x-3)^2}{x}$

Answer: (d) (__________ , __________) & (__________ , __________)

Nature of stationary point _________________________ & ________________________

5. Find the stationary points of each of the following curves and determine the nature of the stationary points. *Use the second derivative test (or the first derivative test).

(a) $y = x^2 + 3x$

Answer: (a) (_________ , _________)

Nature of stationary point ____________________

(b) $y = 2x - x^2$

Answer: (b) (_________ , _________)

Nature of stationary point ____________________

(c) $y = x^3 - 3x^2 + 3x - 7$

Answer: (c) (_________ , _________)

Nature of stationary point ____________________

(d) $y = 4x^3 - 48x$

Answer: (d) (__________ , __________) & (__________ , __________)

Nature of stationary point _________________________ & ________________________

6. A rectangular box has a square base of side x cm.

(a) If the sum of one side of the square and the height is 15 cm, express the volume of the box in terms of x.

(b) Use this expression to determine the maximum volume of the box.

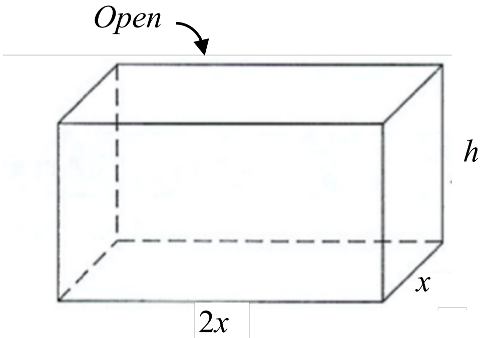
Answer: (a) $V =$ ____________________

(b) Maximum volume $=$ ____________________ cm^3

7. A sector of a circle with radius r cm contains an angle θ radians between the bounding radii.
   (a) Given that the perimeter of the sector is 7 cm, express θ in terms of r and show that the area of the sector is $\frac{1}{2}r(7-2r)$ cm^2.
   (b) Hence, or otherwise, find the maximum area of this sector as r varies.

Answer: (b) Maximum area = ____________________ cm^2

8. An open rectangular tank of length $2x$ metres, width x metres and height h metres is to be constructed from $3.2\ \text{m}^2$ of thin metal.



(a) Express h in terms of x.

(b) Find the value of x which will make the volume of the tank a maximum.

(c) Hence, find this volume, giving your answer correct to 2 decimal places.

Answer: (a) $h =$ ____________________

(b) $x =$ ____________________ cm

(c) Maximum volume $=$ ____________________ cm^3

Differentiation (Rate of Change) – Practice Questions 4

1. The radius of a circle increases at a rate of 0.2 cm/s. Calculate the rate of the increase of the area when the radius is 5 cm. Leave your answer in the form of π.

Answer: Rate of increase of area = ___________________ cm^2/s

2. A cube is expanding in such a way that its sides are changing at a rate of 2 cm/s. Find the rate of change of the total surface area when its volume is 125 cm^3.

Answer: Rate of increase of surface area = ___________________ cm^2/s

3. A spherical ball is being inflated at a rate of 20 cm^3/s. Find the rate of increase of its radius when its surface area is 100π cm^2.

Answer: Rate of the increase of radius = ____________________ cm/s

4. The base of a closed rectangular box is a square of side x cm and its height is 8 cm. Given that the side of the square base increases at a constant rate of 0.05 cm/s. Find (a) the rate of increase of the surface total area and (b) the rate of the volume when the total surface area of the box is 210 cm^2.

Answer: (a) Rate of the increase of total surface area = ___________________ cm^2/s

(b) Rate of the increase of volume = ___________________ cm^3/s

5. The formula for the volume of a sphere of radius r and its surface area are $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$ respectively. When $r = 8$ cm, the volume, V, is increasing at a rate of $10\ \mathrm{m}^3\mathrm{s}^{-1}$. Find the rate of increase of the surface area, A, at this instant.

Answer: Rate of the increase of surface area = ____________________ m^2/s

6. The radius of a circular disc increases at a constant rate of 0.005 cm/s. Find the rate at which the area is increasing when the radius is 15 cm, correct to 3 significant figures.

Answer: Rate of the increase of surface area = ___________________ cm^2/s

7. Variables x and y are related by the equation $y = \frac{x^2-5}{x}$. Given that x and y are functions of t and that $\frac{dy}{dt} = 0.3$, find the corresponding rate of change of x when $x = 2$.

