2023 JC2 H2 MA CT Markers Comments

1 The complex numbers z and w satisfy the equations:

$$2z - ww^* = \sqrt{2}w^2 \text{ and } z + \sqrt{2}w = \frac{1}{2}|w|^2 - \sqrt{2}.$$

(i) Show that $w^2 + 2w + 2 = 0$. [2]

(ii) Hence find all possible pairs of values for z and w. [3]

1	Solution [6] Complex Numbers	
(i)	$2z - ww^* = \sqrt{2}w^2$ $2z = \sqrt{2}w^2 + \lvert w\rvert^2$ Substitute $z = \frac{1}{2}\left[\sqrt{2}w^2 + \lvert w\rvert^2\right]$ into $z + \sqrt{2}w = \frac{1}{2}\lvert w\rvert^2 - \sqrt{2}$ $\left(\frac{1}{2}\right)\left(\sqrt{2}w^2 + \lvert w\rvert^2\right) + \sqrt{2}w = \frac{1}{2}\lvert w\rvert^2 - \sqrt{2}$ $\sqrt{2}w^2 + 2\sqrt{2}w + 2\sqrt{2} = 0$ $w^2 + 2w + 2 = 0$ (Shown)	Majority were able to show the result either via the substitution or elimination method. A number of students did not see that ww* = $\lvert w\rvert^2$, and hence unable to continue or instead wrote w = x + iy and expanded.
(ii)	From GC, $w = -1 \pm \mathrm{i}$ When $w = -1 + \mathrm{i}$, $w^2 = -2\mathrm{i}$, $\lvert w\rvert^2 = 2$ and $z = \frac{\sqrt{2}(-2\mathrm{i}) + 2}{2} = 1 - \sqrt{2}\mathrm{i}$ When $w = -1 - \mathrm{i}$, $w^2 = 2\mathrm{i}$, $\lvert w\rvert^2 = 2$ and $z = \frac{\sqrt{2}(2\mathrm{i}) + 2}{2} = 1 + \sqrt{2}\mathrm{i}$ Thus, $z = 1 - \sqrt{2}\mathrm{i}$ and $w = -1 + \mathrm{i}$ $z = 1 + \sqrt{2}\mathrm{i}$ and $w = -1 - \mathrm{i}$	w can be found via the GC. However some students went on to substitute w = x + iy and solve via comparing real and imaginary parts (which was too much work for 1 mark). Though the method to find z was correct, many made careless mistakes in the computation.

2 The complex numbers z, w and v are such that $z=\sqrt{2}\left(\cos\frac{13\pi}{30}+\mathrm{i}\sin\frac{13\pi}{30}\right)$, $\arg(w)=\frac{\pi}{12}$ and $v=\frac{z^4w}{\mathrm{i}w^*}$.

(i) Find the modulus of v and show that $\arg(v)=-\frac{3}{5}\pi$. [4]

(ii) It is known that v^n is a negative real number. Find the least positive integer value of n. [3]

2	Solution [6] Complex Numbers	
(i)	$\lvert v\rvert=\left\lvert\frac{z^4w}{\mathrm{i}w^*}\right\rvert$ $=\frac{\lvert z^4\rvert\lvert w\rvert}{\lvert\mathrm{i}\rvert\lvert w^*\rvert}$ $=\frac{\lvert z^4\rvert\lvert w\rvert}{(1)\lvert w\rvert}$ $=\lvert z^4\rvert=\sqrt{2}^4=4$ units $z=\sqrt{2}\left(\cos\frac{13\pi}{30}+\mathrm{i}\sin\frac{13\pi}{30}\right)$ $\arg\left(\frac{z^4w}{\mathrm{i}w^*}\right)=4\arg(z)+\arg(w)-(\arg(\mathrm{i})+\arg(w^*))$ $=4\arg(z)+\arg(w)-\arg(i)-(-\arg(w))$ $=4\left(\frac{13\pi}{30}\right)+2\left(\frac{\pi}{12}\right)-\frac{\pi}{2}$ $=\frac{7\pi}{5}=\frac{-3\pi}{5}$ (Principal argument) (Shown)	Most candidates can work out that $\lvert v\rvert=4$ Some candidates are not aware that $\arg(w^*)=-\arg(w)$; $\arg(i)=\frac{\pi}{2}$; $\frac{7\pi}{5}=\frac{-3\pi}{5}$ (Principal argument) since $\frac{7\pi}{5}-2\pi=\frac{-3\pi}{5}$
(ii)	$\arg(v^n)=n\arg(v)=\frac{-3n\pi}{5}$, $\arg(\text{negative real number})=(2k+1)\pi$, where $k\in\mathbb{Z}$ $\frac{-3n\pi}{5}=(2k+1)\pi$ $n=\frac{-5}{3}(2k+1)$ Least positive value of n = 5 when $k=-2$.	Many candidates are wrongly state that $\arg(\text{negative real number})$ $=\mathrm{k}\pi$, where $k\in\mathbb{Z}$

	Note: Setting $\frac{-3n\pi}{5} = k\pi$ is not correct.	

3 The position vectors of distinct points A, B and Q are given by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{q}$ respectively with reference to the origin.

(i) Given $(\mathbf{q}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})=\mathbf{0}$, explain why A, B and Q are collinear. [2]

(ii) It is given further that $\mathbf{q}=\lambda\mathbf{a}+(1-\lambda)\mathbf{b}$ where $0<\lambda<1$, $|\mathbf{b}|=\frac{1}{2}|\mathbf{a}|$, $AQ=\frac{\sqrt{3}}{6}|\mathbf{a}|$ and $\angle AOB=60°$. By finding the value of $(\mathbf{q}-\mathbf{a})\cdot(\mathbf{q}-\mathbf{a})$ in terms of λ and $|\mathbf{a}|$, find the value of λ and write down the ratio of $AQ{:}QB$. [5]

