NANYANG JUNIOR COLLEGE

Chapter 4: Vectors(I)

H2 Mathematics

NAME: ______________________

CLASS: ______________________

Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

Contents

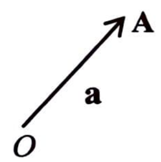
Chapter 4 Vectors

At the end of this chapter, students should be able to

- perform arithmetic operations on vectors and describe their geometric interpretations;
- understand the concept of position vectors, displacement vectors, and direction vectors;
- find the magnitude of a vector;
- find unit vectors and relate with direction cosines and projection vectors;
- evaluate the collinearity of three points;
- apply Ratio Theorem in geometrical applications;
- perform scalar product and vector product, and applying their properties;
- express three-dimensional lines and planes in vector equation form and cartesian equation form;
- understand and evaluate the relationship between two lines (coplanar or skew), between a line and a plane, or between two planes;
- finding the distance between two points, between a point and a line, between two lines, between a point and a plane, between a line and a parallel plane, or between two parallel planes;
- finding the angle between two vectors, between two non-parallel lines, between a line and a non-parallel plane, or between two non-parallel planes.

4.1 Review of Basic Vectors

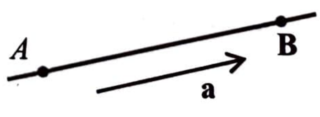
4.1.1 Vector Representation



Vector	**Scalar**
• has *magnitude* and *direction* • represented as $\underset{\sim}{a}$ or $\mathbf{a}$ or $\overrightarrow{OA}$	• has *magnitude* only • usually written as k or λ

Geometrically, we represent a vector by a *directed line segment*. The length of the line segment represents the magnitude of the vector, and the arrow gives the direction of the vector.

4.1.2 Position Vectors and Direction (or Free) Vectors

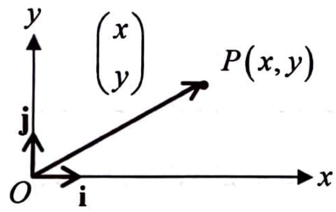
Position Vector	**Direction (or Free) Vector**
• We define the position of a point relative to a *point of reference* (usually the origin O unless stated otherwise). • A position vector is often used to determine the position of a moving object For example, the position vector of a point P with respect to the origin is the vector $\overrightarrow{OP}$ 	• A vector with no fixed point of reference. • A direction vector is often used to give the direction of motion of an object. For example, if $\mathbf{a}$ is the direction vector of the line AB, then any vector $k\mathbf{a}$ is also a direction vector of the line AB, where k is a non-zero real number. 

4.1.3 Vectors Represented in Two- and Three-dimensional Space

In 2-dimensional cartesian space, vectors $\mathbf{i}$ and $\mathbf{j}$ are *perpendicular* vectors with magnitude 1 in the direction of the x- and y- axes respectively. In column vector form, the vectors $\mathbf{i}$ and $\mathbf{j}$ are written as $\mathbf{i}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{j}=\begin{pmatrix}0\\1\end{pmatrix}$ respectively.

The point $P(x,y)$ has position vector $\overrightarrow{OP}$ given by $x\mathbf{i}+y\mathbf{j}$ or $\begin{pmatrix}x\\y\end{pmatrix}$.

Fig. 4.1

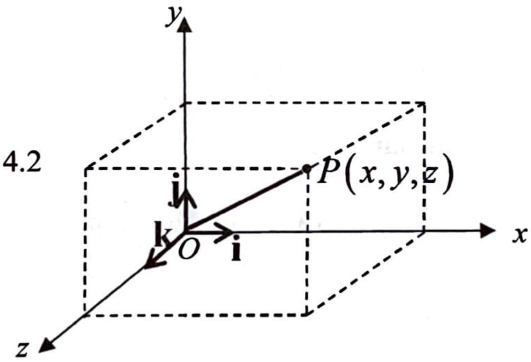


In 3-dimensional cartesian space, we define $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ to be *mutually perpendicular* vectors with magnitude 1 in the direction of the x-, y- and z-axes respectively. In column vector form, the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are written as

$$\mathbf{i}=\begin{pmatrix}1\\0\\0\end{pmatrix},\ \mathbf{j}=\begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } \mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix} \text{ respectively.}$$

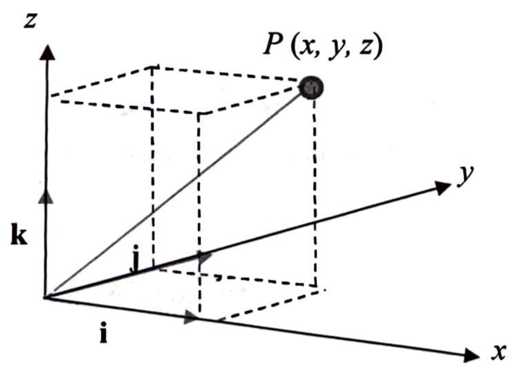
The point $P(x,y,z)$ has position vector $\overrightarrow{OP}$ given by $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ or $\begin{pmatrix}x\\y\\z\end{pmatrix}$ (column vector).

Fig 4.2



AB = AO + OB
= OB − OA

We can also draw it as:



Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

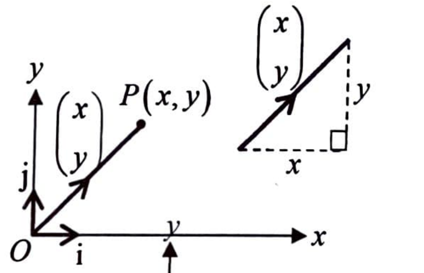
E.g. The point $(3,-1,2)$ has position vector $3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ or $\begin{pmatrix}3\\-1\\2\end{pmatrix}$.

4.1.4 Magnitude of a Vector

The **magnitude** (or length) of a vector $\mathbf{a}$ is denoted by $|\mathbf{a}|$.

> The **magnitude** of the vector $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$, is given by
>
> $$|\mathbf{r}|=\sqrt{x^2+y^2}\,.$$
>
> In 3 dimensions, the vector $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ has magnitude
>
> $$|\mathbf{r}|=\sqrt{x^2+y^2+z^2}\,.$$

Fig 4.3



For example, the vector $\mathbf{a}=\begin{pmatrix}3\\-1\\2\end{pmatrix}$ has magnitude

$$|\mathbf{a}|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}\,.$$

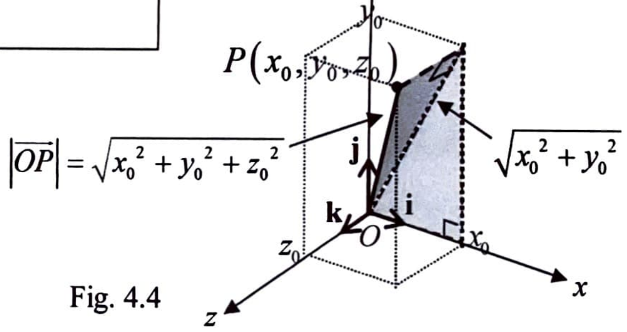


Fig. 4.4

Note: Two vectors **a** and **b** are *equal* ($\mathbf{a}=\mathbf{b}$) if and only if they have the *same magnitude and direction.*

