Polar Coordinates (Solutions)

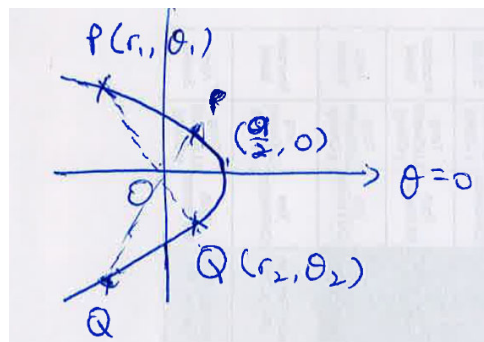
1. [04/AJC/FM/I/6]

A curve C has equation, in polar coordinates, $r = \dfrac{a}{1+\cos\theta}$, $0 \le \theta \le 2\pi$, where a is a positive constant.

Let the pole be denoted by O, the points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ on the curve C are such that POQ is a straight line and $0 < \theta_1 < \pi$.

If the length of the chord PQ is $\dfrac{8a}{3}$, find the coordinates of point P. **[5]**

[Solution]



$r = \dfrac{a}{1+\cos\theta}, 0 \le \theta < 2\pi$

$r_1 = \dfrac{a}{1+\cos\theta_1}, r_2 = \dfrac{a}{1+\cos\theta_2}$

$r_1 + r_2 = \dfrac{8a}{3}$

$\dfrac{a}{1+\cos\theta_1} + \dfrac{a}{1+\cos\theta_2} = \dfrac{8a}{3}$

Since POQ is a straight line, $0 < \theta_1 < \pi$, then $\theta_2 = \pi + \theta_1$

$\dfrac{a}{1+\cos\theta_1} + \dfrac{a}{1+\cos(\pi+\theta_1)} = \dfrac{8a}{3}$

$\dfrac{1}{1+\cos\theta_1} + \dfrac{1}{1-\cos\theta_1} = \dfrac{8}{3}$

$\dfrac{1}{1-\cos^2\theta_1} = \dfrac{4}{3}$

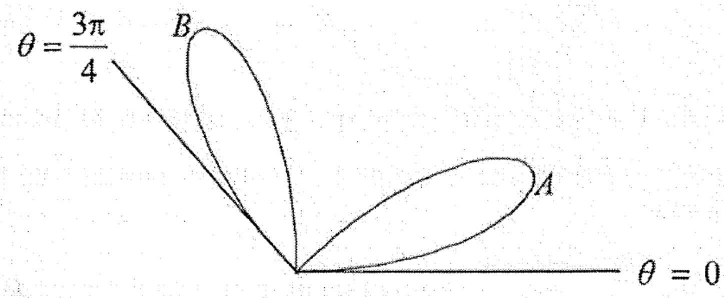
$\sin\theta_1 = \dfrac{\sqrt{3}}{2}$ (Since $0 < \theta_1 < \pi, \sin\theta_1 > 0$)

$\theta = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$

Coordinates of $P = \left(\tfrac{2}{3}a, \tfrac{1}{3}\pi\right)$ or $\left(2a, \tfrac{2}{3}\pi\right)$.

2. [04/NJC/FM/I/7]

The diagram shows part of a polar curve C with equation $r = a\sin b\theta$, where $b \in \mathbb{Z}^+$.



State, with a reason, the least possible value of b. **[2]**

(i) Prove that $r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = a^2\left(1+\left(b^2-1\right)\cos^2 b\theta\right)$. Using the value of b found above, show that the arc length of one loop of C is at most πa. **[3]**

(ii) A and B are two points on C furthest from the pole. Show that the cartesian coordinates of A is $\left(a\cos\frac{\pi}{8}, a\sin\frac{\pi}{8}\right)$. Hence, find the cartesian equation of the line AB in the form $y = mx + c$. **[3]**

[least $b = 4$; $y = -\left(\cot\frac{3\pi}{8}\right)x + a\cot\frac{3\pi}{8}\cos\frac{\pi}{8} + a\sin\frac{\pi}{8}$]

[Solution]

Since $b \in \mathbb{Z}^+$, and $r = 0$ when $\theta = \frac{3\pi}{4}$, least possible $b = 4$

(i) $$r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = a^2 \sin^2 b\theta + a^2 b^2 \cos^2 b\theta = a^2\left(1 - \cos^2 b\theta\right) + a^2 b^2 \cos^2 b\theta$$
$$= a^2\left[1 + \left(b^2 - 1\right)\cos^2 b\theta\right]$$

Arc length of one loop $= \int_0^{\frac{\pi}{4}} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = a\int_0^{\frac{\pi}{4}} \sqrt{1 + 15\cos^2 4\theta}\ \mathrm{d}\theta$

$$\le a\int_0^{\frac{\pi}{4}} \sqrt{1+15}\ \ \mathrm{d}\theta = 4a\int_0^{\frac{\pi}{4}} \mathrm{d}\theta = \pi a$$

(ii) At A, $\theta = \frac{\pi}{8}, r = a$. Cartesian coordinates of A $= \left(a\cos\frac{\pi}{8}, a\sin\frac{\pi}{8}\right)$

At B, $\theta = \frac{3\pi}{4} - \frac{\pi}{8} = \frac{5\pi}{8}, r = a$. Cartesian coordinates of B $= \left(a\cos\frac{5\pi}{8}, a\sin\frac{5\pi}{8}\right)$

Gradient of line AB $= \dfrac{a\sin\frac{5\pi}{8} - a\sin\frac{\pi}{8}}{a\cos\frac{5\pi}{8} - a\cos\frac{\pi}{8}} = \dfrac{2\cos\frac{3\pi}{8}\sin\frac{2\pi}{8}}{-2\sin\frac{3\pi}{8}\sin\frac{2\pi}{8}} = -\cot\frac{3\pi}{8}$

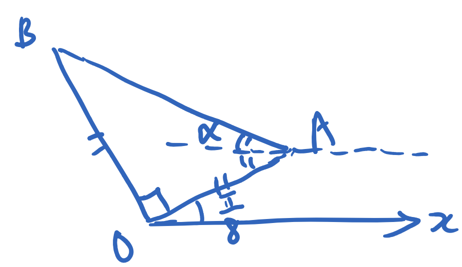
Thus AB : $\quad y - a\sin\frac{\pi}{8} = -\cot\frac{3\pi}{8}\left(x - a\cos\frac{\pi}{8}\right)$

$$y = -x\cot\frac{3\pi}{8} + a\cot\frac{3\pi}{8}\cos\frac{\pi}{8} + a\sin\frac{\pi}{8}$$ [Note answer is not simplified]

Alternative

Note that triangle AOB is right-angled at O,

and is also isosceles. Thus $\alpha = \frac{\pi}{8}$.



AB : $\quad y - a\sin\frac{\pi}{8} = -\tan\frac{\pi}{8}\left(x - a\cos\frac{\pi}{8}\right)$

$$y = -x\tan\frac{\pi}{8} + 2a\sin\frac{\pi}{8}$$

3. [04/TJC/FM/I/5]

The curve C has polar equation $r = \dfrac{1}{\sqrt{\sin\theta}}$, for $0 < \theta < \pi$.

(i) Show that C has the following characteristics:

(a) $r \geq 1$ for the given range of θ;

(b) the curve C is symmetrical about the line $\theta = \dfrac{\pi}{2}$;

(c) the curve C has an asymptote $\theta = 0$.
[Hint: consider the x- and y- coordinates of C as $\theta \to 0$.] **[3]**

(ii) Give a sketch of the curve. **[1]**

(iii) Find the area of thé closed region bounded by C and the line with cartesian equation $y = \dfrac{1}{\sqrt{2}}$. **[5]**

[Solution]

(i) **(a)** For $0 < \theta < \pi$, $0 < \sin\theta \leq 1$. Thus $0 < \sqrt{\sin\theta} \leq 1$ and $r = \dfrac{1}{\sqrt{\sin\theta}} \geq 1$.