3	Solution [7] Abstract Vectors	
(i)	$(\mathbf{q}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})=\mathbf{0}$ where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{q}$ are distinct $\Rightarrow(\mathbf{q}-\mathbf{a})//(\mathbf{b}-\mathbf{a})$ since $\mathbf{q}-\mathbf{a}\neq\mathbf{0}$, $\mathbf{b}-\mathbf{a}\neq\mathbf{0}$ $\Rightarrow(\mathbf{q}-\mathbf{a})=k(\mathbf{b}-\mathbf{a})$ for some $k\in\mathbb{R}$ $\Rightarrow\overrightarrow{AQ}=k\overrightarrow{AB}$ AQ is parallel to AB with common point A. The points A, B and Q are collinear.	Most students were able to deduce that the vector $\overrightarrow{AQ}$ and $\overrightarrow{AB}$ are parallel as their cross product is the zero vector. Students should be specific in naming the vectors properly and not stating that $(\mathbf{q}-\mathbf{a})$ is parallel to $(\mathbf{b}-\mathbf{a})$. Also, expanding the cross product will not be useful in deriving the parallel property of the vectors as needed.
(ii)	Consider $(\mathbf{q}-\mathbf{a})\cdot(\mathbf{q}-\mathbf{a})$ $=(\lambda\mathbf{a}+(1-\lambda)\mathbf{b}-\mathbf{a})\bullet(\lambda\mathbf{a}+(1-\lambda)\mathbf{b}-\mathbf{a})$ $=((\lambda-1)\mathbf{a}+(1-\lambda)\mathbf{b})\bullet((\lambda-1)\mathbf{a}+(1-\lambda)\mathbf{b})$ $=(\lambda-1)^2\lvert\mathbf{a}\rvert^2+2(\lambda-1)(1-\lambda)\mathbf{a}\bullet\mathbf{b}+(1-\lambda)^2\lvert\mathbf{b}\rvert^2$ $=(\lambda-1)^2\lvert\mathbf{a}\rvert^2-2(\lambda-1)(\lambda-1)\mathbf{a}\bullet\mathbf{b}+(\lambda-1)^2\lvert\mathbf{b}\rvert^2$ $=(\lambda-1)^2\lvert\mathbf{a}\rvert^2-2(\lambda-1)^2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 60°+(\lambda-1)^2\lvert\mathbf{b}\rvert^2$	There were students who were not able to quote the result that $\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$ and hence deduce that it should be $(\mathbf{q}-\mathbf{a})\cdot(\mathbf{q}-\mathbf{a})=\lvert\mathbf{q}-\mathbf{a}\rvert^2$. Very few students were able to correctly expand $(\mathbf{q}-\mathbf{a})\cdot(\mathbf{q}-\mathbf{a})$ to obtain $\frac{3(\lambda-1)^2}{4}\lvert\mathbf{a}\rvert^2$ due to the following **errors**:

	$= (\lambda-1)^2\lvert\mathbf{a}\rvert^2 - (\lambda-1)^2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert + (\lambda-1)^2\lvert\mathbf{b}\rvert^2$ $= (\lambda-1)^2\left[\lvert\mathbf{a}\rvert^2 - \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert + \lvert\mathbf{b}\rvert^2\right]$ $= (\lambda-1)^2\left[\lvert\mathbf{a}\rvert^2 - \lvert\mathbf{a}\rvert\left(\frac{1}{2}\lvert\mathbf{a}\rvert\right) + \left(\frac{1}{2}\lvert\mathbf{a}\rvert\right)^2\right]$ $= \frac{3(\lambda-1)^2}{4}\lvert\mathbf{a}\rvert^2$ $(\mathbf{q}-\mathbf{a})\cdot(\mathbf{q}-\mathbf{a}) = \frac{3(\lambda-1)^2}{4}\lvert\mathbf{a}\rvert^2$ $\overrightarrow{AQ}\cdot\overrightarrow{AQ} = \frac{3(\lambda-1)^2}{4}\lvert\mathbf{a}\rvert^2$ $\left(\frac{\sqrt{3}}{6}\lvert\mathbf{a}\rvert\right)^2 = \frac{3(\lambda-1)^2}{4}\lvert\mathbf{a}\rvert^2$ $\frac{3}{36} = \frac{3(\lambda-1)^2}{4}$ $(\lambda-1)^2 = \frac{1}{9}$ $\Rightarrow \lambda - 1 = \pm\frac{1}{3}$ $\therefore \lambda = \frac{2}{3}$ or $\frac{4}{3}$ (rejected) Thus, for $\mathbf{q} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ $\mathbf{q} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ $\mathbf{q} = \frac{2\mathbf{a}+\mathbf{b}}{\mathbf{3}}$ By ratio theorem, $AQ : QB = 1 : 2$	(i) writing $(\mathbf{q}-\mathbf{a})\cdot(\mathbf{q}-\mathbf{a})$ $= q^2 - 2aq + a^2$ instead of $\lvert\mathbf{q}\rvert^2 - 2\mathbf{q}\bullet\mathbf{a} + \lvert\mathbf{a}\rvert^2$ (ii) replacing $\mathbf{b} = \frac{1}{2}\mathbf{a}$ in expression for vector $\mathbf{q}$ (note that $\lvert\mathbf{b}\rvert = \frac{1}{2}\lvert\mathbf{a}\rvert$ does not imply $\mathbf{b} = \frac{1}{2}\mathbf{a}$) (iii) wrongly expanding the expression $\lvert(\lambda-1)\mathbf{a} + (1-\lambda)\mathbf{b}\rvert$ as $\lvert(\lambda-1)\mathbf{a}\rvert + \lvert(1-\lambda)\mathbf{b}\rvert$. Note that in general, $\lvert\mathbf{a}+\mathbf{b}\rvert \neq \lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert$

4 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{1}{x^2 - 1},\ x \in \mathbb{R},\ x \neq -1, 1$$

(i) Show that f is not a one-one function. [1]

Another function g is defined by

$$\mathrm{g} : x \mapsto \frac{1}{x^2 - 1},\ x < a,\ x \neq b\,,$$

where $b < a$.

(ii) State the maximum value of a such that g^{-1} exists and write down the value of b. [2]

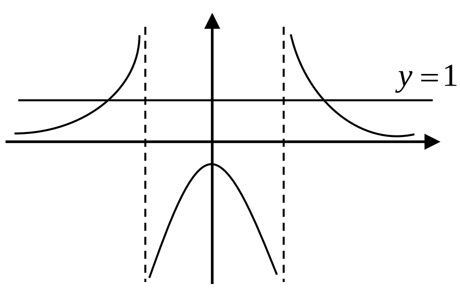
For the rest of the question, take $a = -1$.