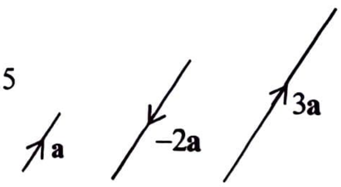
4.1.5 Multiplication of a Vector by a Scalar

Let $\mathbf{a}$ be a non-zero vector. The vector $\lambda\mathbf{a}$, where λ is a real number, is parallel to $\mathbf{a}$, and its magnitude is $|\lambda|$ times that of $|\mathbf{a}|$. If

(i) $\lambda>0$, then $\lambda\mathbf{a}$ is in the same direction as that of $\mathbf{a}$.
(ii) $\lambda<0$, then $\lambda\mathbf{a}$ is in the opposite direction to that of $\mathbf{a}$.
(iii) $\lambda=0$, we simply get the zero vector **0**. (written as $\underset{\sim}{0}$) ↓ $\begin{pmatrix}0\\0\\0\end{pmatrix}$

For example if $\mathbf{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, then

Fig. 4.5



(i) $\begin{pmatrix}-2\\-4\\-6\end{pmatrix}=-2\begin{pmatrix}1\\2\\3\end{pmatrix}=-2\,\mathbf{a}$. So $\begin{pmatrix}-2\\-4\\-6\end{pmatrix}$ is in the opposite direction as $\mathbf{a}$ but twice its length.

(ii) $\begin{pmatrix}3\\6\\9\end{pmatrix}=3\begin{pmatrix}1\\2\\3\end{pmatrix}=3\,\mathbf{a}$. So $\begin{pmatrix}3\\6\\9\end{pmatrix}$ is in the same direction as $\mathbf{a}$ and is thrice its length.

4.1.6 Vector Addition and Subtraction

To add two vectors $\mathbf{a}$ and $\mathbf{b}$, we can use either the triangle law or the parallelogram law of vector addition.

4.1.6.1 Triangle law of Vector Addition

Let $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{QR}$. Then $\mathbf{a}+\mathbf{b} = \overrightarrow{PR}$, i.e. $\overrightarrow{PQ}+\overrightarrow{QR} = \overrightarrow{PR}$

For example, the vectors $\overrightarrow{PQ} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\overrightarrow{QR} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$, so

$$\overrightarrow{PR} = \overrightarrow{PQ}+\overrightarrow{QR} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}3\\-1\\2\end{pmatrix} = \begin{pmatrix}4\\1\\5\end{pmatrix}.$$

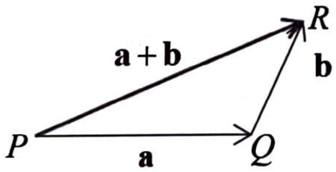


Fig 4.6

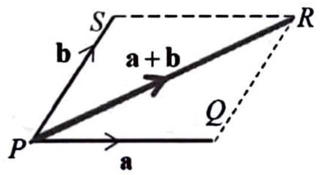
For example, the vectors $\overrightarrow{AB} = \mathbf{u}+2\mathbf{v}$ and $\overrightarrow{BC} = 2\mathbf{u}+\mathbf{v}$, then

$$\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC} = (\mathbf{u}+2\mathbf{v})+(2\mathbf{u}+\mathbf{v}) = 3\mathbf{u}+3\mathbf{v}$$

4.1.6.2 Parallelogram Law of Vector Addition

Let $\mathbf{a}$ and $\mathbf{b}$ be adjacent directed sides of the parallelogram as shown. Then $\mathbf{a}+\mathbf{b}$ is given by the directed diagonal $\overrightarrow{PR}$.

Fig 4.7



<u>ThinkZone:</u>
The triangle and the parallelogram laws of vector addition are equivalent. Can you see why?

If $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, …, $\mathbf{a}_n$ are represented by the sides of an open polygon taken in order, then the line that closes the polygon in the opposite sense represents the vector $\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\ldots+\mathbf{a}_n$, i.e. $\overrightarrow{OA_n} = \mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\ldots+\mathbf{a}_n$. This is known as the polygon law of vector addition.

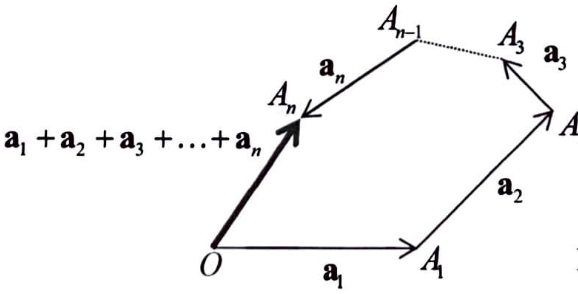


Fig 4.8

4.1.6.3 Vector Subtraction

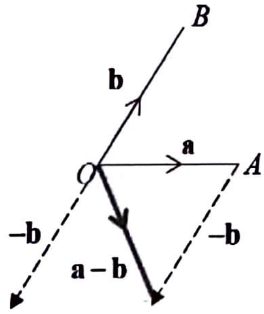
By definition, $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$.

$\overrightarrow{OA}-\overrightarrow{OB}=\overrightarrow{OA}+\left(-\overrightarrow{OB}\right)$

$=\overrightarrow{BO}+\overrightarrow{OA}$

$=\overrightarrow{BA}$ by triangle law of vector addition

Fig 4.9



Usually, we will find $\overrightarrow{BA}$ by writing $\overrightarrow{BA}=\overrightarrow{OA}-\overrightarrow{OB}$, which is the same as $\overrightarrow{BA}=\overrightarrow{BO}+\overrightarrow{OA}$.

Notation: $\left|\overrightarrow{BA}\right|$ represents the length of $\overrightarrow{BA}$ which is also the distance between point B and A. It can also be written as BA (without arrow).

4.1.6.4 Displacement Vector

One of the most common uses of vector addition is seen in displacement vectors.

Displacement is one example of a vector quantity.

For instance, a car journey from a Town A to a Town B may be represented by a displacement vector $\overrightarrow{AB}$. Its magnitude is the distance between A and B. Its direction is that of the straight line segment joining A to B.

Suppose a man drives from A to B and then from B to C.

On another occasion he drives directly from A to C.
Since the result of these two journeys is the same,
we may write $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.
The displacement $\overrightarrow{AC}$ is the *resultant* (vector sum) of the displacements $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

Fig 4.10

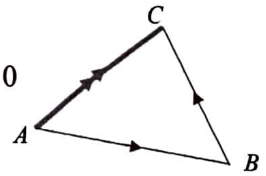
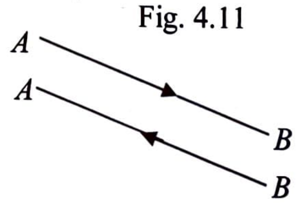


Fig. 4.11



Consider now a journey from A to B, followed by the return journey from B to A . Since the result of these two journeys is a zero displacement , it is reasonable to write, $\overrightarrow{AB}+\overrightarrow{BA}=\mathbf{0}$ and $\overrightarrow{BA}=-\overrightarrow{AB}$.