(b) Since $\sin(\pi - \theta) = \sin\theta$, $\dfrac{1}{\sqrt{\sin(\pi-\theta)}} = \dfrac{1}{\sqrt{\sin\theta}}$.

C is symmetrical about the line $\theta = \dfrac{\pi}{2}$.

(a) $\theta \to 0$, $x = \dfrac{1}{\sqrt{\sin\theta}}\cos\theta \to \infty$, $y = \dfrac{\sin\theta}{\sqrt{\sin\theta}} = \sqrt{\sin\theta} \to 0$

Thus $y = 0$ is an asymptote to C and thus $\theta = 0$ is an asymptote.

(iii) $y = \dfrac{1}{\sqrt{2}} \Rightarrow \sqrt{\sin\theta} = \dfrac{1}{\sqrt{2}} \Rightarrow \sin\theta = \dfrac{1}{2}$

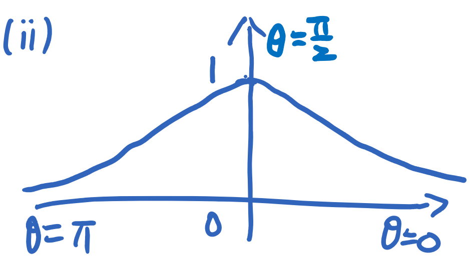
Thus $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$.

$$\text{Area} = 2\left[\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\sin\theta}\,\mathrm{d}\theta - \frac{1}{2}\left(\sqrt{2}\cdot\frac{\sqrt{3}}{2}\right)\frac{1}{\sqrt{2}}\right]$$

$$= \left[-\ln(\operatorname{cosec}\theta + \cot\theta)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \frac{\sqrt{3}}{2}$$

$$= \ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}$$

(ii)



4 The curve C_1 has polar equation

$$r = a + a\cos 2\theta \text{ for } 0 \le \theta < 2\pi \text{, where } a \text{ is a positive constant.}$$

The curve C_2 is obtained by rotating C_1 through an angle of $\frac{\pi}{2}$ anticlockwise about the pole.

(i) Find a polar equation for C_2 in terms of $\cos 2\theta$. [1]

(ii) On the same diagram, sketch the curves C_1 and C_2, indicating the polar coordinates of the points of intersections of the curves. [2]

(iii) Find the exact area of the regions that lie within both C_1 and C_2. [4]

(iv) A design for a necklace is made by taking the combined graphs of C_1 and C_2, but with the perimeter of the regions described in part **(iii)** removed. Find the perimeter of the necklace. [2]

[CJC/FM/2017/Promo/Q4]

[Solution]

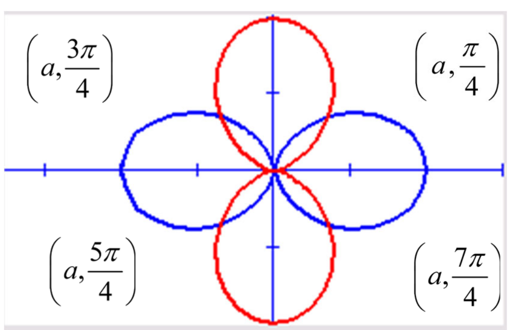
4	[CJC\Promo\P\Q4]

(i)

A polar equation for C_2 is

$$r = a + a\cos 2\left(\theta + \frac{\pi}{2}\right) = a + a\cos(2\theta + \pi) = a - a\cos(2\theta)$$

(ii)



(iii)

Area $= 8 \cdot \frac{1}{2}\int_0^{\frac{\pi}{4}} a^2 (1 - \cos 2\theta)^2 \, d\theta$ by symmetry

$$= 4a^2 \int_0^{\frac{\pi}{4}} \left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= 4a^2 \int_0^{\frac{\pi}{4}} \left(\frac{3}{2} - 2\cos 2\theta + \frac{\cos 4\theta}{2}\right) d\theta$$

$$= 4a^2 \left[\frac{3}{2}\theta - \sin 2\theta + \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{3}{2}\pi - 4\right) a^2 \text{ units}^2$$

	(iv) $r = a(1+\cos 2\theta) \Rightarrow \dfrac{\mathrm{d}r}{\mathrm{d}\theta} = -2a\sin 2\theta$ $\text{Perimeter} = 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{r^2 + \left(\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$ $= 4a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{(1+\cos 2\theta)^2 + (-2\sin 2\theta)^2}\,\mathrm{d}\theta$ $\approx 13.710a$ $\approx 13.7a$ units

5 Show that the gradient of tangent of the polar curve with equation $r=\dfrac{1}{1+\cos\theta}$ is $-\cot\dfrac{\theta}{2}$. [5]

[EJC/FM/2017/Promo/Q8b]

[Solution]

5	$r=\dfrac{1}{1+\cos\theta}\Rightarrow\dfrac{dr}{d\theta}=\dfrac{\sin\theta}{(1+\cos\theta)^2}$ Note that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta+r\cos\theta}{\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta-r\sin\theta}=\dfrac{\left(\dfrac{\sin\theta}{(1+\cos\theta)^2}\right)\sin\theta+\left(\dfrac{1}{1+\cos\theta}\right)\cos\theta}{\left(\dfrac{\sin\theta}{(1+\cos\theta)^2}\right)\cos\theta-\left(\dfrac{1}{1+\cos\theta}\right)\sin\theta}$ $=\dfrac{\sin^2\theta+\cos\theta(1+\cos\theta)}{\sin\theta\cos\theta-\sin\theta(1+\cos\theta)}=\dfrac{1+\cos\theta}{-\sin\theta}=\dfrac{2\cos^2\dfrac{\theta}{2}}{-2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}=-\cot\dfrac{\theta}{2}$ Alternatively: $x=r\cos\theta=\dfrac{\cos\theta}{1+\cos\theta}\Rightarrow\dfrac{dx}{d\theta}=\dfrac{-\sin\theta(1+\cos\theta)-(-\sin\theta)\cos\theta}{(1+\cos\theta)^2}=\dfrac{-\sin\theta}{(1+\cos\theta)^2}$ $y=r\sin\theta=\dfrac{\sin\theta}{1+\cos\theta}\Rightarrow\dfrac{dy}{d\theta}=\dfrac{\cos\theta(1+\cos\theta)-(-\sin\theta)\sin\theta}{(1+\cos\theta)^2}=\dfrac{1+\cos\theta}{(1+\cos\theta)^2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\dfrac{\mathrm{d}y}{\mathrm{d}\theta}}{\dfrac{\mathrm{d}x}{\mathrm{d}\theta}}=\dfrac{1+\cos\theta}{-\sin\theta}=\dfrac{2\cos^2\dfrac{\theta}{2}}{-2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}=-\cot\dfrac{\theta}{2}$

6 The curve C has equation $2x^2 - xy + 2y^2 = 150$.

(a) Show that a polar equation for C can be expressed in the form

$$r^2 = \frac{P}{Q + R\sin 2\theta},$$

where P, Q and R are integers to be found and $-\pi < \theta \le \pi$. [3]

(b) Hence, find the polar coordinates of the points on C which are the furthest from the pole O. [3]

[JJC/FM/2017/Promo/Q2]

[Solution]

	(a) $2x^2 - xy + 2y^2 = 150$ $2\left(x^2 + y^2\right) - xy = 150$ $2r^2 - (r\cos\theta)(r\sin\theta) = 150$ $r^2\left(2 - \sin\theta\cos\theta\right) = 150$ $r^2 = \dfrac{150}{2 - \dfrac{\sin 2\theta}{2}}$ $= \dfrac{300}{4 - \sin 2\theta}$ (Shown) $\therefore P = 300, Q = 4, R = -1$ **(b)** $r^2 = \dfrac{300}{4 - \sin 2\theta}$ For r to be maximum, $\sin 2\theta = 1$ $2\theta = \dfrac{\pi}{2}, -\dfrac{3\pi}{2}$ $\theta = \dfrac{\pi}{4}, -\dfrac{3\pi}{4}$ Max. $r = \sqrt{\dfrac{300}{4-1}} = 10$ Polar coordinates are $\left(10, \dfrac{\pi}{4}\right)$ and $\left(10, -\dfrac{3\pi}{4}\right)$.