(iii) Find $\mathrm{g}^{-1}(x)$. Sketch the graphs of g and g^{-1} on a single clearly labelled diagram, showing the geometrical relation between the graphs. [4]

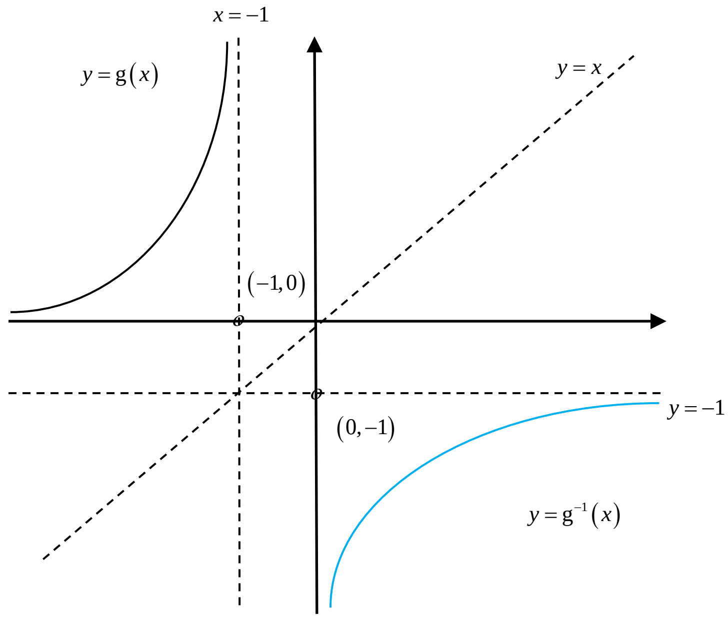
A function h is defined by

$$\mathrm{h} : x \mapsto x^2,\ x \geq 0.$$

(iv) Explain why the composite function hg exists and define hg in a similar form. [2]

4	Solution [9] Functions	
(i)	 Since the line $y = 1$ cuts the graph of f more than once, Therefore f is not a one-one function.	Students generally had the right idea for how to solve this, but many did not use the standard explanation and made mistakes as a result. Common mistakes: - No graph drawn - General $y = k$ instead of a specific value - Claims that **any** line would cut twice, rather than a specific value. - Omitting conclusion.

	Alternatively Consider $x_1=-2$ and $x_2=2$. Then $\mathrm{f}\left(x_1\right)=\dfrac{1}{(-2)^2-1}=\dfrac{1}{3}$, and $\mathrm{f}\left(x_2\right)=\dfrac{1}{(2)^2-1}=\dfrac{1}{3}$. Since $x_1 \neq x_2$, but $\mathrm{f}\left(x_1\right)=\mathrm{f}\left(x_2\right)$, then f is not one-one.	Some students attempted the alternative method and were generally successful.
(ii)	$a=0$ $b=-1$	Generally well done.
(iii)	let $y=\dfrac{1}{x^2-1}$ $yx^2-y=1$ $yx^2=1+y$ $x^2=\dfrac{1+y}{y}$ $x=-\sqrt{\dfrac{1+y}{y}}$ or $x=\sqrt{\dfrac{1+y}{y}}$, rejected since $x<0$. $\mathrm{g}^{-1}(x)=-\sqrt{\dfrac{1+x}{x}}$	Students generally had a good idea about how to approach this question. Common mistakes: - Forgot ± when taking square root. - Incorrectly rejected negative root. - Forgot to explicitly state $\mathrm{g}^{-1}(x)$ - Forgot to swap back x and y for $\mathrm{g}^{-1}(x)$

		Students were usually able to draw y = g(x). Some forgot to draw in the asymptotes. Some students drew the graph on (-∞,0) instead of (-∞,-1). The graph of y = $g^{-1}(x)$ was not as well drawn. Asymptotes were often incorrect. A number of students drew the graph on the domain of (-∞,-1) instead of (0,∞). Labelling of the graph was not good. Students have taken to labelling y = g(x) as g. No marks were deducted, but students should be mindful to write in the equation of the graph or the label of the curve, rather than just the name of the function.

(iv)	For composite function hg to exist, $R_g \subseteq D_h$ Since: $R_g = (0, \infty)$ $D_h = [0, \infty)$ Then hg exists. $\text{hg} : x \mapsto \dfrac{1}{(x^2-1)^2}, x < -1$.	Students generally had some idea of how to complete this question, but were unable to do so accurately. Common errors: - Incorrect R_g, or not stating R_g. - Not stating the domain of hg - Taking D_h as D_{hg}. - Algebraic errors, e.g. $\dfrac{1}{(x^2-1)^2} = \dfrac{1}{x^4-1}$ Students' set notation and functions notation was not good.

5 (a) The amount of time a student spends in studying Mathematics follows this pattern:

On Day 1, he studies for 20 minutes. For each subsequent day, he increases the amount of time spent by 4 minutes. He keeps doing so until he first exceeds 2.5 hours in a day. After that, he reduces the amount of time spent by 10% each day until he is studying for less than 20 minutes in a day.

(i) Determine the first day on which he will study for less than 20 minutes. [4]

(ii) Hence, calculate the total time spent on studying starting from the first day until the day he studies for less than 20 minutes, giving your answer to the nearest minute. [2]

(b) The student changes his study plan. He decides that the time, measured in minutes, spent on studying will decrease following a geometric progression with first term *a* and common ratio *r*. He notices that the time spent on the first, second and fourth day will form consecutive terms of an arithmetic progression as well. Determine the ratio of the time that he will spend studying on the sixth day to that for the twelfth day. [4]

<table>
<tr><td>5</td><td>Solution [10] AP GP</td><td></td></tr>
<tr><td>(a)
(i)</td><td>Day 1, $n = 1$: 20 mins
Day 2, $n = 2$: 24 mins
Day 3, $n = 3$: 28 mins
…
Day n, $n = n$: $20+(n-1)(4)>150$
$n>33.5$