Note: A **zero vector** is any vector with *zero magnitude.* It is denoted by $\mathbf{0}$. We write it as $\underset{\sim}{0}$ or $\begin{pmatrix}0\\0\\0\end{pmatrix}$.

4.1.7 Laws of Vector Algebra

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be any three given vectors and λ, μ be any two real scalars. The following laws apply to vectors:

1. $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$ (Commutative law of vector addition)

2. $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$ (Associative law of vector addition)

3. $\lambda(\mu\,\mathbf{a}) = (\lambda\mu)\mathbf{a}$ (Associative law of scalar multiplication)

4. $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a}+\lambda\mathbf{b}$
5. $(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{a}$

} (Distributive laws)

Caution: $\mathbf{ab}$ and $\dfrac{\mathbf{a}}{\mathbf{b}}$ are both meaningless. We *do not multiply or divide vectors*!

For example $\mathbf{a}=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, we get

$$\mathbf{a}+\mathbf{b}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}2\\3\\4\end{pmatrix}.$$

$$\mathbf{b}+\mathbf{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\3\\4\end{pmatrix}.$$

Thus $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$.

4.1.8 Unit Vector

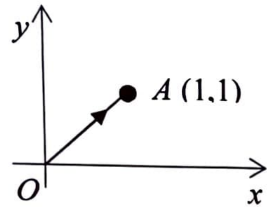
A **unit vector** is any vector which has a magnitude of 1 unit.

We denote a unit vector in the direction of $\mathbf{a}$ by $\hat{\mathbf{a}}$. Thus $\hat{\mathbf{a}}$ is in the same direction as $\mathbf{a}$ and $|\hat{\mathbf{a}}|=1$. Vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ defined in Section 4.1.3 are unit vectors in the direction of the x-axis, y-axis and z-axis respectively since $|\mathbf{i}|=1$, $|\mathbf{j}|=1$ and $|\mathbf{k}|=1$.

> The unit vector $\hat{\mathbf{a}}$ in the direction of $\mathbf{a}$ is given by $\hat{\mathbf{a}}=\dfrac{\mathbf{a}}{|\mathbf{a}|}$.
>
> In general, if $\mathbf{a}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, then $\hat{\mathbf{a}}=\dfrac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix}x\\y\\z\end{pmatrix}$.

to make it a unit vector: $\frac{\underset{\sim}{a}}{\sqrt{2}}$

Consider this



Question: Is $\overrightarrow{OA}$ a unit vector? NO.

$|\overrightarrow{OA}| = \sqrt{1^2+1^2} = \sqrt{2}$

Example 4.1 With respect to the origin O, the position vector of the point B is given by $\mathbf{b} = -\mathbf{i}+\mathbf{j}-2\mathbf{k}$. Write down a unit vector parallel to $\mathbf{b}$ and a vector of magnitude 2 units and parallel to $\mathbf{b}$.

Solution:

$\hat{\underset{\sim}{b}} = \frac{1}{\sqrt{(-1)^2+1^2+(-2)^2}}\begin{pmatrix}-1\\1\\-2\end{pmatrix}$

$= \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\-2\end{pmatrix}$ → unit vector parallel to $\underset{\sim}{b}$

vector required $= \frac{2}{\sqrt{6}}\begin{pmatrix}-1\\1\\-2\end{pmatrix}$ which is a unit vector in the same direction as $\underset{\sim}{b}$.

another vector can be $\frac{-1}{\sqrt{6}}\begin{pmatrix}-1\\1\\-2\end{pmatrix}$ and $\frac{-2}{\sqrt{6}}\begin{pmatrix}-1\\1\\-2\end{pmatrix}$ respectively.
→ opposite direction

ThinkZone:

What is the geometrical interpretation of $k\hat{\mathbf{a}}$, where k is a scalar?

It represents a vector parallel to $\hat{\mathbf{a}}$ with length $|k|$.

Self-Review 4.1 Find the possible vector(s) **v** if $|\mathbf{v}|$= 24 units and $\hat{\mathbf{v}} = \frac{2}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$. $[16\mathbf{i}-16\mathbf{j}-8\mathbf{k}]$

Example 4.2 A airplane A flies, in still air, in a fixed direction given by the vector **i** – 2**j** + 2**k**. If the airplane maintains a constant speed of 900 kmh^{-1}, find its velocity in still air in the form $k(\mathbf{i}-2\mathbf{j}+2\mathbf{k})$ where k is a real positive constant.

Solution:

Velocity of airplane A

$$=900\underbrace{\left(\frac{\mathbf{i}-2\mathbf{j}+2\mathbf{k}}{|\mathbf{i}-2\mathbf{j}+2\mathbf{k}|}\right)}_{\text{unit vector}}$$

$$=900\left(\frac{\mathbf{i}-2\mathbf{j}+2\mathbf{k}}{\sqrt{1^2+(-2)^2+2^2}}\right)=300(\mathbf{i}-2\mathbf{j}+2\mathbf{k})$$

Alternatively, we can also see it as:

ThinkZone:

Speed is a scalar quantity.
Velocity is a vector quantity, which possesses both magnitude and direction.

A vector divided by its own magnitude will give a unit vector in the same direction as that vector.

$\begin{pmatrix}2/3\\-2/3\\-1/3\end{pmatrix} = \frac{1}{24}\begin{pmatrix}x\\y\\z\end{pmatrix}$

The concept of vectors is one of the most useful ideas in applied mathematics.

$\lvert k(\mathbf{i}-2\mathbf{j}+2\mathbf{k})\rvert = 900$ $k\sqrt{1^2+(-2)^2+2^2} = 900$ $3k = 900$ $k = 300$ Velocity of plane $= 300(\mathbf{i}-2\mathbf{j}+2\mathbf{k})$	

4.1.9 Direction Cosines

Consider the point $P(x, y, z)$. Then $\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\left|\overrightarrow{OP}\right| = r = \sqrt{(x^2 + y^2 + z^2)}$.

The direction of the vector $\overrightarrow{OP}$ is determined by the angles $\overrightarrow{OP}$ makes with the directions of the axes OX, OY, OZ. If these angles are α, β, γ respectively, then

$$\cos\alpha = \frac{x}{r};\ \cos\beta = \frac{y}{r};\ \cos\gamma = \frac{z}{r}$$

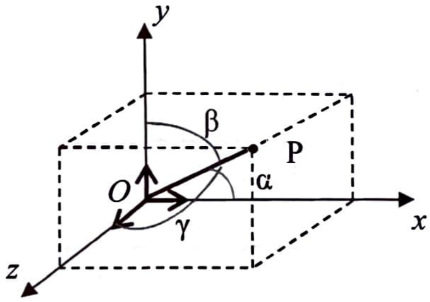


Fig 4.12

where $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called **direction cosines** of $\overrightarrow{OP}$.