7 The straight line with polar equation

$$r_1 = \frac{1}{a\sin\theta + b\cos\theta},$$

is a tangent to the circle with the polar equation,

$$r_2 = 2c\cos\theta,$$

where a, b and c are real numbers, $a^2 + b^2 \neq 0$ and $c \neq 0$.

By first finding the Cartesian equations of the respective polar equations, find the possible value(s) of c in terms of a and/or b. [8]

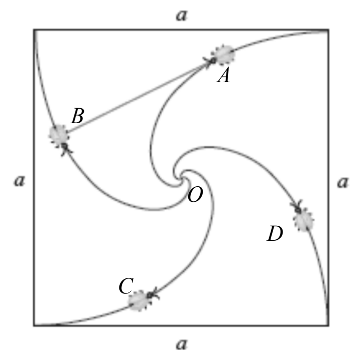
[MJC/FM/2017/Promo/Q2]

[Solution]

	(i) $r_1 = \frac{1}{a\sin\theta + b\cos\theta}$ $\Rightarrow r_1 a\sin\theta + r_1 b\cos\theta = 1$ $\Rightarrow ay + bx = 1$ $\Rightarrow y = \frac{1-bx}{a}$, when $a \neq 0$ $r_2 = 2c\cos\theta$ $\Rightarrow r_2^2 = 2cr_2\cos\theta$ $\Rightarrow x^2 + y^2 = 2cx$ $\Rightarrow (x-c)^2 + y^2 = c^2$ <u>Case 1 (when *a* is non-zero)</u> Substituting $y = \frac{1-bx}{a}$ into $(x-c)^2 + y^2 = c^2$ gives: $(x-c)^2 + \left(\frac{1-bx}{a}\right)^2 = c^2$ $\Rightarrow a^2\left(x^2 - 2cx + c^2\right) + 1 - 2bx + b^2x^2 - a^2c^2 = 0$ $\Rightarrow \left(a^2 + b^2\right)x^2 - 2\left(a^2c + b\right)x + 1 = 0$ - (*) Since the line is a tangent to the circle, there is only one solution. Thus the discriminant to (*) is zero. $D = \left[-2\left(a^2c + b\right)\right]^2 - 4\left(a^2 + b^2\right)(1) = 0$ $\Rightarrow \left(a^2c + b\right)^2 - \left(a^2 + b^2\right) = 0$ $\Rightarrow a^4c^2 + 2a^2bc + b^2 - a^2 - b^2 = 0$ $\Rightarrow a^4c^2 + 2a^2bc - a^2 = 0$

	Since a is non-zero, $a^2c^2+2bc-1=0$ $c=\frac{-2b\pm\sqrt{(2b)^2+4a^2}}{2a^2}=\frac{-b\pm\sqrt{b^2+a^2}}{a^2}$ <u>Case 2 (when a is zero)</u> The Cartesian Equation of the line reduces to $0y+bx=1\Rightarrow x=\frac{1}{b}$ This implies that the line is tangential to the circle at its non-zero x-intercept $(2c,0)$, since equation is $(x-c)^2+y^2=c^2$. Thus, $\frac{1}{b}=2c\Rightarrow c=\frac{1}{2b}$

8



In the above diagram, four bugs A, B, C and D are placed at the four corners of a square with side of length a. The bugs crawl counter clockwise towards the centre of the square, O, along the spiral paths. Bug A starts from the corner $\left(\frac{a}{2}, \frac{a}{2}\right)$. The line joining the bug A to the bug B is tangent to the path of the bug A.

(i) Taking O as the origin and the coordinates of the bug A to be (x, y), explain why the Cartesian coordinates of the bug B are $(-y, x)$. **[1]**

It is given that $OA = r$ and OA makes an angle θ with the x-axis.

(ii) Show that the gradient of the line AB is $\dfrac{\tan\theta - 1}{\tan\theta + 1}$. **[2]**

(iii) By expressing $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$ in terms of r, θ and $\dfrac{dr}{d\theta}$, show that $\dfrac{dr}{d\theta} = -r$. **[4]**

(iv) It is known that the polar equation of the path of the bug A is in the form $r = ke^{-\theta}$, show that $k = \dfrac{\sqrt{2}a}{2}e^{\frac{\pi}{4}}$, assuming the pole is at the centre of the square. **[2]**

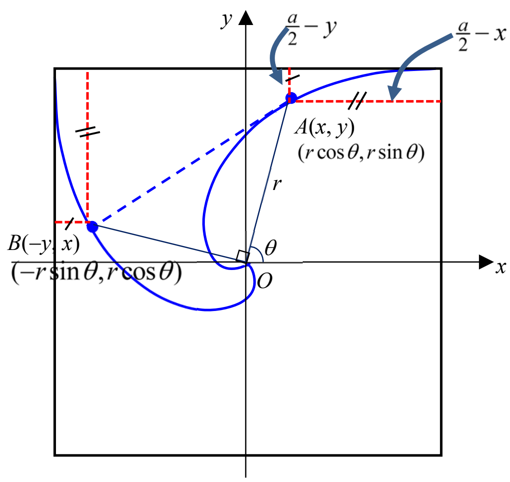
(v) Find the exact distance travelled by the bug A for $\dfrac{\pi}{4} \le \theta \le 2\pi$. **[4]**

[SAJC/FM/2017/Promo/Q10]

[Solution]

8

(i) When the bug A is at the point (x, y), it is at a horizontal distance $\frac{a}{2} - x$ and vertical distance $\frac{a}{2} - y$ from the top right corner. Since the bugs are moving at the same speed, the bug B should be at the corresponding distance from the top left corner. Hence the Cartesian coordinates of B are $(-y, x)$.



(ii) Let $OA = r$.

If OA makes an angle θ with the x-axis then
$x = r\cos\theta, \quad y = r\sin\theta$

The coordinates of $B(-r\sin\theta, r\cos\theta)$
Gradient of AB:

$$m = \frac{r\sin\theta - r\cos\theta}{r\sin\theta + r\cos\theta} = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$$

Dividing throughout by cos θ, $\quad m = \dfrac{\tan\theta - 1}{\tan\theta + 1}$ (Shown)

(iii) $x = r\cos\theta, \quad y = r\sin\theta$

Differentiating wrt θ,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta \quad \text{by product rule}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} = \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r}{\frac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta}$$

Since $\quad m=\dfrac{\mathrm{d}y}{\mathrm{d}x}$

$$\frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta+r}{\frac{\mathrm{d}r}{\mathrm{d}\theta}-r\tan\theta}=\frac{\tan\theta-1}{\tan\theta+1}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan^2\theta+\cancel{\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta}+\cancel{r\tan\theta}+r=\cancel{\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta}-\frac{\mathrm{d}r}{\mathrm{d}\theta}-r\tan^2\theta+\cancel{r\tan\theta}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}\left(\tan^2\theta+1\right)=-r\left(\tan^2\theta+1\right)$$

$$\therefore \qquad \frac{\mathrm{d}r}{\mathrm{d}\theta}=-r$$

(iv) Given that the polar equation of the path is $r=ke^{-\theta}$.