Least $n=34$

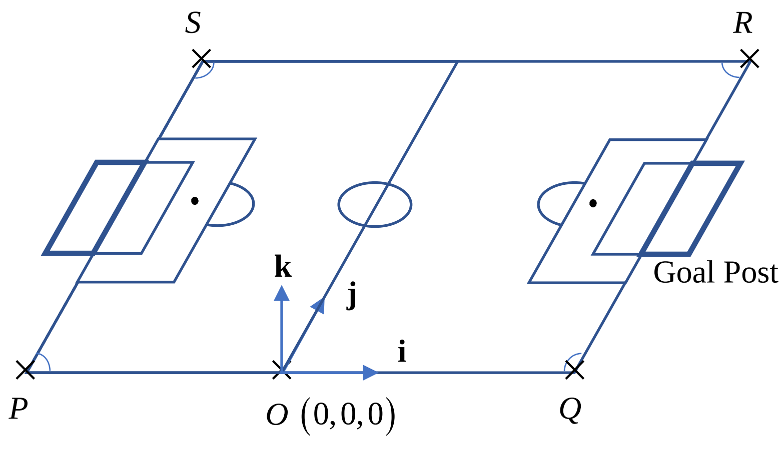
On the 34th day, the study time for mathematics exceeds 150 mins i.e. 2.5 hours.</td><td></td></tr>
<tr><td></td><td>Day 34, $n = 34$: Time spent $=20+(34-1)(4)$
$=152$ mins

Day 35, $n = 35$: $(0.9)^1(152)$ ($k = 1$)
Day 36, $n = 36$: $(0.9)^2(152)$ ($k = 2$)
Day 37, $n = 37$: $(0.9)^3(152)$ ($k = 3$)
…
Day 34 + k: $(0.9)^k(152)$ ($k = k$)</td><td>Generally, students who did not manage to solve part (a)(i) and (a)(ii) properly displayed a common missing trait; <u>absence in creating</u> a clearly listed number pattern. This resulted in students applying the</td></tr>
</table>

	For $(0.9)^k\left(152\right)<20$ $k>19.25$ i.e. minimum $k=20$ Thus, he will study less than 20 mins on Day 54.	wrong values of first term especially when applying the AP or GP term of sum formula.
(a) (ii)	For the first 34 days, total time taken studying: $T_1=\frac{34}{2}\left[2\left(20\right)+\left(34-1\right)\left(4\right)\right]$ $=2924$ For the next 20 days, total time taken studying: $T_2=\frac{\left[(0.9)(152)\right]\left(1-0.9^{20}\right)}{1-0.9}$ $=1201.683157$ Thus, total time spent $T=2924+1201.683157$ $=4125.68$ $=4126$ mins	
(b)	Given that the first term of GP is a and the common ratio is r. Given that first, second and fourth day would follow an arithmetic progression Then $ar-a=ar^3-ar$. $r^3-2r+1=0$ Using GC, $r=1$ (rejected since the study time is decreasing) $r=-1.618$ (rejected since study time cannot be negative) $r=0.61803$	For (b), some students had forgotten how to approach such question which uses the phrase "…are consecutive terms or AP (or GP)…". Students ought to know that such phrase is common and they need to use the characteristics of an AP (consistent common difference between terms) to create the necessary equation to solve. Many students either didn't know they can use GC to solve an equation leaving to inexact answers, or thought they must solve

	$\dfrac{\text{Time spent on Day 6}}{\text{Time spent on Day 12}} = \dfrac{ar^5}{ar^{11}}$ $= \dfrac{0.61803^5}{0.61803^{11}}$ $= 17.9$ i.e. 17.9 : 1	equations algebraically all the time. All students ought to know that the question will suggest if an exact solution is needed.

6



A rectangular soccer field *PQRS* is modelled by the *x*-*y* plane with the origin *O*, midway between the points *P* and *Q* as shown in the diagram. The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually perpendicular such that $\mathbf{i}$ is parallel to *PQ* and $\mathbf{j}$ is parallel to *QR*.

There are two players A and B on the field. Player A kicks the soccer ball (which has negligible size compared to the size of the field) from a location with coordinates $(1,\ 1,\ 0)$, in the direction of $10\mathbf{i}+10\mathbf{j}+\mathbf{k}$. Assume that the ball travels in a straight line.

(i) Find the acute angle between the direction of travel of the soccer ball and the ground. [2]

At this instant, Player B is at the point with coordinates $(23,\ 27.24,\ 0)$ and remains stationary as he watches the ball.

(ii) Show that when Player B's location is closest to the path taken by the soccer ball, the ball is at the point with coordinates $(25,\ 25,\ 2.4)$. [3]

When the ball was at the point with coordinates $(25,\ 25,\ 2.4)$, Player B leapt and headed the ball towards the goal. Due to Player B's action, the direction of the ball changed.

(iii) Given that the soccer ball now travels in a straight line l in the direction of $10\mathbf{i}-\mathbf{k}$, find the position vector of its point of contact with the ground, *C*. [3]

The soccer ball then bounced off the ground along a straight line l' in the same plane as l and the normal to the ground, such that the angle between l and the normal equals the angle between the normal and l'.

(iv) Find a Cartesian equation of l'. [4]

6	Solution [12] Vectors (Lines and Planes)	
(i)	$\cos(90°-\alpha)$ ($\alpha=$ angle between normal and direction) $=\sin\theta$ ($\theta=$ angle between line and ground). $=\dfrac{\left\lvert\begin{pmatrix}10\\10\\1\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\sqrt{10^2+10^2+1}\sqrt{1}}$ $=\dfrac{1}{\sqrt{201}}$ $\theta=4.0°$ (correct to 1 decimal point)	(i) Many students thought that the xy plane has a normal vector of $\begin{pmatrix}1\\1\\0\end{pmatrix}$, or confused in the formula for finding angle between line and plane. (ii) ~~(i)~~ "show that Player B's location is closest to a line of travel" – requires concept of finding foot of perpendicular from a point to a line.
(ii)	Vector equation of the path taken by the ball: $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}10\\10\\1\end{pmatrix},\ \lambda\in\mathbb{R}$ Closest location of ball to B is located at the foot of perpendicular from B to the path. Let F be the foot of perpendicular, then: $\overrightarrow{OF}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}10\\10\\1\end{pmatrix}$, for some value of λ $=\begin{pmatrix}1+10\lambda\\1+10\lambda\\\lambda\end{pmatrix}$ Now, using the fact that $\overrightarrow{BF}\perp\mathbf{d}$ of the path of ball:	(ii), (iii) & (iv): for students who didn't do well for these parts, they showed problems in understanding the question. It could suggest that the students might already have adequate knowledge in vectors, and probably be able to solve questions that test students directly on concepts but not able to find the link to apply said concepts to a contextual question. (iii) ~~(ii)~~ "finding point of contact between a line of travel with the ground" - intersection between a line and a plane.