Note that:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2+y^2+z^2}{r^2} = 1$$

The ratios $x : y : z$ are called the **direction ratios** of the vector $\overrightarrow{OP}$.

Example 4.3

(a) Find the angles which the vector $3\mathbf{i} - 4\mathbf{k}$ makes with the positive directions of the x, y and z-axes.

(b) Find the vector of magnitude of 2 which makes an angle 60° and 150° with the positive directions of the x and y-axes respectively.

Solution:	**ThinkZone:**
(a) Let α, β, γ are the angles which $3\mathbf{i} - 4\mathbf{k}$ makes with the positive directions of the x, y and z-axes respectively $\cos\alpha = \frac{3}{\sqrt{3^2+(-4)^2}} = \frac{3}{5} \Rightarrow \alpha \approx 53.1^\circ$. $\cos\beta = \frac{0}{\sqrt{3^2+(-4)^2}} = 0 \Rightarrow \beta = 90^\circ$. $\cos\gamma = \frac{-4}{\sqrt{3^2+(-4)^2}} = -\frac{4}{5} \Rightarrow \gamma \approx 143.1^\circ$. So the vector $3\mathbf{i} - 4\mathbf{k}$ makes angles 53.1°, 90° and 143.1° with the positive directions of the x, y and z-axes respectively. (b) Let the required vector be $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $\lvert\mathbf{v}\rvert = 2$. So	Is $3\mathbf{i} - 4\mathbf{k}$ a 2- or 3-dimensional vector?

$\frac{x}{\|\mathbf{v}\|} = \cos\alpha = \cos 60^\circ = \frac{1}{2} \Rightarrow x = \frac{1}{2}(2) = 1$. $\frac{y}{\|\mathbf{v}\|} = \cos\beta = \cos 150^\circ = -\frac{\sqrt{3}}{2} \Rightarrow y = -\frac{\sqrt{3}}{2}(2) = -\sqrt{3}$. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ $\Rightarrow \cos^2\gamma = 1 - \left(\frac{1}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)^2 = 0 \Rightarrow \cos\gamma = 0$ So $\frac{z}{\|\mathbf{v}\|} = \cos\gamma = 0 \Rightarrow z = 0$. Therefore $\mathbf{v} = \mathbf{i} - \sqrt{3}\mathbf{j}$.	The angles α, β, γ are *not independent*. They are related by the equation: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

4.1.10 Parallel Vectors

$\cos^2\alpha + \cos\beta^2 + \cos\gamma^2 = \frac{a^2+b^2+c^2}{r^2} = 1$

> Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero vectors.
> Then $\mathbf{a}$ and $\mathbf{b}$ are **parallel** if and only if $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$.

For example $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}$ are parallel vectors since $\begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

Example 4.4 Find the values of a and b if the vectors $4\mathbf{i} + a\mathbf{j} + 6\mathbf{k}$ and $2\mathbf{i} + 4\mathbf{j} + b\mathbf{k}$ are parallel.

Solution:	ThinkZone:
Since $\begin{pmatrix} 4 \\ a \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ b \end{pmatrix}$ are parallel, $\begin{pmatrix} 4 \\ a \\ 6 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ 4 \\ b \end{pmatrix}$ for some $\lambda \in \mathbb{R}\setminus\{0\}$, Hence $4 = 2\lambda$ — (1) $a = 4\lambda$ — (2) $6 = \lambda b$ — (3) From (1), $\lambda = 2$ Substituting $\lambda = 2$ into (2) and (3) $a = 4(2) = 8$ and $6 = 2b \Rightarrow b = 3$	

Example 4.5 Find the vector(s) $\mathbf{v}$ if $\mathbf{v}$ is parallel to the vector $8\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and is equal in magnitude to the vector $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

Solution:	ThinkZone:

Since $\mathbf{v}$ is parallel to the vector $8\mathbf{i}+\mathbf{j}+4\mathbf{k}$, so $\mathbf{v}=\lambda\begin{pmatrix}8\\1\\4\end{pmatrix}$ for some $\lambda \in \mathbb{R}\setminus\{0\}$. $\lvert\mathbf{v}\rvert=\left\lvert\lambda\begin{pmatrix}8\\1\\4\end{pmatrix}\right\rvert=\left\lvert\begin{pmatrix}1\\-2\\2\end{pmatrix}\right\rvert$ $\lvert\lambda\rvert\sqrt{8^2+1^2+4^2}=\sqrt{1^2+(-2)^2+2^2}$ $9\lvert\lambda\rvert=3$ $\Rightarrow \lvert\lambda\rvert=\frac{1}{3} \Rightarrow \lambda=\pm\frac{1}{3}$ Hence $\mathbf{v}=\frac{1}{3}\begin{pmatrix}8\\1\\4\end{pmatrix}$ or $-\frac{1}{3}\begin{pmatrix}8\\1\\4\end{pmatrix}$.	What are two important information given in the question?

Example 4.6 If $\mathbf{a}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}-4\mathbf{j}+5\mathbf{k}$ and $\mathbf{c}=-\mathbf{i}-4\mathbf{j}+4\mathbf{k}$, find the value of q such that $\mathbf{a}+2\mathbf{b}+q\mathbf{c}$ is parallel to the x-axis.

Solution:	**ThinkZone:**
$\mathbf{a}+2\mathbf{b}+q\mathbf{c}=\begin{pmatrix}2\\-3\\1\end{pmatrix}+2\begin{pmatrix}2\\-4\\5\end{pmatrix}+q\begin{pmatrix}-1\\-4\\4\end{pmatrix}=\begin{pmatrix}6-q\\-11-4q\\11+4q\end{pmatrix}$ $\mathbf{a}+2\mathbf{b}+q\mathbf{c}$ is parallel to the x-axis $\Rightarrow$ $\mathbf{a}+2\mathbf{b}+q\mathbf{c}$ is parallel to $\mathbf{i}$. Hence $\begin{pmatrix}6-q\\-11-4q\\11+4q\end{pmatrix}=\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$ for some $\lambda \in \mathbb{R}\setminus\{0\}$. Thus $q+\lambda=6$ $11+4q=0$ Solving the above equations, we get $q=-\frac{11}{4}$ and $\lambda=\frac{35}{4}$.	What are possible direction vectors for x-axis? Can we use $2\mathbf{i}$ instead?

4.1.10.1 An Important Result

If $\mathbf{a}$ and $\mathbf{b}$ are non-zero, non-parallel vectors such that $\lambda\mathbf{a}=\mu\mathbf{b}$ for some $\lambda,\ \mu \in \mathbb{R}$, then $\lambda=\mu=0$.

For example, $\lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}=\mu\begin{pmatrix}2\\1\\4\end{pmatrix}$ implies that $\lambda=\mu=0$.

Suppose $\lambda \neq 0$, then $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = k \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ where $k = \dfrac{\mu}{\lambda}$

$$3 = 2k \quad \Rightarrow \quad k = \frac{3}{2}$$
$$-1 = k \quad \Rightarrow \quad k = -1$$
$$2 = 4k \quad \Rightarrow \quad k = \frac{1}{2}$$

which is a contradiction. Thus $\lambda = 0$. By a similar argument, $\mu = 0$.