Bug A starts from the corner $\left(\frac{a}{2},\frac{a}{2}\right)$, so when $\theta=\dfrac{\pi}{4}$, $\; r=a\sin\dfrac{\pi}{4}=\dfrac{\sqrt{2}a}{2}$

Therefore $\dfrac{\sqrt{2}a}{2}=ke^{-\frac{\pi}{4}} \quad\Rightarrow\quad k=\dfrac{\sqrt{2}a}{2}e^{\frac{\pi}{4}}$

So the equation of the polar curve is $r=\dfrac{\sqrt{2}a}{2}e^{\frac{\pi}{4}-\theta}$.

(v) Distance travelled by the A for $\dfrac{\pi}{4}\le\theta\le 2\pi$ is

$$L=\int_{\frac{\pi}{4}}^{2\pi}\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta=\int_{\frac{\pi}{4}}^{2\pi}\sqrt{r^2+(-r)^2}\,\mathrm{d}\theta$$

$$=\int_{\frac{\pi}{4}}^{2\pi}\sqrt{2}\,r\,\mathrm{d}\theta$$

$$=\int_{\frac{\pi}{4}}^{2\pi}\sqrt{2}\left(\frac{\sqrt{2}a}{2}\mathrm{e}^{\frac{\pi}{4}-\theta}\right)\mathrm{d}\theta$$

$$=\int_{\frac{\pi}{4}}^{2\pi}a\mathrm{e}^{\frac{\pi}{4}-\theta}\,\mathrm{d}\theta$$

$$=a\left[-\mathrm{e}^{\frac{\pi}{4}-\theta}\right]_{\frac{\pi}{4}}^{2\pi}$$

$$=a\left(1-\mathrm{e}^{-\frac{7\pi}{4}}\right)$$

9 A curve C has polar equation given by

$$r^2 = a\cos^2\theta + b\sin^2\theta, \text{ where } a \text{ and } b \text{ are non-zero constants.}$$

(a) State the condition(s) on a and b so that C has tangent(s) through the pole. [1]

(b) State the condition(s) on a and b so that C will never come back to the pole. [1]

(c) Sketch C, where $0 \le \theta \le 2\pi$, for the following cases:

(i) $\sqrt{a} > \sqrt{b}$, [2]

(ii) $a > 0$ and $b < 0$. [2]

(d) Given that $a = 4$ and $b = 2$, find the exact area of the region enclosed by C. [3]

(e) Write down the polar equation of the curve if C is

(i) rotated 90° anti-clockwise about the pole, [1]

(ii) reflected about the line $y = x$. [1]

[TJC/FM/2017/Promo/Q8]

[Solution]

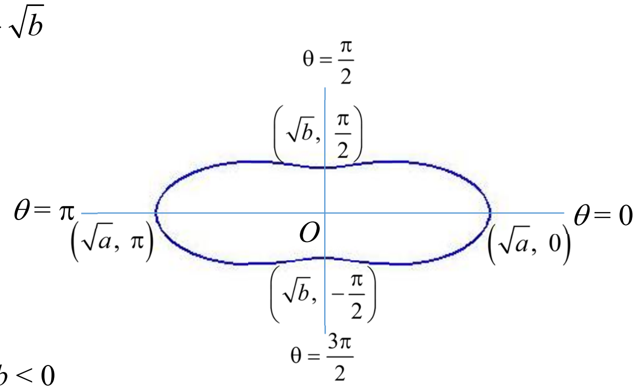
9 [TJC\Promo\P\Q8]

(a) For $r = 0$, $a\cos^2\theta + b\sin^2\theta = 0 \Rightarrow \tan^2\theta = -\dfrac{a}{b}$

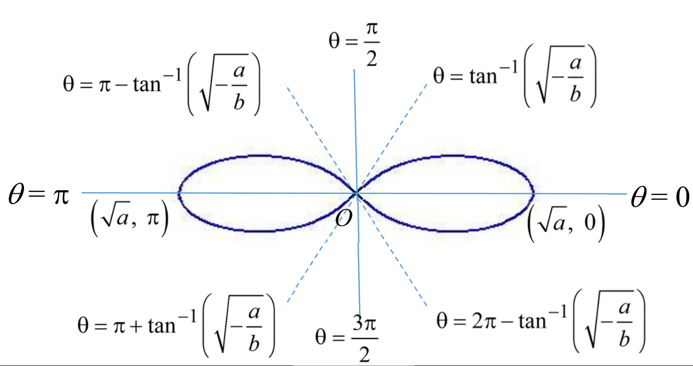
For real solutions, $\dfrac{a}{b} < 0$ which means a and b must be of opposite signs.

(b) For $r > 0$ for all θ, $a\cos^2\theta + b\sin^2\theta > 0 \Rightarrow a > 0$ and $b > 0$

(c) **(i)** $\sqrt{a} > \sqrt{b}$



(ii) $a > 0$ and $b < 0$



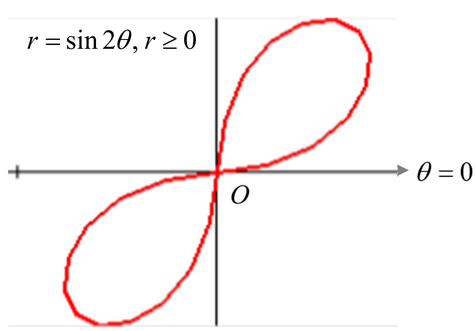
	(d) Area $= 4\times\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}} (4\cos^2\theta + 2\sin^2\theta)\,\mathrm{d}\theta$ $= 2\int_0^{\frac{\pi}{2}} 4\left(\frac{1+\cos 2\theta}{2}\right) + 2\left(\frac{1-\cos 2\theta}{2}\right)\,\mathrm{d}\theta$ $= 2\int_0^{\frac{\pi}{2}} (3+\cos 2\theta)\,\mathrm{d}\theta$ $= 2\left[3\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 3\pi$ units^2 **(e)** **(i)** Replace θ with $\left(\theta - \frac{\pi}{2}\right)$, the new polar curve is $r^2 = a\cos^2\left(\theta - \frac{\pi}{2}\right) + b\sin^2\left(\theta - \frac{\pi}{2}\right) = a\sin^2\theta + b\cos^2\theta$ **(ii)** Replace θ with $\left(\frac{\pi}{2} - \theta\right)$, the new polar curve is $r^2 = a\sin^2\theta + b\cos^2\theta$

10 Sketch the curve $r = \sin 2\theta$, for $r \geq 0$, $0 \leq \theta < 2\pi$. [2]

Describe the curves $r = \sin 2n\theta$, where $r \geq 0$, n are positive integers and show that the area enclosed by such a curve is independent of n. [6]

[VJC/FM/2018/P1/4]

[Solution]



Curves $r = \sin 2n\theta$ has $2n$ identical loops.

Area enclosed $= 2n \times 2 \times \frac{1}{2}\int_0^{\frac{\pi}{4n}} \sin^2 2n\theta \,\mathrm{d}\theta$

$= n\int_0^{\frac{\pi}{4n}} (1-\cos 4n\theta)\,\mathrm{d}\theta$

$= n\left[\theta - \frac{\sin 4n\theta}{4n}\right]_0^{\frac{\pi}{4n}}$

$= \frac{\pi}{4}$ sq. units

Hence, the area enclosed by such a curve is independent of n. (shown)

11 The curve D has polar equation

$$r = 6\sin\frac{1}{2}\theta \text{, where } 0 \le \theta < 2\pi .$$

(i) Sketch D, indicating clearly all key features and symmetries of the curve. [2]