	$\overrightarrow{BF} \bullet \begin{pmatrix}10\\10\\1\end{pmatrix}=0$ $\left[\begin{pmatrix}1+10\lambda\\1+10\lambda\\\lambda\end{pmatrix}-\begin{pmatrix}23\\27.24\\0\end{pmatrix}\right]\bullet\begin{pmatrix}10\\10\\1\end{pmatrix}=0$ $\begin{pmatrix}10\lambda-22\\10\lambda-26.24\\\lambda\end{pmatrix}\bullet\begin{pmatrix}10\\10\\1\end{pmatrix}=0$ $100\lambda-220+100\lambda-262.4+\lambda=0$ $201\lambda=482.4$ $\lambda=2.4$ $\overrightarrow{OF}=\begin{pmatrix}1\\1\\0\end{pmatrix}+2.4\begin{pmatrix}10\\10\\1\end{pmatrix}$, for some value of λ $=\begin{pmatrix}25\\25\\2.4\end{pmatrix}$ Coordinates of F is $(25,\ 25,\ 2.4)$.	
(iii)	$l:\mathbf{r}=\begin{pmatrix}25\\25\\2.4\end{pmatrix}+\mu\begin{pmatrix}10\\0\\-1\end{pmatrix},\ \mu\in\mathbb{R}$ Equation of the ground : $\mathbf{r}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ (x-y plane) At contact, i.e. intersection: $\left[\begin{pmatrix}25\\25\\2.4\end{pmatrix}+\mu\begin{pmatrix}10\\0\\-1\end{pmatrix}\right]\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ $\begin{pmatrix}25+10\mu\\25\\2.4-\mu\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ $2.4-\mu=0$ $\mu=2.4$	(iv) ~~(iii)~~ the phrasing used is obtained from the 2017 A Level Specimen Paper. It could be construed as misleading as some students thought of using scalar product and angle involved to solve, but it simply needed students to see that it is a classical question of finding reflected line about a plane / line. One possible way to improve in this area, is to (obviously) try to understand the context AND THEN draw out a

	$\overrightarrow{OC}=\begin{pmatrix}49\\25\\0\end{pmatrix}$	<u>simple diagram</u> to see what concepts in vectors needs to be applied. Usually, such questions involved 2 objects: point and point, point and line, point and plane, line and line, line and plane, plane and plane, etc. After which, the appropriate method can then be used to solve the problem.
(iv)	Let the initial location of the ball be *D*, $D(25,\ 25,\ 2.4)$ Let *N* be the foot of perpendicular from *D* to the ground. By observation, $N(25,\ 25,\ 0)$. Let *D'* be the point of reflection of the point *D* about the ground. By ratio theorem, $\overrightarrow{CN}=\dfrac{\overrightarrow{CD}+\overrightarrow{CD'}}{2}$ $\overrightarrow{CD'}=2\overrightarrow{CN}-\overrightarrow{CD}$ $\overrightarrow{CD'}=2\begin{pmatrix}-24\\0\\0\end{pmatrix}-\begin{pmatrix}-24\\0\\2.4\end{pmatrix}$ $\overrightarrow{CD'}=\begin{pmatrix}-24\\0\\-2.4\end{pmatrix}$ $=-2.4\begin{pmatrix}10\\0\\1\end{pmatrix}$ $l':\mathbf{r}=\begin{pmatrix}49\\25\\0\end{pmatrix}+t\begin{pmatrix}10\\0\\1\end{pmatrix},\ t\in\mathbb{R}$	

	$l': \dfrac{x-49}{10} = z,\ y = 25$	

7 A fair six-sided die and a fair four-sided die are rolled together. The outcome of each die is noted. Let the random variable X denote the absolute difference between the outcomes of the 2 dice in a roll.

(i) Construct a probability distribution table for X. [2]

(ii) Show that $\mathrm{E}(X)=\dfrac{11}{6}$ and find the variance of X. [3]

<table>
<tr><td>7</td><td>Solution [5] DRV</td><td></td></tr>
<tr><td>(i)</td><td>Table of outcomes:

	1	**2**	**3**	**4**	**5**	**6**
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2

Probability distribution table is

x	0	1	2	3	4	5
$\mathrm{P}(X=x)$	$\frac{4}{24}$	$\frac{7}{24}$	$\frac{6}{24}$	$\frac{4}{24}$	$\frac{2}{24}$	$\frac{1}{24}$
</td><td>Generally well done.
Most responses are able to identify all 6 outcomes correctly while a handful missed one or 2 of the outcome.
Attempts to find probability without using the table of outcomes generally are successful.</td></tr>
<tr><td>(ii)</td><td>

$$\mathrm{E}(X)=(0)\left(\frac{4}{24}\right)+(1)\left(\frac{7}{24}\right)+(2)\left(\frac{6}{24}\right)+(3)\left(\frac{4}{24}\right)+(4)\left(\frac{2}{24}\right)+(5)\left(\frac{1}{24}\right)=\frac{11}{6}\ \text{(shown)}$$

$$\mathrm{E}(X^2)=(0)^2\left(\frac{4}{24}\right)+(1)^2\left(\frac{7}{24}\right)+(2)^2\left(\frac{6}{24}\right)+(3)^2\left(\frac{4}{24}\right)+(4)^2\left(\frac{2}{24}\right)+(5)^2\left(\frac{1}{24}\right)=\frac{31}{6}$$

$$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2=\frac{31}{6}-\left(\frac{11}{6}\right)^2=\frac{65}{36}$$
</td><td>Most who got part (i) correct generally gets the E(X) correct as well.

Attempts to find Var(X) is generally successful except for some who forgot the formula for Var(X). Few had calculation errors.</td></tr>
</table>

8 In a code breaker game, the game master chooses from a set of eight characters {A, B, C, D, E, 1, 2, 3} including five letters and three digits. He chooses 4 characters without repetition and arranges them in a line for the challenger to decode.