See Example **4.28** for application of this result.

4.1.11 Collinearity

Three or more points are **collinear** if they lie in a straight line.
To prove that three points *A, B* and *C* are collinear, we can show any of the following:

- $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel
- $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel
- $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are parallel

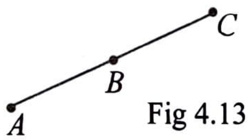


Fig 4.13

ThinkZone:
Can you give another pair of direction vectors such that *A, B* and *C* are collinear?

Note that there is a ***common point*** between the 2 pairs of parallel vectors.

Example 4.7 Show that the points $A(3,2,4)$, $B(2,4,6)$ and $C(0,8,10)$ are collinear. Find the ratio $AB:BC$.

Solution:

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix}$ and $\overrightarrow{bc} = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix}$

Thus $\overrightarrow{BC} = 2\overrightarrow{AB}$

since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel and B is a common point, the three points are collinear.

$\overrightarrow{BC} = 2\overrightarrow{AB}$

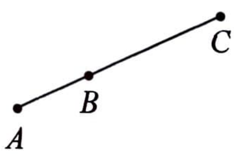
$\Rightarrow \left|\overrightarrow{BC}\right| = 2\left|\overrightarrow{AB}\right|$

$BC = 2AB$

$\dfrac{AB}{BC} = \dfrac{1}{2}$

Hence $AB : BC = 1:2$

ThinkZone:



Warning: Do not write $\dfrac{\overrightarrow{AB}}{\overrightarrow{BC}} = \dfrac{1}{2}$. Why?

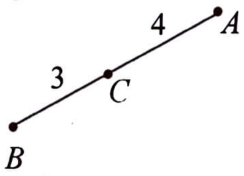
Example 4.8 If $3\overrightarrow{OA}+4\overrightarrow{OB}-7\overrightarrow{OC}=\mathbf{0}$, prove that A, B and C are collinear . Find the ratios $BC:CA$ and $AC:AB$.

Solution:

$$3\overrightarrow{OA}+4\overrightarrow{OB}-7\overrightarrow{OC}=\mathbf{0}$$
$$3\overrightarrow{OA}-3\overrightarrow{OC}+4\overrightarrow{OB}-4\overrightarrow{OC}=\mathbf{0}$$
$$3\left(\overrightarrow{OA}-\overrightarrow{OC}\right)+4\left(\overrightarrow{OB}-\overrightarrow{OC}\right)=\mathbf{0}$$
$$3\overrightarrow{CA}+4\overrightarrow{CB}=\mathbf{0}$$
$$\overrightarrow{CA}=-\frac{4}{3}\overrightarrow{CB}$$

Hence $\overrightarrow{CA}//\overrightarrow{CB}$, with C being a common point.

$\Rightarrow$ A, B and C are collinear.



$\overrightarrow{CA}=-\frac{4}{3}\overrightarrow{CB}\Rightarrow\overrightarrow{CA}=\frac{4}{3}\overrightarrow{BC}$

$\frac{BC}{CA}=\frac{3}{4}$

$BC:CA=3:4$ and $AC:AB=4:7$

ThinkZone:

Why is $-7\overrightarrow{OC}$ written as $-3\overrightarrow{OC}-4\overrightarrow{OC}$?

What is significance of $\overrightarrow{CA}$ and $\overrightarrow{CB}$ having opposite signs?

Alternatively,

$\overrightarrow{CA}=-\frac{4}{3}\overrightarrow{CB}\Rightarrow CA=\frac{4}{3}CB$

$\Rightarrow\frac{CB}{CA}=\frac{3}{4}$

Self-Review 4.2

(a) Determine which of the following sets of points are collinear:

(i) $(1,-1,2)$, $(4,1,7)$, $(-2,-3,-2)$ [No]

(ii) $(2,6,-6)$, $(3,2,0)$, $(4,-2,6)$ [Yes]

(b) Find a and b if the points $(1,-2,2)$, $(2,2,1)$, $(a,b,-1)$ are collinear. [$a=4$, $b=10$]

4.1.12 The Ratio Theorem

Let O be the reference point and AB a line segment with $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$, and $\lambda,\ \mu\in\mathbb{R}^{+}$.

If a point P divides AB in the ratio $\lambda:\mu$, then $\overrightarrow{OP}=\frac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu}$.

In particular, if P is the mid-point of AB, then $\lambda=\mu$ and so $\overrightarrow{OP}=\frac{\mathbf{a}+\mathbf{b}}{2}$. This is called the **Midpoint Theorem**.

$\overrightarrow{OP}=\frac{\mu(\mathbf{a}+\mathbf{b})}{2\mu}$ (if $\lambda=\mu$) $=\frac{\mathbf{a}+\mathbf{b}}{2}$

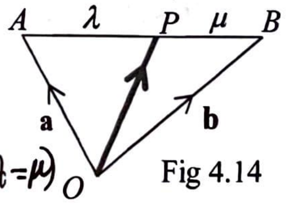


Fig 4.14

Proof:

Since $AP:PB=\lambda:\mu$, $\overrightarrow{AP}=\frac{\lambda}{\lambda+\mu}\overrightarrow{AB}$.

By triangle law of vector addition,

Fig. 4.15

$$
\begin{aligned}
\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP} \\
&= \overrightarrow{OA} + \frac{\lambda}{\lambda+\mu}\overrightarrow{AB} \\
&= \mathbf{a} + \frac{\lambda}{\lambda+\mu}(\mathbf{b}-\mathbf{a}) = \frac{\lambda \mathbf{a} + \mu \mathbf{a} + \lambda \mathbf{b} - \lambda \mathbf{a}}{\lambda + \mu} \\
&= \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda+\mu}
\end{aligned}
$$

Note: The Ratio Theorem formula is available in MF26.

For example, if a point P divides a line segment AB in the ratio 3:1, then

$$
\begin{aligned}
\overrightarrow{OP} &= \frac{1\overrightarrow{OA} + 3\overrightarrow{OB}}{1+3} \\
&= \frac{\overrightarrow{OA} + 3\overrightarrow{OB}}{4} \\
&= \frac{\mathbf{a} + 3\mathbf{b}}{4}
\end{aligned}
$$

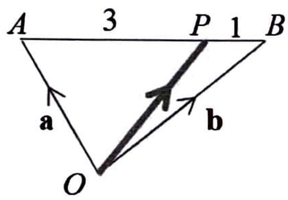


Fig 4.16

In some cases, a third point P may be introduced to be collinear to the first two points A and B, but lie outside the line segment AB. In such a scenario, we say that P lies on AB produced, or BA produced, depending on which side of line segment is being extended.

P lies on AB produced.

Fig 4.17A

P lies on BA produced.