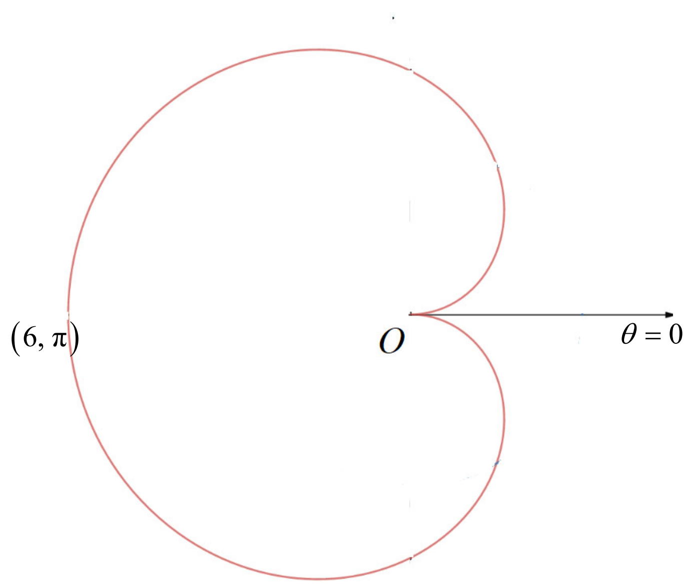
(ii) Find the arc length of D. [2]

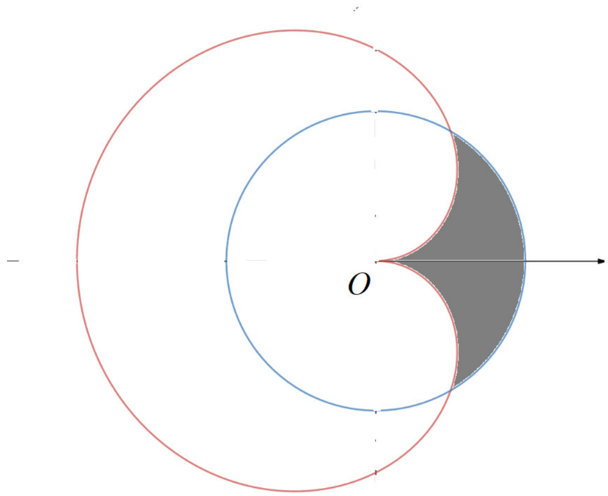
The locus of points (r, θ) satisfying $6\sin\frac{1}{2}\theta \le r \le 3$ forms a region R.

(iii) Find the exact area of R. [5]

[NJC/FM/2018/P2/4]

[Solution]

(i)	
(ii)	Arc length of C $= \displaystyle\int_0^{2\pi} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$ $= \displaystyle\int_0^{2\pi} \sqrt{\left(6\sin\frac{1}{2}\theta\right)^2 + \left(3\cos\frac{1}{2}\theta\right)^2}\ \mathrm{d}\theta$ ≈ 29.065 ≈ 29.1

(iii)	 R is one of the regions bounded by C and the circle with polar equation $r=3$, as shown in the diagram. To find the point of intersection $6\sin\frac{1}{2}\theta=3$ $\sin\frac{1}{2}\theta=\frac{1}{2}$ $\frac{1}{2}\theta=\frac{\pi}{6}$ or $\frac{1}{2}\theta=\frac{5\pi}{6}$ $\theta=\frac{\pi}{3}$ or $\theta=\frac{5\pi}{3}$ Area of R $=2\left[\text{area of sector with angle }\frac{\pi}{3}-\int_0^{\frac{\pi}{3}}\frac{1}{2}\left(6\sin\frac{\theta}{2}\right)^2\,\mathrm{d}\theta\right]$ $=2\left[\frac{1}{2}\left(\frac{\pi}{3}\right)3^2-\int_0^{\frac{\pi}{3}}18\sin^2\frac{\theta}{2}\,\mathrm{d}\theta\right]$ $=2\left[\frac{3\pi}{2}-9\int_0^{\frac{\pi}{3}}(1-\cos\theta)\,\mathrm{d}\theta\right]$ $=3\pi-18\left[\theta-\sin\theta\right]_0^{\frac{\pi}{3}}$ $=3\pi-18\left(\frac{\pi}{3}-\sin\frac{\pi}{3}\right)$ $=9\sqrt{3}-3\pi$

12 The curve Γ has polar equation

$$r = 1+\cos\theta, \quad 0 \le \theta \le 2\pi.$$

The circle with equation $(x-2)^2 + y^2 = 4$ intersects Γ at O, A and B, where O is the pole.

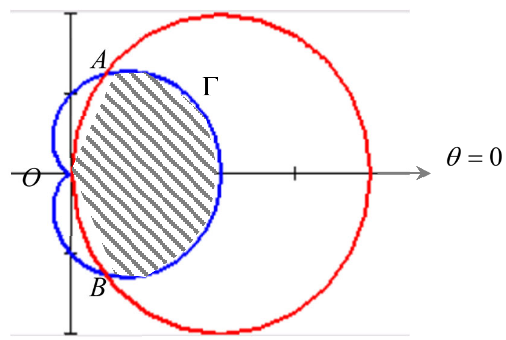
Determine the perimeter of the sector OAB, where OA and OB are straight line segments and AB is an arc on Γ that lies within the circle.

Leave your answer in an exact surd form. [8]

[VJC/FM/2019/MCT/6]

[Solution]

To find A, B: (convert both equations to the same kind, either polar or Cartesian)



$r = 1+\cos\theta, \quad 0 \le \theta \le 2\pi.$

$(x-2)^2 + y^2 = 4$

$\Rightarrow x^2 - 4x + y^2 = 0$

$\Rightarrow r^2 - 4r\cos\theta = 0$

$\Rightarrow r = 4\cos\theta$

At A and B, $4\cos\theta = 1+\cos\theta$. $\quad \therefore \cos\theta = \frac{1}{3}.$

$\therefore OA = OB = 1 + \frac{1}{3} = \frac{4}{3}$

Arc length of $AB = 2\int_0^{\alpha} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$, where $\cos\alpha = \frac{1}{3}$.

$$= 2\int_0^{\alpha} \sqrt{(1+2\cos\theta+\cos^2\theta)+(-\sin\theta)^2}\,\mathrm{d}\theta$$

$$= 2\sqrt{2}\int_0^{\alpha} \sqrt{1+\cos\theta}\ \mathrm{d}\theta$$

$$= 2\sqrt{2}\int_0^{\alpha} \sqrt{2}\cos\frac{\theta}{2}\ \mathrm{d}\theta$$

$$= 4\left[2\sin\frac{\theta}{2}\right]_0^{\alpha}$$

$$= 8\sin\frac{\alpha}{2}$$

$$= 8\sqrt{\frac{1-\cos\alpha}{2}}$$

$$= \frac{8}{\sqrt{3}}$$

$\therefore$ the required perimeter is $\frac{8}{3} + \frac{8}{\sqrt{3}}$ $\left(\text{equivalently } \frac{8}{3}\left(1+\sqrt{3}\right)\right)$

13 The curve T_1 has polar equation $r = 3 + 2\cos 3\theta$, $0 \le \theta < 2\pi$.

(i) Sketch T_1, indicating all the key features and equations of lines of symmetry of the curve. [3]

(ii) A piece of wire of length 32 units long is used to bend into the shape of T_1. State, with a reason, whether the wire is long enough to do so. [2]

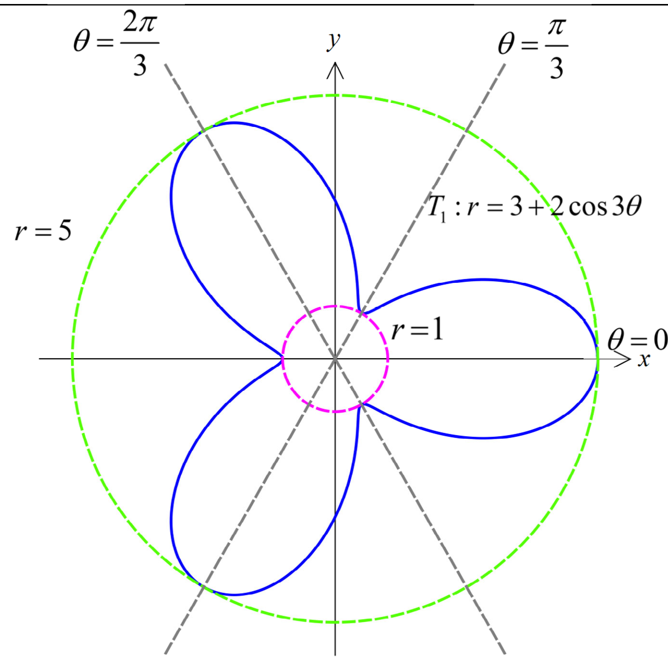
Another curve T_2 has polar equation $r = 2 + \cos 3\theta$, $0 \le \theta < 2\pi$.