(i) Find the probability that the game master includes the 3 digits for his code. [2]

(ii) After a few tries, the challenger figures out that the game master has included all 3 digits in the code. Find the probability that the digits in the code are arranged in ascending order. [3]

<table>
<tr><td>8</td><td>Solution [5] Probability</td><td></td></tr>
<tr><td>(i)</td><td>$$\mathrm{P}(\text{3 digits all included}) = \frac{n(\text{3 digits included})}{n(\text{any 4 character code})}$$ $$= \frac{\binom{5}{1}\times 4!}{\binom{8}{4}\times 4!} = \frac{\binom{5}{1}}{\binom{8}{4}}$$ $$= \frac{5}{70} = \frac{1}{14}$$</td><td>Main issue with most unsuccessful attempts is the P&C part. A lot of responses are not able to correctly give the number of ways to have 3 digits included in the 4-character code.
Another issue is that students miscount the number of letters given by the question.</td></tr>
<tr><td></td><td>Alternatively,
$\mathrm{P}(\text{3 digits all included})$ $$= \frac{5}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}\times 4$$ $$= \frac{1}{14}$$</td><td>Most using this alternative method forgot in include the cases due to arrangement or simply calculated the amount of arrangements wrongly.</td></tr>
<tr><td>(ii)</td><td>$$\mathrm{P}\left(\begin{array}{c}\text{digits in code arranged}\\ \text{in ascending order}\end{array}\middle|\,\text{3 digits included in code}\right)$$ $$= \frac{\mathrm{P}(\text{3 digits in code arranged in ascending order})}{\mathrm{P}(\text{3 digits in code})}$$ $$= \frac{\dfrac{\binom{5}{1}\times\binom{4}{1}}{\binom{8}{4}\times 4!}}{\dfrac{1}{14}} = \frac{20}{1680}\div\frac{1}{14}$$ $$= \frac{1}{6}$$</td><td>Most responses did not note down that the probability is a conditional probability. However, the working shows evidence of working on a reduced set involving 3 digits and 1 letter. As many did not realise that it is a conditional probability, most responses found only the numerator and that itself is also complicated by lack of P&C skills.
Some responses used $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)$</td></tr>
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	$\binom{5}{1}$ ways to choose 1 of the 5 letters. $\binom{4}{1}$ ways to insert the chosen letter among the digits.	without even justifying why events A and B are independent.
	Alternatively, $\text{P}\left(\begin{array}{c}\text{digits in code arranged}\\ \text{in ascending order}\end{array}\middle\vert \text{3 digits included in code}\right)$ $=\dfrac{\text{n}\left(\text{3 digits in code arranged in ascending order}\right)}{\text{n}\left(\text{3 digits in code}\right)}$ $=\dfrac{\binom{5}{1}\times\binom{4}{1}}{\binom{5}{1}\times 4!}$ $=\dfrac{1}{6}$	
	Alternatively, $\text{P}\left(\begin{array}{l}\text{digits in code arranged in ascending}\\ \qquad\qquad\text{order} \mid \text{3 digits included in code}\end{array}\right)$ $=\dfrac{\text{n}\left(\text{3 digits in code arranged in ascending order}\right)}{\text{n}\left(\text{3 digits in code}\right)}$ which is the ratio of 3 digits arranged in ascending order and 3 digits arranged without restriction $=\dfrac{1}{3!}=\dfrac{1}{6}$	

9 4 brothers invited 6 other boys to have a basketball match among themselves.

(a) Find the number of ways they could form 2 teams of 5. [2]

After the match, the boys went to a restaurant for food. They were seated at a round table with 10 identical chairs.

(b) Find the number of ways where at least one of the brothers was not seated next to another brother. [5]

9	Solution [6] P & C	
(a)	Number of ways $= \dfrac{\binom{10}{5}\binom{5}{5}}{2!}$ $= 126$	Many students did not realise the need to divide $\binom{10}{5}\binom{5}{5}$ by 2! to avoid double counting.
(b)	Case 1: 4 brothers separated into 2 units consisting of 1 and 3 brothers in respective units Number of ways $= \binom{4}{3} \times 3! \times \dfrac{6!}{6} \times \binom{6}{2} \times 2!$ $= 86400$ Case 2: 4 brothers separated into 3 units consisting of 1, 1, 2 brothers in respective units Number of ways $= \binom{4}{2} \times 2! \times \dfrac{6!}{6} \times \binom{6}{3} \times 3!$ $= 172800$ Case 3: 4 brothers all separated Number of ways $= \dfrac{6!}{6} \times \binom{6}{4} \times 4!$ $= 43200$ Total number of ways $= 86400 + 172800 + 43200$ $= 302400$	Most students either list all possible cases or use the negation method for this part. For the direct listing and counting of possible cases, there was a mistake in citing the case that 3 brothers are all separated as this is equivalent to the case when all 4 brothers are separated. Also, for case 1 and 2, many students did not include the multiplication of the term $\binom{4}{3}$ & $\binom{4}{2}$ in counting process.
	Alternatively, Unrestricted arrangements = $\dfrac{10!}{10} = 362880$ Unwanted Case 1: 4 brothers separated into 2 units consisting 2 brothers each Number of ways $= \dfrac{\binom{4}{2} \times 2! \times 2!}{2!} \times \dfrac{6!}{6} \times \binom{6}{2} \times 2!$ $= 43200$ Unwanted Case 2: 4 brothers all seated together	For students who use the negation method, most of them were able to apply $\dfrac{n!}{n}$ for circular arrangement. However, while most were able to cite Case 2 in their solution, they were not able to cite Case 1 and calculate number of ways for it

	Number of ways $= 4! \times \frac{7!}{7}$ $= 17280$ Total number of ways $= 362880 - 43200 - 17280$ $= 302400$	correctly.