Fig 4.17B

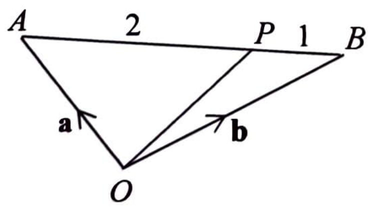
Remark: It is always good to draw a simple diagram prior to applying the Ratio Theorem.

Example 4.9 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$. Find in terms of $\mathbf{a}$ and $\mathbf{b}$,

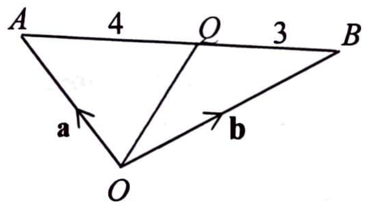
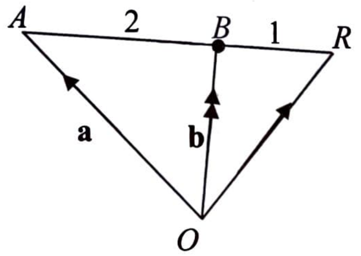
(i) the position vector of P on AB such that $\overrightarrow{AP} = 2\overrightarrow{PB}$,

(ii) the position vector of Q which divides AB in the ratio $4:3$,

(iii) the position vector of R on AB produced such that $2AR = 3AB$.

Solution:	ThinkZone:
(i)  $\overrightarrow{AP} = 2\overrightarrow{PB} \Rightarrow AP:PB = 2:1$ By Ratio Theorem, $\overrightarrow{OP} = \dfrac{\mathbf{a}+2\mathbf{b}}{3}$	Do not write $\dfrac{\overrightarrow{AP}}{\overrightarrow{PB}}$.

The concept of vectors is one of the most useful ideas in applied mathematics.

(ii)  By Ratio Theorem, $\overrightarrow{OQ}=\dfrac{3\mathbf{a}+4\mathbf{b}}{7}$	
(iii)  $2AR=3AB \Rightarrow \dfrac{AR}{AB}=\dfrac{3}{2}$ $\overrightarrow{OB}=\dfrac{\overrightarrow{OA}+2\overrightarrow{OR}}{3}$ $\Rightarrow \overrightarrow{OR}=\dfrac{1}{2}\left(3\overrightarrow{OB}-\overrightarrow{OA}\right)$ $=\dfrac{1}{2}(3\mathbf{b}-\mathbf{a})$	What do you understand by '*R* on *AB* produced'?

Self-Review 4.3 The position vectors of B and C referred to a point O are **b** and **c** respectively. Find an expression for the position vector of L, the point on BC between B and C such that $BL : LC = 2:1$. If the point A with position vector **a** (referred to O) is such that O is the midpoint to AL, prove that ~~3a + b + 2c = 0~~ $[\overrightarrow{OL} = \frac{1}{3}(\mathbf{b}+2\mathbf{c})]$

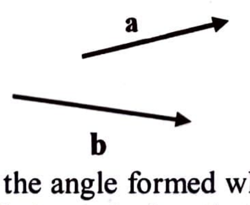
$3\underset{\sim}{a} + \underset{\sim}{b} + 2\underset{\sim}{c} = \underset{\sim}{0}$.

4.2 Vector Multiplication – The Scalar and Vector Products

4.2.1 Angle between Vectors

Given two vectors $\mathbf{a}$ and $\mathbf{b}$, we define the angle θ between $\mathbf{a}$ and $\mathbf{b}$ as the angle formed when the two vectors are seen emanating from the same source.

Fig 4.18A



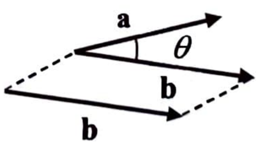
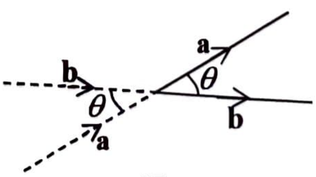


Fig. 4.18B

One can prove that the angle formed when the two vectors are both pointing towards the same point is the same as the angle formed when the two vectors are pointing out from that same point.

Fig. 4.19



4.2.2 The Scalar Product

The **scalar product** (or **dot product**) of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a}\bullet\mathbf{b}$, is defined as

$$\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \text{ where } \theta \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b} \ (0° \le \theta \le 180°).$$

>0 if θ is acute
<0 if θ is obtuse

The scalar product of two vectors in *cartesian form* is computed as follows:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \bullet \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

For example $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} = (2)(0)+(-1)(3)+(4)(5) = 17$.

4.2.2.1 Properties of the Scalar Product

1. $\mathbf{a}\bullet\mathbf{b}$ is a scalar (i.e. $\mathbf{a}\bullet\mathbf{b} \in \mathbb{R}$) If either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then $\mathbf{a}\bullet\mathbf{b} = 0$.	E.g. $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} = (2)(0)+(3)(3)+(4)(5) = 29$
2. $\mathbf{a}\bullet\mathbf{b} = \mathbf{b}\bullet\mathbf{a}$ (*commutative*)	$\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ $= |\mathbf{b}||\mathbf{a}|\cos\theta$ $= \mathbf{b}\bullet\mathbf{a}$
3. $\mathbf{a}\bullet(\mathbf{b}+\mathbf{c}) = \mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{c}$ (*distributive over addition*)	E.g. $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet \left[\begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $= 17 + 9 = 26$
4. $(\lambda\mathbf{a})\bullet\mathbf{b} = \mathbf{a}\bullet(\lambda\mathbf{b}) = \lambda(\mathbf{a}\bullet\mathbf{b})$ where $\lambda \in \mathbb{R}$ $\mathbf{a}\bullet\mathbf{b}\bullet\mathbf{c}$ is meaningless. Do you know why?	E.g. $2\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet 2\begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} = 2\left[\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}\right]$
5. $\mathbf{a}\bullet\mathbf{a} = |\mathbf{a}|^2$	By definition, $\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. Hence $\mathbf{a}\bullet\mathbf{a} = |\mathbf{a}||\mathbf{a}|\cos 0° = |\mathbf{a}|^2$ Similarly, $(\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{b}) = |\mathbf{a}-\mathbf{b}|^2$. This is an important and useful result.
6. If $\mathbf{a},\mathbf{b} \neq \mathbf{0}$, then $\mathbf{a}\bullet\mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}$	$0 = \mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90°$

Example 4.10 The angle between two vectors **p** and **q** is $\cos^{-1}\frac{4}{21}$. If $\mathbf{p}=6\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ and $\mathbf{q}=-2\mathbf{i}+\lambda\mathbf{j}-4\mathbf{k}$, find the positive value(s) of λ.

Solution:	**ThinkZone:**
$\mathbf{p}\cdot\mathbf{q}=\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\cos\theta$ $\begin{pmatrix}6\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}-2\\\lambda\\-4\end{pmatrix}=\sqrt{6^2+3^2+(-2)^2}\sqrt{(-2)^2+\lambda^2+(-4)^2}\cos\left(\cos^{-1}\frac{4}{21}\right)$ $-12+3\lambda+8=(7)\left(\sqrt{20+\lambda^2}\right)\frac{4}{21}$ $\Rightarrow\left[\frac{3}{4}(-4+3\lambda)\right]^2=20+\lambda^2$ $65\lambda^2-216\lambda-176=0$ $(\lambda-4)(65\lambda+44)=0$ $\Rightarrow \lambda=4$ or $-\frac{44}{65}$. Since $\lambda>0, \therefore \lambda=4$.	