(iii) Use calculus to evaluate the exact area enclosed in between T_1 and T_2. [3]

[RI/FM/2019/MCT/3]

[Solution]

13 (i)



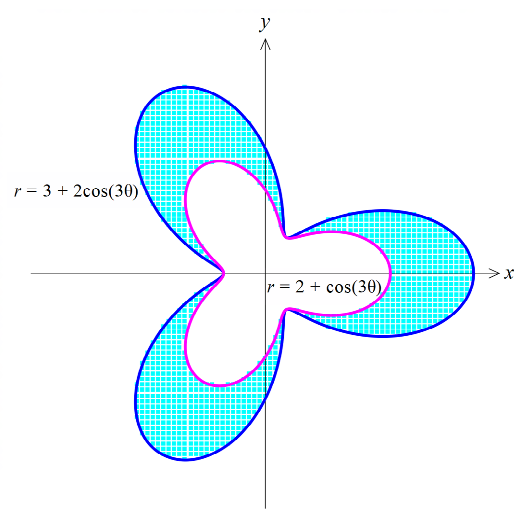
(ii)

$$\text{Length of } T_1 = 6\int_0^{\frac{\pi}{3}}\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta \quad \left(\text{or } \int_0^{2\pi}\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta\right)$$

$$\approx 32.19578989 = 32.2 \text{ (3 sf) } > 32.$$

So the wire is not long enough to be bent into the shape of T_1.

(iii)



Required area

$$= 6\times\frac{1}{2}\int_0^{\frac{\pi}{3}}(3+2\cos 3\theta)^2-(2+\cos 3\theta)^2\ \mathrm{d}\theta$$

$$= 3\int_0^{\frac{\pi}{3}}5+8\cos 3\theta+3\cos^2 3\theta\ \mathrm{d}\theta = 3\int_0^{\frac{\pi}{3}}5+8\cos 3\theta+\frac{3}{2}(\cos 6\theta+1)\ \mathrm{d}\theta$$

$$= 3\int_0^{\frac{\pi}{3}}\frac{13}{2}+8\cos 3\theta+\frac{3}{2}\cos 6\theta\ \mathrm{d}\theta$$

$$= 3\left[\frac{13\theta}{2}+\frac{8}{3}\sin 3\theta+\frac{3}{12}\sin 6\theta\right]_0^{\frac{\pi}{3}} = 3\left(\frac{13\pi}{6}\right)=\frac{13\pi}{2}\ \text{units}^2.$$

14 [It is given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.]

A polar curve C has equation $r = \sin 2\theta$ where $0 \le \theta \le \frac{\pi}{2}$. O is the pole and the tangent at a point $P(r,\theta)$ on the curve is parallel to the line $\theta = \frac{\pi}{2}$.

(i) Sketch C. [2]

(ii) Show that $\cos\theta = \sqrt{\frac{2}{3}}$ at P. [3]

(iii) Find the area bounded by C, the line $\theta = 0$ and the tangent at P. [5]

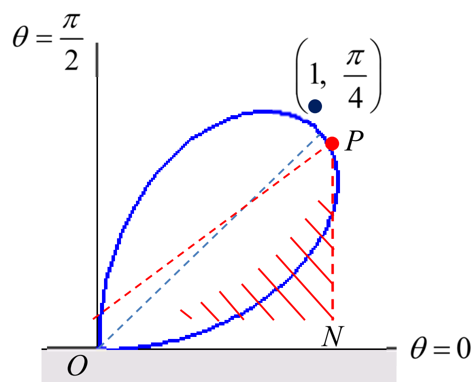
[TJC/FM/2018/P1/6]

[Solution]

(i) Let $r = 0$ where $0 \le \theta \le \frac{\pi}{2}$

$\sin 2\theta = 0 \Rightarrow \theta = 0, \frac{\pi}{2}$ (tangents)

r is maximum when $\sin 2\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$



(ii) $x = r\cos\theta = \sin 2\theta\cos\theta = \frac{1}{2}(\sin 3\theta + \sin\theta)$

$\frac{dx}{d\theta} = \frac{1}{2}(3\cos 3\theta + \cos\theta) = \frac{1}{2}\left(3\left(4\cos^3\theta - 3\cos\theta\right) + \cos\theta\right) = 2\cos\theta\left(3\cos^2\theta - 2\right)$

At P, $\frac{dy}{dx}$ is undefined $\Rightarrow \frac{dx}{d\theta} = 0 \Rightarrow \cos\theta = 0$ or $\cos\theta = \pm\sqrt{\frac{2}{3}}$

Since $0 < \theta < \frac{\pi}{2}$ at P, $\cos\theta = \sqrt{\frac{2}{3}}$

(iii) Area of triangle OPN $= \frac{1}{2}(r\sin\theta)(r\cos\theta)$

$$= \frac{1}{2}r^2\sin\theta\cos\theta = \frac{1}{2}(2\sin\theta\cos\theta)^2\sin\theta\cos\theta$$

$$= 2\left(\frac{1}{\sqrt{3}}\right)^3\left(\sqrt{\frac{2}{3}}\right)^3 = \frac{4\sqrt{2}}{27}$$

Required area $= \frac{4\sqrt{2}}{27} - \int_0^{\cos^{-1}\sqrt{\frac{2}{3}}} \frac{1}{2}r^2\,d\theta$

$$= \frac{4\sqrt{2}}{27} - \frac{1}{2}\int_0^{\cos^{-1}\sqrt{\frac{2}{3}}} \sin^2 2\theta\,d\theta = 0.0949 \quad \text{using GC}$$

15 **(a)** The Archimedean spiral, S, which was first studied by the Greek mathematician Archimedes in the 3rd century BC, has polar equation given by

$$r = a + b\theta$$

where a and b are non-negative real constants and $\theta \geq 0$.

The Archimedean spiral has the property that any ray from the pole intersects successive turnings of the spiral at points with a constant separation distance d, hence also the name "arithmetic spiral".

(i) Prove the above property and state the value of d. [2]

(ii) For the case when $a = b$, prove that the angle which the tangent to S at the point (r, θ) makes with the initial line is given by

$$\tan^{-1}(1+\theta) + \theta$$

and hence write down the cartesian equation of the tangent to S at the point where $\theta = 0$. [5]

(b) Let A and B be two points on a polar curve corresponding to $\theta = \alpha$ and $\theta = \beta$ respectively. The area of the curved surface generated when the arc AB is rotated completely about the initial line is given by the integral

$$2\pi \int_{\alpha}^{\beta} r \sin\theta \left\{ r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta} \right)^2 \right\}^{\frac{1}{2}} \mathrm{d}\theta .$$

Use the above integral to derive the formula for the surface area of a sphere of radius a, explaining your working clearly. [3]

[NYJC/FM/2017/P2/4]

[Solution]