10 The production manager of a food manufacturing company claims that the mean mass of its best-selling nutrition fruit bars is 125 grams, as stated on the packets. For the most recent batch of production where each fruit bar has its own production serial number, the manager uses his computer programme to randomly select 60 of these serial numbers and check the mass of each of the selected fruit bar. The masses, x grams, of the random sample of 60 fruit bars are summarized as follows.

$$\sum(x-125)=-21.0 \qquad \sum(x-125)^2=117.975$$

(i) State what it means for a sample to be random in this context. [1]

(ii) Calculate the unbiased estimates of the population mean and variance of the mass of the fruit bars. [2]

(iii) Test, at the 5% level of significance, the manager's claim that the mean mass of fruit bars is 125 grams. You should state the hypotheses and define any symbols used. [4]

(iv) Determine, with reasons, if there is a need for the production manager to know the distribution of the masses of the fruit bars. [1]

10	Solution [12] Hypothesis Testing	
(i)	A random sample in this context means that each of the fruit bar in the batch of production has an <u>equal probability</u> of being selected and every fruit bar in the sample is <u>independently</u> selected.	Many students did not state the need for every fruit bar in the sample to be independently selected.
(ii)	The unbiased estimator for the population mean is $\bar{x}=\frac{\sum(x–125)}{60}+125=124.65$ The unbiased estimator for the population variance is $s^2=\frac{1}{59}\left[117.975–\frac{(-21.0)^2}{60}\right]=1.875$ or $\frac{15}{8}$.	Common mistakes: $\bar{x}=\frac{\sum(x–125)}{60}=-\frac{21}{60}$ as we still need to 'add back' 125 as in solution step. A considerable number of students wrongly used the formula: $s^2=\frac{n}{n-1}\sum\left(x-\bar{x}\right)$ Note that in the case here, $\bar{x}\neq 125$
(iii)	Let μ be the population mean mass, in gram of the fruit bars produced by the food company and $\overline{X}$ be the sample mean mass of the randomly selected fruit bars. To test $\mathrm{H}_0:\mu=125$ against $\mathrm{H}_1:\mu\neq 125$ at the 5% level of significance.	Correct mistakes include: 1. Not explaining what μ represents; 2. Stating alternative hypothesis to be $\mathrm{H}_1:\mu<125$ 3. Did not include the

	Under H_0, $\overline{X} \sim N\left(125, \frac{1.875}{60}\right)$ approximately. The test statistics $Z = \dfrac{\overline{X}-125}{s/\sqrt{60}} \sim N(0,\ 1)$. or $Z = \dfrac{\overline{X}-125}{\sqrt{15/8}/\sqrt{60}} \sim N(0,1)$ Using GC, we perform a two-tail z-test and note that the value of test statistic ~~z = 1.979899~~ and the *p*-value = 0.047715 $z = -1.979899$ Since *p*-value =0.047715 < 0.05, we reject H_0 and conclude that there is sufficient evidence at 5% level of significance that the mean mass of the fruit bars is **not** 125 grams.	phrase ' Under H_0 ' in explaining distribution of $\overline{X}$. 4. Wrongly stating $\overline{X} \sim N\left(124.65, \frac{1.875}{60}\right)$ 5. Wrong test statistic: $Z = \dfrac{125-\overline{X}}{s/\sqrt{60}} \sim N(0,\ 1)$. 6. Carelessness in keying correct value of *s* in using GC to find the p-value. 7. Making the wrong conclusion eg we reject H_0 and conclude that there is sufficient evidence at 5% level of significance that the mean mass of the fruit bars is 125 grams.
(iv)	There is no need for the production manager to know the population distribution of the masses of the fruit bars. Since the sample size $n = 60$ is large, $\overline{X}$ follows a normal distribution approximately by the Central Limit Theorem.	Many students were able to provide the answer with correct reasoning ie sample size $n = 60$ is large. However not many stated that it is the mean $\overline{X}$ that follows normal distribution approximately

11. In a national examination, candidates offering Mathematics take Paper 1 and Paper 2. The scores of candidates for Paper 1 and Paper 2 represented by variables X and Y respectively are independent and normally distributed with means and variances as follows:

Paper Score	Mean	Variance
X	67	278.89
Y	μ	σ^2

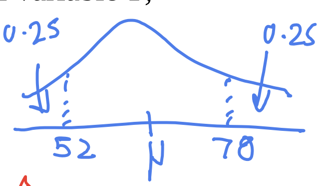
It is known that $\mathrm{P}(Y<52)=\mathrm{P}(Y>78)=0.25$.

(i) Determine the value of μ and show that the value of σ is 19.3, correct to 3 significant figures. [2]

(ii) To obtain Distinction for the subject, a candidate will need to have a total score of more than 145 for the 2 papers. Find the probability that a randomly selected candidate will obtain a Distinction. [3]

(iii) Candidates who obtain a score of more than k for Paper 1 are awarded Distinction for Paper 1. It is observed that less than 30% of the candidates are awarded Distinction for Paper 1. Find the minimum value of k to obtain Distinction for Paper 1, giving your answer correct to the nearest whole number. [3]

(iv) In another study, the exam board is reviewing the performance of candidates with Paper 2 score more than Paper 1 score by at least 10. In a random sample of 10 candidates, determine the probability that the exam board will have at least 4 of such candidates. [4]

11	Solution [12] Normal Distribution	
(i)	Since $\mathrm{P}(Y<52)=\mathrm{P}(Y>78)=0.25$, by symmetrical property of normal distribution of variable Y, mean $\mu=\dfrac{52+78}{2}=65$  Next, $\mathrm{P}(Y<52)=0.25$ $\Rightarrow \mathrm{P}\left(\dfrac{Y-65}{\sigma}<\dfrac{52-65}{\sigma}\right)=0.25$ $\Rightarrow \mathrm{P}\left(Z<\dfrac{52-65}{\sigma}\right)=0.25$ Must Show $\Rightarrow \dfrac{52-65}{\sigma}=-0.67448975 ---(*)$ $\Rightarrow \sigma=19.27382886=19.3$ (3 s.f.) **Need to show standardization working (*) clearly**	A number of candidates did not realize that the mean is the average of 52 and 78, which would simplify the working. You cannot assume that $\sigma=19.3$ and show that $\mathrm{P}(Y<52)=0.25$, and then claim that therefore $\sigma=19.3$ is correct.