Example 4.11 (H2 Math N2010/I/1 modified) The position vectors **a** and **b** are such that $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert$. Show that $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=0$.

Solution:	**ThinkZone:**
$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$ $=\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot(-\mathbf{b})+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot(-\mathbf{b})$ $=\lvert\mathbf{a}\rvert^2-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}-\lvert\mathbf{b}\rvert^2$ $=0\ \ (\because \lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert)$	What is the geometrical significance of this result? Is the converse true? That is if $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=0$ would it imply $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert$?

Example 4.12 (Application of Scalar Product to Work Done)
The 'work done' by a constant force **F** acting on a particle and causing it to move from point A to point B is given by the scalar product $\mathbf{F}\bullet\mathbf{d}$ where $\mathbf{d}=\overrightarrow{AB}$, is called the displacement vector.
Find the work done by the constant force **i** – 2**j** + 2**k** which acts on a particle and causing it to move from the point $(1,0,-2)$ to the point $(3,4,5)$.

Suppose a system of forces $\mathbf{F}_1,\mathbf{F}_2,\ldots,\mathbf{F}_n$ act on a particle and causing it to undergo a displacement given by the vector **d**. Show that the sum of the work done by the individual forces acting on the particle is equal to the work done by the resultant force on the particle.

Solution:	**ThinkZone:**
Displacement vector, $\mathbf{d}=\begin{pmatrix}3\\4\\5\end{pmatrix}-\begin{pmatrix}1\\0\\-2\end{pmatrix}=\begin{pmatrix}2\\4\\7\end{pmatrix}$.	

Work done by the force $\mathbf{F} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

$= \mathbf{F} \bullet \mathbf{d}$

$$= \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

$= 2 - 8 + 14$

$= 8$ units

Sum of work done by the system of forces

$= \mathbf{F}_1 \bullet \mathbf{d} + \mathbf{F}_2 \bullet \mathbf{d} + \ldots + \mathbf{F}_n \bullet \mathbf{d}$

$$= \underbrace{(\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n)}_{\mathbf{F}_R} \bullet \mathbf{d} = \mathbf{F}_R \bullet \mathbf{d}$$

= work done by resultant force

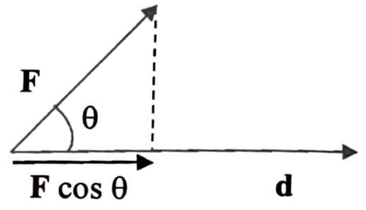
where $\mathbf{F}_R$ is the resultant force.

Work done

$= \mathbf{F} \bullet \mathbf{d}$

$= |\mathbf{F}||\mathbf{d}|\cos\theta = (|\mathbf{F}|\cos\theta)|\mathbf{d}|$

= Force in the direction of displacement × length of Displacement



The unit for work done is 'Joule'.

Use repeatedly the distributive property of scalar product $(\mathbf{a} + \mathbf{b}) \bullet \mathbf{d} = \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{d}$

Self-Review 4.4 Three vectors **a**, **b** and **c** are such that $\mathbf{a} \neq \mathbf{b} \neq \mathbf{c} \neq \mathbf{0}$ and **a**, **b** and **c** are not mutally perpendicular.

(a) If $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{b} \bullet (\mathbf{a} - \mathbf{c})$, prove that $\mathbf{c} \bullet (\mathbf{a} + \mathbf{b}) = 0$.

(b) If $(\mathbf{a} \bullet \mathbf{b})\mathbf{c} = (\mathbf{b} \bullet \mathbf{c})\mathbf{a}$, prove that **c** and **a** are parallel.

4.2.2.2 Finding Angle between Two Vectors

Example 4.13 Given that the position vectors of three points A, B and C are $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = \alpha\mathbf{i} + \mathbf{j}$ respectively.

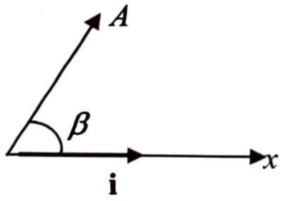
(i) Determine the angle between $\overrightarrow{OA}$ and the x-axis.

(ii) Determine the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

(iii) Find the possible value(s) of α such that $\measuredangle ABC = 60°$.

Solution:

(i)



Let β denotes the angle between $\overrightarrow{OA}$ and the x-axis.

$$\overrightarrow{OA} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |\overrightarrow{OA}| \left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right| \cos\beta$$

ThinkZone

Why do we choose **i**?

Would the angle be different if 2**i** is used?

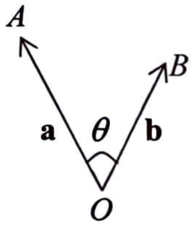
$$\begin{pmatrix}2\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=\left|\begin{pmatrix}2\\-1\\3\end{pmatrix}\right|\left|\begin{pmatrix}1\\0\\0\end{pmatrix}\right|\cos\beta$$

$$2=\sqrt{14}\cos\beta$$

$$\cos\beta=\frac{2}{\sqrt{14}}$$

$$\beta=57.7^\circ \qquad \text{(1 d.p.)}$$

(ii)



Let θ denotes the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$$\overrightarrow{OA}\bullet\overrightarrow{OB}=\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos\theta$$

$$\begin{pmatrix}2\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}0\\1\\-2\end{pmatrix}=\left|\begin{pmatrix}2\\-1\\3\end{pmatrix}\right|\left|\begin{pmatrix}0\\1\\-2\end{pmatrix}\right|\cos\theta$$

$$-7=\sqrt{14}\sqrt{5}\cos\theta$$

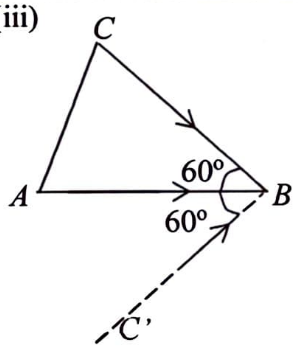
$$\cos\theta=\frac{-7}{\sqrt{70}}$$

$$\theta=146.8^\circ \text{ (1 d.p.)}$$

Does $\overrightarrow{OA}\cdot\overrightarrow{BO}$ gives a different angle?

How do we determine which vectors to use?

(iii)



$$\overrightarrow{AB}=\underset{\sim}{b}-\underset{\sim}{a}=\begin{pmatrix}0\\1\\-2\end{pmatrix}-\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}-2\\2\\-5\end{pmatrix}$$

$$\overrightarrow{CB}=\underset{\sim}{b}-\underset{\sim}{c}=\begin{pmatrix}0\\1\\-2\end{pmatrix}-\begin{pmatrix}\alpha\\1\\0\end{pmatrix}=\begin{pmatrix}-\alpha\\0\\-2\end{pmatrix}$$

$$\overrightarrow{AB}\bullet\overrightarrow{CB}=\left|\overrightarrow{AB}\right|\left|\overrightarrow{CB}\right|\cos\measuredangle ABC$$

Will $\overrightarrow{BA}\bullet\overrightarrow{BC}$ give the same answer?