15	(a)(i) $d = a + b(\theta + 2\pi) - [a + b\theta] = 2b\pi$ which is a constant. (ii) Parametric equations of S are $x = r\cos\theta = a(1+\theta)\cos\theta$ and $y = r\sin\theta = a(1+\theta)\sin\theta$. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}\theta}\Big/\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \dfrac{\sin\theta + (1+\theta)\cos\theta}{\cos\theta - (1+\theta)\sin\theta}$ $= \dfrac{\tan\theta + (1+\theta)}{1-(1+\theta)\tan\theta}$ $= \tan(\theta+\phi)$ where $\phi \doteq \tan^{-1}(1+\theta)$ So angle which the tangent to S at the point (r,θ) makes with the initial line is $\theta + \phi = \theta + \tan^{-1}(1+\theta)$. Gradient of the tangent to S at the point where $\theta = 0$ is $\tan(\tan^{-1}1) = 1$. Also, when $\theta = 0, r = a$. Thus the cartesian equation of the tangent is $y = x - a$. (b) Consider the circle $r = a$. Surface area of sphere $= 2\pi\int_0^\pi r\sin\theta\left\{r^2 + \left(\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right\}^{\frac{1}{2}}\mathrm{d}\theta$ $= 2\pi\int_0^\pi a\sin\theta\{a^2\}^{\frac{1}{2}}\mathrm{d}\theta$ $\left(\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = 0\right)$ $= -2\pi a^2[\cos\theta]_0^\pi$ $= 4\pi a^2$

16 The curve T has polar equation $r = \sqrt{2} - \sin\theta,\ 0 \le \theta < 2\pi.$

(i) Sketch T. Show on your diagram, the exact polar coordinates of the point of intersection of T with the initial line. [2]

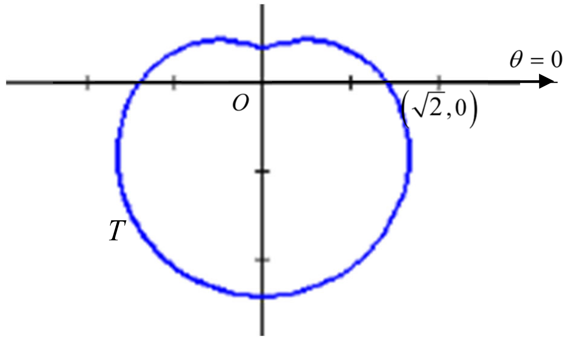
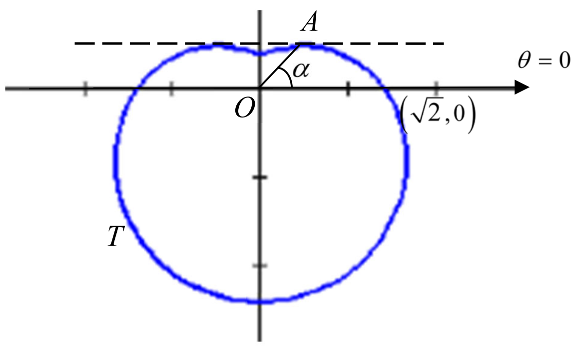
(ii) The tangent to T at the point A where $\theta = \alpha,\ 0 < \alpha < \dfrac{\pi}{2}$ is parallel to the initial line.

Show that $\alpha = \dfrac{\pi}{4}$ (do not merely verify) and find the exact distance from A to the pole. [4]

(iii) Find the exact area of the region bounded by T, the tangent in part **(ii)** and the half-line $\theta = \dfrac{\pi}{6}$. [5]

[HCI et al/FM/2018/P1/6]

[Solution]

<table>
<tr><td>16(i)</td><td>
</td></tr>
<tr><td>(ii)</td><td>

$y = r\sin\theta = \left(\sqrt{2} - \sin\theta\right)\sin\theta = \sqrt{2}\sin\theta - \sin^2\theta$

$\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = \sqrt{2}\cos\theta - 2\sin\theta\cos\theta$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = 0$

$\therefore \dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow \sqrt{2}\cos\theta - 2\sin\theta\cos\theta = 0$

$\cos\theta\left(\sqrt{2} - 2\sin\theta\right) = 0$

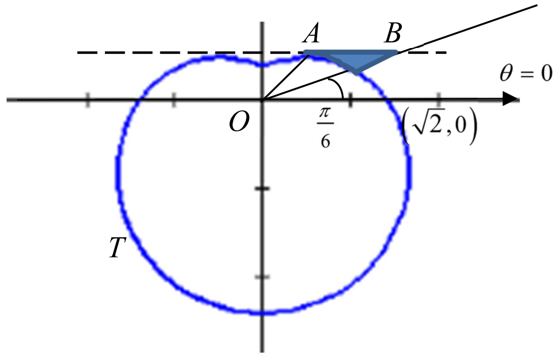
$\cos\theta = 0 \;\text{ or }\; \sin\theta = \dfrac{\sqrt{2}}{2}$

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \;\text{ or }\; \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

At A, $\alpha = \dfrac{\pi}{4}$ since $\theta = \dfrac{\pi}{4}$ is the only value in the first quadrant.

Distance from A to the pole $= \sqrt{2} - \sin\dfrac{\pi}{4} = \sqrt{2} - \dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ units</td></tr>
</table>

(iii)



A is $\left(\frac{1}{\sqrt{2}}\cos\frac{\pi}{4}, \frac{1}{\sqrt{2}}\sin\frac{\pi}{4}\right) \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$

Cartesian equation for half-line $\theta = \frac{\pi}{6}: y = \left(\tan\frac{\pi}{6}\right)x = \frac{1}{\sqrt{3}}x$

The tangent and the half-line $\theta = \frac{\pi}{6}$ intersect at $x = \frac{\sqrt{3}}{2}$

Exact area required

$= \text{Area of triangle } OAB - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} r^2 \, \mathrm{d}\theta$

$= \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\sqrt{2} - \sin\theta\right)^2 \mathrm{d}\theta$

$= \frac{\sqrt{3}-1}{8} - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(2 - 2\sqrt{2}\sin\theta + \sin^2\theta\right) \mathrm{d}\theta$

$= \frac{\sqrt{3}-1}{8} - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(2 - 2\sqrt{2}\sin\theta + \frac{1-\cos 2\theta}{2}\right) \mathrm{d}\theta$

$= \frac{\sqrt{3}-1}{8} - \frac{1}{2}\left[\frac{5}{2}\theta + 2\sqrt{2}\cos\theta - \frac{\sin 2\theta}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{\sqrt{3}-1}{8} - \frac{1}{2}\left(\left(\frac{5\pi}{8} + 2 - \frac{1}{4}\right) - \left(\frac{5\pi}{12} + \sqrt{2}\sqrt{3} - \frac{\sqrt{3}}{8}\right)\right)$

$= \frac{\sqrt{6}}{2} + \frac{\sqrt{3}}{16} - 1 - \frac{5\pi}{48}$ units^2

17 The rotating blades of a fan can be modelled by the curve C with polar equation

$$r = a\left|\cos 2\theta\right|$$

where $-\pi < \theta \le \pi$ and a is a positive constant.

(i) The region R not containing the pole is bounded between the part of C for which $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ and the line with polar equation $r = \frac{\sqrt{3}a}{4\cos\theta}$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$. Find, in terms of a, the exact area of R. [4]

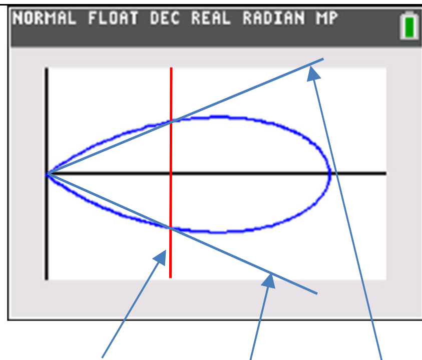
(ii) The fan is rotated through an acute angle α radians about its centre. The rotated fan can be modelled by the curve with polar equation $r = a\left|\sin 2\theta\right|$.