(ii)	$\mathrm{E}(X+Y)=\mathrm{E}(X)+\mathrm{E}(Y)$ $=67+65=132$ $\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)$ $=278.89+19.3^2=651.38$ Thus, we have $X+Y\sim \mathrm{N}(132,\ 651.38)$ Using GC, $\mathrm{P}(X+Y>145)=0.305$. Hence the probability that a randomly selected Maths candidate will get a Distinction grade is 0.305.	Most candidates could do this part.
(iii)	$X\sim \mathrm{N}(67,\ 278.89)$ $\mathrm{P}(X>k)<0.3$ $1-\mathrm{P}(X\le k)<0.3$ $\mathrm{P}(X\le k)>0.7$ $k>75.7574$ $\leftarrow$ **Need to show this step clearly** Least value of *k* is 76, to the nearest whole number.	Many candidates are penalized as they failed to include the INEQUALITY $k>75.7574$ in the working.
(iv)	$\mathrm{E}(Y-X)=\mathrm{E}(Y)-\mathrm{E}(X)$ $=65-67=-2$ $\mathrm{Var}(Y-X)=\mathrm{Var}(Y)+\mathrm{Var}(X)$ $=19.3^2+278.89=651.38$ Thus, we have $Y-X\sim \mathrm{N}(-2,\ 651.38)$ Using GC, we then have $\mathrm{P}(Y-X\ge 10)=0.3191132653$. Next we define the new random variable *W* to be the number of students out of 10 such that their Paper 2 score is 10 more than their Paper 1 score. $W\sim \mathrm{B}(10, 0.3191132653)$ Hence,	Most candidates are aware that they have to set up a binomial distribution to solve the problem. Some candidates wrongly use binompdf instead of binomcdf in the last step. Need to understand &

	$\mathrm{P}(W \geq 4) = 1 - \mathrm{P}(W \leq 3)$ $= 1 - 0.5980499499$ $= 0.402$ (3.s.f)	appreciate the difference between probability distribution function and cumulative distribution function.

12 On average, a type of printer is known to produce errors on 100p % of the pages it prints, where $0 < p < 1$. A sample of 1000 pages is taken, and the number of pages with printing errors, X, is counted.

(i) State in context an assumption needed for X to be well-modelled by a binomial distribution. [1]

For the rest of the question, you may assume X follows a binomial distribution.

(ii) Given $\mathrm{P}(X = 6) = \mathrm{P}(X = 7)$, show the value of p is $\frac{1}{143}$. [2]

(iii) Hence find the expectation and variance of X. [2]

(iv) Given the sample contained fewer than 7 errors, find the probability there were no errors in the sample. [3]

The manufacturer takes n random samples of 1000 pages, where n is large, and the number of errors in each sample was recorded.

(v) State the distribution for the mean number of errors in a sample. Justify your answer. [2]

(vi) It is known that the probability of the mean number of errors in these samples being more than 7.5 is less than 0.05, determine the least value of n. [3]

12	Solution [13] Binomial Distribution + Sampling	
(i)	The event of a page containing an error is <u>independent</u> of the event that any other page contains an error. OR The probability that an error occurs in a page has a <u>constant</u> value of p.	The use of correct words is important here and many students did not pay attention to that. Instead of writing the "*event* of …. is independent", many wrote the "*probability* of …. is independent". Instead of writing the "probability of …. is *constant*", some wrote the "probability of …. is *equal*". Note that constant ≠ equal.
(ii)	$X \sim B(1000, p)$	As this is a "show" question, substantial amount of workings is needed. Just by writing (*) and then give the final result or use GC directly was not sufficient. Need to

	$\mathrm{P}(X=6)=\mathrm{P}(X=7)$ $\binom{1000}{6}p^{6}(1-p)^{994}=\binom{1000}{7}p^{7}(1-p)^{993}$ (*) $\frac{1000!}{6!994!}(1-p)=\frac{1000!}{7!993!}p$ $7(1-p)=994p$ $p=\frac{7}{1001}=\frac{1}{143}$	demonstrate how $\binom{1000}{6}$ and $\binom{1000}{7}$ were being simplified.
(iii)	$\mathrm{E}(X)=np=\frac{1000}{143}=6.99$ $\mathrm{Var}(X)=np(1-p)=6.94$	
(iv)	$\mathrm{P}(X=0 \mid X<7)=\frac{\mathrm{P}(\{X=0\}\cap\{X<7\})}{\mathrm{P}(X\le 6)}$ $=\frac{\mathrm{P}(X=0)}{\mathrm{P}(X\le 6)}$ $=\frac{8.95998\times10^{-4}}{0.450234}$ $=0.00199007=0.00199$ (3 s.f.)	Majority recognized that this is conditional probability but were unable to write down the numerators and denominators correctly. A common mistake: $\mathrm{P}(X=0 \mid X<7)$ $=\frac{\mathrm{P}(X=0)\cap \cancel{P(X<7)}}{\cancel{\mathrm{P}(X<7)}}$ · Intersection is used on events, not on probabilities.
(v)	Since n > 30 is large, $\overline{X}$ will be approximately normally distributed by the Central Limit theorem. $\mathrm{E}(\overline{X})=\mathrm{E}(X)=6.99301$ $\mathrm{Var}(\overline{X})=\frac{1}{n}\mathrm{Var}(X)=\frac{6.94410}{n}$ Hence, $\overline{X}\sim \mathrm{N}\left(6.99,\frac{6.94}{n}\right)$ approx..	Majority did not give the mean and variance of the normal distribution though students were able to identify the normal distribution and reason with CLT.
(vi)	Given $\mathrm{P}(\overline{X}>7.5)<0.05$, we have $\mathrm{P}\left(Z>\frac{7.5-6.99301}{\sqrt{\frac{6.94410}{n}}}\right)<0.05$	Majority was able to formulate but had difficulties in solving for n. Many made mistakes with the inequality sign and used

	Then, $\dfrac{7.5-6.99301}{\sqrt{\dfrac{6.94410}{n}}}>1.64485$ $\Rightarrow 0.50699>1.64485\sqrt{\dfrac{6.94410}{n}}$ $\Rightarrow \sqrt{n}>8.54939$ $\Rightarrow n>73.09198$ Thus, the least value of n is 74.	-1.64485. Some started with correct inequality but switched into equality in latter steps. There were no reasons provided as to why final answer was rounded up rather than to nearest whole number. This issue could be avoided by using inequality throughout. Many did not use sufficient decimal places in the mean and variance and resulted in loss of accuracy in the final answer.