Answer: $\frac{dx}{dt} =$ ___________________

Differentiation (Trigonometric, Exponential & Logarithmic Functions) – Practice Questions 5

1. Differentiate the following with respect to x:

(a) $4\sin 7x - x$

Answer: (a) ____________________

(b) $\cos 7x + \sin 3x$

Answer: (b) ____________________

(c) $\cos\left(\frac{\pi}{3} - \frac{3}{2}x\right)$

Answer: (c) ____________________

2. Differentiate the following with respect to x:

(a) $\sec x$

Answer: (a) ____________________

(b) $\operatorname{cosec} 2x$

Answer: (b) ____________________

(c) $\sec^2 3x$

Answer: (c) ____________________

3. Differentiate the following with respect to x:

(a) $x \tan 2x$

Answer: (a) ____________________

(b) $5 \sin 3x \cos 2x$

Answer: (b) ____________________

(c) $3 \tan x \cos^2 4x$

Answer: (c) ____________________

4. Differentiate the following with respect to x:

(a) $\ln(4 - x^2)$

Answer: (a) ____________________

(b) $\ln\frac{5x-3}{3x+7}$

Answer: (b) ____________________

(c) $\ln\frac{2x+3}{3x-4}$

Answer: (c) ____________________

(d) $\ln \frac{1}{(3x-5)^3}$

Answer: (d) ____________________

(e) $\ln \sqrt{6x^2 - 3}$

Answer: (e) ____________________

5. Find $\frac{\mathrm{dy}}{\mathrm{d}x}$ for the following.

(a) $y = \log_5 \sin x$

Answer: (a) $\frac{\mathrm{dy}}{\mathrm{d}x}$ = ______________________

(b) $y = \log_3 4x^{\frac{1}{2}}$

Answer: (b) $\frac{\mathrm{dy}}{\mathrm{d}x}$ = ______________________

(c) $e^y = 2x^3 + 7$

Answer: (c) $\frac{dy}{dx}$ = ______________________

(d) $e^y = \sec x$

Answer: (d) $\frac{dy}{dx}$ = ______________________

6. Differentiate the following with respect to x:

(a) e^{-4x}

Answer: (a) ___________________

(b) $e^{x} + \frac{1}{e^{x}}$

Answer: (b) ___________________

(c) $e^{\sin 2x}$

Answer: (c) ___________________

(d) $\mathrm{e}^{\tan x}$

Answer: (d) ____________________

(e) $\frac{1}{2}x^2\mathrm{e}^{\sin x}$

Answer: (e) ____________________

(f) $\frac{x \cos x}{e^x}$

Answer: (f) ___________________

Differentiation – Exam Questions

1. The equation of a curve is $y = 2 - x - \frac{2x+3}{x-3}$.

   (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

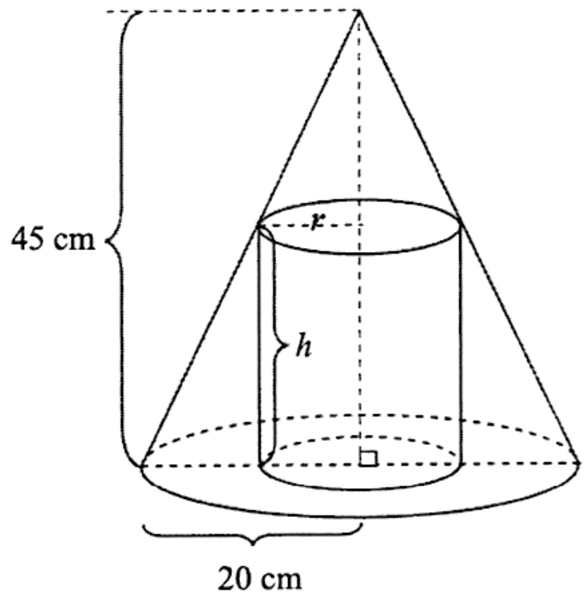
   (b) Find the x-coordinate(s) of the stationary point(s) of the curve.

(c) Determine the nature of each stationary point.

2. A cylindrical ice block of base radius r cm is melting in such a way that the total surface area, A cm^2, is decreasing at a constant rate of 72 cm^2/s. Given that the height is twice the radius and assuming that the ice block retain its shape, calculate the rate of change of r when $r = 5$ cm.

Answer: ____________________

3. The diagram shows a solid cylinder of radius r cm and height h cm inscribed in a hollow cone of height 45 cm and base radius 20 cm. The cylinder rests on the base of the cone and the circumference of the top surface of the cylinder touches the curved surface of the cone.



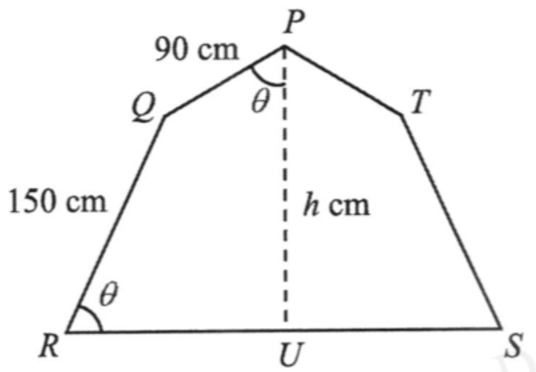
(a) Show that the volume, V cm^3, of the cylinder is given by $V = 45\pi r^2 - \frac{9}{4}\pi r^3$.

(b) Given that r can vary, find the maximum volume of the cylinder, leaving your answer in terms of π.

(c) Hence, show that the cylinder occupies at most $\frac{4}{9}$ of the volume of the cone.

4. Differentiate $\ln(2x^2 + 1)$ with respect to x.

5. The diagram shows the side view *PQRST* of a tent. The tent rests with *RS* on horizontal ground. *PQRST* is symmetrical about the vertical *PU*, where *U* is the midpoint of *RS*. Angle *QPU* = Angle *QRU* = θ radians and the lengths of *PQ* and *QR* are 9l cm and 150 cm respectively. The vertical height of *P* from the ground is h cm.