Without the aid of the graphing calculator, determine the exact value of α in terms of π, justifying your answer with relevant working. [2]

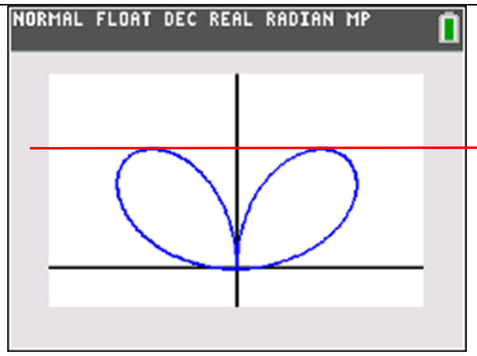
(iii) A thin straight wire, which can be modelled by a line ℓ, rests horizontally on top of two blades of the rotated fan in **(ii)**. Find the cartesian equation of ℓ. [4]

[NYJC/FM/2019/MYE/P1/6]

[Solution]

17(i)	 $r=\frac{\sqrt{3}a}{4\cos\theta}$ $\theta=-\frac{\pi}{6}$ $\theta=\frac{\pi}{6}$ For $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$, $a\left\|\cos 2\theta\right\|=\frac{\sqrt{3}a}{4\cos\theta}\Rightarrow\cos 2\theta=\frac{\sqrt{3}}{4\cos\theta}$ By observation or GC, $\theta=\pm\frac{\pi}{6}$. Let A and B be respectively the points of intersection between $r=\frac{\sqrt{3}a}{4\cos\theta}$ and C ($0\le\theta\le\frac{\pi}{4}$) and between $r=\frac{\sqrt{3}a}{4\cos\theta}$ and the initial line. $OB=\ r=\frac{\sqrt{3}a}{4\cos 0}=\frac{\sqrt{3}}{4}a$. $AB=\ r=\left(\frac{\sqrt{3}a}{4\cos\frac{\pi}{6}}\right)\sin\frac{\pi}{6}=\frac{1}{4}a$. Area of R $=2\left\{\frac{1}{2}\int_0^{\frac{\pi}{6}}a^2\cos^2 2\theta\ \mathrm{d}\theta-\frac{1}{2}\left(\frac{1}{4}a\right)\left(\frac{\sqrt{3}}{4}a\right)\right\}$ $=\frac{1}{2}a^2\int_0^{\frac{\pi}{6}}(1+\cos 4\theta)\ \mathrm{d}\theta-\frac{\sqrt{3}}{16}a^2$ $=\frac{1}{2}a^2\left[\theta+\frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{6}}-\frac{\sqrt{3}}{16}a^2$ $=\frac{\pi}{12}a^2$
(ii)	$r=a\left\|\sin 2\theta\right\|=a\left\|\cos\left(2\theta-\frac{\pi}{2}\right)\right\|=a\left\|\cos 2\left(\theta-\frac{\pi}{4}\right)\right\|$. Thus $\alpha=\frac{\pi}{4}$.

(iii)



$r = a\left|\sin 2\theta\right| \Rightarrow r^2 = a^2 \sin^2 2\theta$

$\qquad = 4a^2 \sin^2\theta \cos^2\theta$

$r^2 = 4a^2 \left(\dfrac{y}{r}\right)^2 \left(\dfrac{x}{r}\right)^2$

$\Rightarrow \left(r^2\right)^3 = 4a^2x^2y^2$

$\Rightarrow \left(x^2 + y^2\right)^3 = 4a^2x^2y^2 \quad (y > 0)$ ------ (1)

Differentiate w.r.t. x gives

$3\left(x^2 + y^2\right)^2 \left(2x + 2y\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) = 4a^2\left(2x^2y\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2xy^2\right)$

Put $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ into the above and simplifying gives

$6x\left(x^2 + y^2\right)^2 = 8a^2xy^2 \Rightarrow \left(x^2 + y^2\right)^2 = \dfrac{4a^2y^2}{3}$ ------ (2) since $x \neq 0$.

$\dfrac{(1)}{(2)} : x^2 + y^2 = 3x^2 \Rightarrow y^2 = 2x^2$ ------ (3)

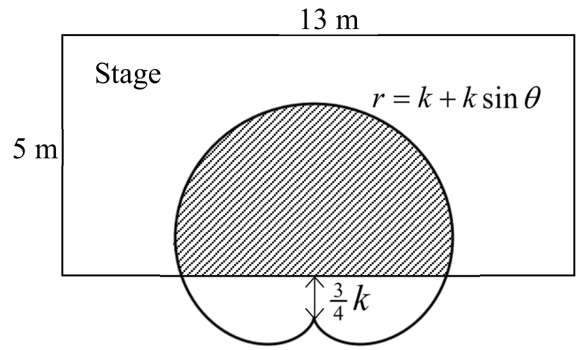
Substitute (3) into (2) gives

$\left(x^2 + 2x^2\right)^2 = \dfrac{4a^2\left(2x^2\right)}{3} \Rightarrow x^2 = \dfrac{8}{27}a^2$.

So $y^2 = 2\left(\dfrac{8}{27}a^2\right) \Rightarrow y = \dfrac{4}{3\sqrt{3}}a$ since $y > 0$,

which is the cartesian eqn of ℓ.

18 Jenny is holding her vocal live performance on a rectangular stage of length 13 m and breadth 5 m. The technical crew uses a microphone with a cardioid pickup pattern so that it minimizes the pickup of noise from the audience. The crew places the microphone at a distance of $\frac{3}{4}k,\ k \geq 0$, from the front of the stage as shown in the figure.



The boundary of the optimal pickup region is given by the cardioid $r = k + k\sin\theta$, where r is measured in meters and the microphone is at the pole. The optimal pickup region on the stage is indicated by the shaded area in the figure. Find

(a) the furthest distance, in terms of k, that Jenny can be on the stage from the microphone so that she is within the optimal pickup region, and [2]

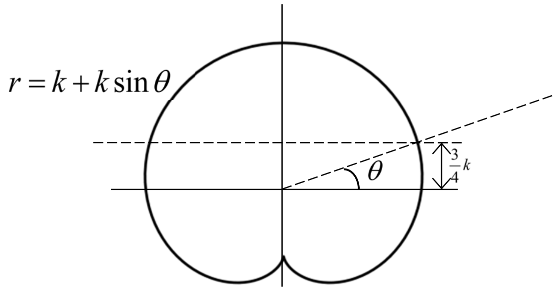
(b) the minimum value of k if the optimal pickup region that Jenny has on stage is at least 75% of the stage area. [6]

[ACJC/FM/2017/P2/3]

[Solution]

3	Solution: (i) $\frac{dr}{d\theta} = 0 \Rightarrow k\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\frac{d^2r}{d\theta^2} = -k\sin\theta$ When $\theta = \frac{\pi}{2}$, $\frac{d^2r}{d\theta^2} = -k < 0$ $\therefore$ maximum r=2k occurs when $\theta = \frac{\pi}{2}$ Furthest distance, $r = 2k$

(ii)



When $y = \frac{3}{4}k,\quad r\sin\theta = \frac{3}{4}k$

$$\Rightarrow (k + k\sin\theta)\sin\theta = \frac{3}{4}k$$

$$\Rightarrow 4\sin^2\theta + 4\sin\theta - 3 = 0$$

$$\Rightarrow \sin\theta = \frac{1}{2} \text{ or } -\frac{3}{2} \text{ (rejected since } 0 < \theta < \frac{\pi}{2})$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

$$\text{Area} = 2\left[\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} k^2(1+\sin\theta)^2\,\mathrm{d}\theta - \frac{1}{2}\left(\frac{3}{4}k\right)\left(\frac{\frac{3}{4}k}{\tan\frac{\pi}{6}}\right)\right]$$

$$= k^2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\sin\theta)^2\,\mathrm{d}\theta - \frac{9\sqrt{3}}{16}k^2$$

$$= 2.54507k^2$$

Given: Area $\geq 0.75\,(13\times 5)$

$$\Rightarrow 2.54507k^2 \geq 0.75\,(13\times 5)$$

$$\Rightarrow k \geq 4.38$$