(a) Explain clearly why $h = 90\cos\theta + 150\sin\theta$.

(b) Express h in the form of $R\cos(\theta - \alpha)$, where $R > 0$ and α is an acute angle.

(c) Find the greatest possible value of h and the value of θ at which this occurs.

(d) Find the values of θ when $h = 160$.

6. Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan^3 5x) = 15\sec^4 5x - 15\sec^2 5x$.

7. Answer the following parts:

(a) Given that $y = \frac{3x}{\sqrt{5-4x}}$, express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the form $\frac{ax+b}{\sqrt{(5-4x)^n}}$, where a, b and n are real constants.

(b) Hence find the equation of the normal to the curve $y = \frac{3x}{\sqrt{5-4x}}$ at the point on the curve where $x = 1$.

8. Given that $y = \operatorname{cosec} x \tan x$,

(a) show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \sec^2 x$, and

(b) determine where y is decreasing for $0 \leq x \leq 2\pi$.

9. A piece of wire has a fixed length of k cm long is bent to form a rectangle. Show that the area of the rectangle is a maximum when it is a square.

10. A curve has the equation $y = \ln \frac{1-x}{x^2}$, $x < 1$ and $x \neq 0$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$,

(b) A particle moves along the curve in such a way that the y-coordinate of the particle is changing at a constant rate of 9 units per second.

Find the rate at which the x-coordinate of the particle is increasing at the instant when $y = \ln 2$.

11. The equation of a curve is $y = 3(3x^6 - 1)$.

(a) Find the coordinates of the stationary point on the curve.

(b) Determine the nature of the stationary point.

12. The curve has a gradient function $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^{2x}+a}{e^{2x}}$, where a is a constant.

The gradient of the normal to the curve at the point $(0, -1)$ is $\frac{1}{2}$.

(a) Find the value of a.

(b) Explain why the curve has no stationary point.

13. Given that $y = \sin 4x + \cos^4 x$, find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = \frac{\pi}{4}$.

14. Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left[4\sin^2\left(\frac{x}{2}+\pi\right)\right] = k\sin x$ where k is a constant.

15. It is given that $\mathrm{f}(x) = \ln \sqrt[3]{\frac{5+x}{5-x}}$.

(a) Find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$.

(b) Hence determine the range of values of x for which both $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$ are positive.

16. The equation of a curve is $y = kx\sqrt{2x+3}$ where k is a constant

(a) Obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the form $\frac{ak(x+b)}{\sqrt{2x+3}}$ where a and b are real constants.

(b) A point moves along the same curve in such a way that when $x = 3$, the rate of increase of y with respect to time is thrice the rate of increase of x with respect to the time. Find the value of k.

17. The function f is defined by $\mathrm{f}(x) = xe^{-x} + ke^{\frac{3}{5}x}$, where k is a constant. Given that $\mathrm{f}'(0) = 5$, find the value of k.

18. The surface area of an object, A cm^2, is given by $A = 300x - \left(\frac{\pi+4}{8}\right)x^2$. Given that x cm can vary, find the stationary value of A and determine whether it is a maximum or minimum.

19. A curve has the equation $y = -2\left(1 - \frac{1}{4}x\right)^2 + p$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Hence, find the equation of the line that is parallel to the tangent of the curve at $x = 1$ and passes through the origin.

(b) The normal to the curve at $x = 1$ passes through the point $\left(-1, 1\frac{1}{2}\right)$. Find the value of the constant p.

20. D

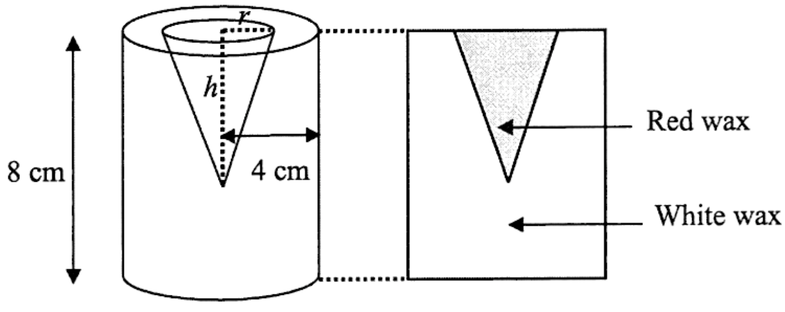


Diagram *A* Diagram *B* (cross-sectional view)

Diagram *A* shows a cylindrical candle stick of height 8 cm and radius 4 cm. The candle stick is made of white wax and red wax as shown in diagram B. The red wax forms an inverted cone of radius *r* cm and height *h* cm.

(a) Given that the sum of the radius and height of the cone is to remain constant at 5 cm, express *h* in terms of *r*.

(b) Show that the volume, *V*, of the white wax is given by

$V = 128\pi - \frac{5}{3}\pi r^2 + \frac{1}{3}\pi r^3$.

(c) Find the value of r for which V is stationary.