Alternatively,

$$\overrightarrow{BA}\bullet\overrightarrow{BC}=\begin{pmatrix}2\\-2\\5\end{pmatrix}\bullet\begin{pmatrix}\alpha\\0\\2\end{pmatrix}=\left|\begin{pmatrix}2\\-2\\5\end{pmatrix}\right|\left|\begin{pmatrix}\alpha\\0\\2\end{pmatrix}\right|\cos\angle ABC$$

$$2\alpha+10=\sqrt{33}\sqrt{\alpha^2+4}\left(\frac{1}{2}\right)$$

$$17\alpha^2-160\alpha-268=0$$

$$\alpha=10.9 \text{ or } -1.45$$

$\begin{pmatrix}-2\\2\\-5\end{pmatrix}\bullet\begin{pmatrix}-\alpha\\0\\-2\end{pmatrix}=\left\|\begin{pmatrix}-2\\2\\-5\end{pmatrix}\right\|\left\|\begin{pmatrix}-\alpha\\0\\-2\end{pmatrix}\right\|\cos 60°$ $2\alpha+10=\sqrt{33}\sqrt{\alpha^2+4}\left(\frac{1}{2}\right)$ $4(2\alpha+10)^2=33(\alpha^2+4)$ $17\alpha^2-160\alpha-268=0$ $\alpha=10.9$ or -1.45 Hence there are two distinct values of α such that $\measuredangle ABC=60°$.	

Note:

> $\mathbf{a}\bullet\mathbf{b}>0 \Rightarrow \theta$ is acute.
> $\mathbf{a}\bullet\mathbf{b}<0 \Rightarrow \theta$ is obtuse.
> All angles in degrees are to be corrected to 1 d.p. unless otherwise stated in the question.

Self-Review 4.5 In a triangle *ABC*, $\overrightarrow{AB}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\overrightarrow{BC}=-\mathbf{i}+4\mathbf{j}$. Find the cosine of the angle *ABC*. Find the vector $\overrightarrow{AC}$ and use it to calculate the angle *BAC*. [$-\sqrt{\frac{7}{34}}, 6\mathbf{j}+3\mathbf{k}, 33.2°$]

4.2.2.3 Testing for Perpendicular Vectors

We saw on Section 4.2.2.1 that for two non-zero vectors **a** and **b** such that $\mathbf{a}\bullet\mathbf{b}=0$, then we have $0=\mathbf{a}\bullet\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta \Rightarrow \cos\theta=0 \Rightarrow \theta=90°$. Therefore we have the following result:

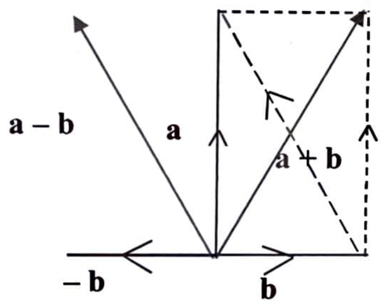
> Let **a** and **b** be non-zero vectors. Then $\mathbf{a}\bullet\mathbf{b}=0 \Leftrightarrow$ **a** and **b** are perpendicular.

E.g. $\begin{pmatrix}2\\-1\\4\end{pmatrix}$ and $\begin{pmatrix}5\\2\\-2\end{pmatrix}$ are perpendicular vectors since $\begin{pmatrix}2\\-1\\4\end{pmatrix}\bullet\begin{pmatrix}5\\2\\-2\end{pmatrix}=10+(-2)+(-8)=0$.

Example 4.14 Referred to the origin *O*, the position vectors of the points *A* and *B* are $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $\mathbf{i}-3\mathbf{j}+5\mathbf{k}$ respectively. Show that *OA* is perpendicular to *AB*.

Solution:	**ThinkZone:**
$\overrightarrow{AB}=\begin{pmatrix}1\\-3\\5\end{pmatrix}-\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}-1\\-4\\2\end{pmatrix}$ $\overrightarrow{OA}\bullet\overrightarrow{AB}=\begin{pmatrix}2\\1\\3\end{pmatrix}\bullet\begin{pmatrix}-1\\-4\\2\end{pmatrix}=-2-4+6=0$. Hence, *OA* is perpendicular to *AB*.	

Example 4.15 If **a** and **b** are perpendicular vectors, show that $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$. Give a geometrical interpretation of this result.

Solution:	**ThinkZone:**
Applying $\mathbf{a}\bullet\mathbf{a}=|\mathbf{a}|^2$: $|\mathbf{a}+\mathbf{b}|^2=(\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}+\mathbf{b})$ $=\mathbf{a}\bullet\mathbf{a}+\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}$ $=\mathbf{a}\bullet\mathbf{a}+2\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{b}$ $=\mathbf{a}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}$ since $\mathbf{a}\bullet\mathbf{b}=0$ (**a** and **b** are perpendicular vectors) $=\mathbf{a}\bullet\mathbf{a}-2\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{b}$ $=(\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})$ $=|\mathbf{a}-\mathbf{b}|^2$ Taking square root gives $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$  This result says that the diagonals of a rectangle have the same length.	Why do we introduce $-2\mathbf{a}\bullet\mathbf{b}$? Why do we take the positive square root only?

Self-Review 4.6 Given two non-zero vectors **a** and **b**, show that if $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are perpendicular, then $|\mathbf{a}|=|\mathbf{b}|$. Give a geometrical interpretation of this result.

Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

4.2.3 The Vector (or Cross) Product

The **vector product** (or **cross product**) of two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a}\times\mathbf{b}$, is defined by

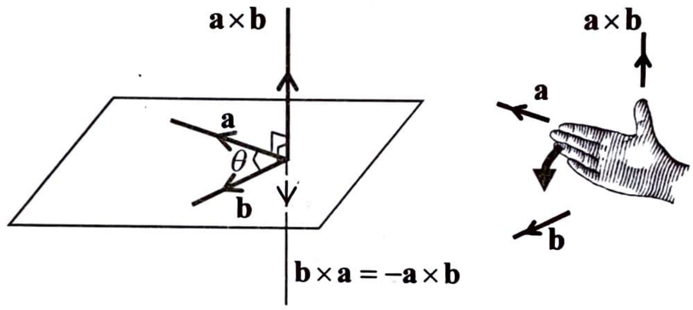
$$\boxed{\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\ \hat{\mathbf{n}}}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

The vector product $\mathbf{a}\times\mathbf{b}$

- has magnitude given by
  $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ $(0°\le\theta\le180°)$,
- is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. That is, $\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b}) = 0$ and $\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b}) = 0$,
- has direction given by the *right-hand rule*:
  place palm flat with fingers in the direction of $\mathbf{a}$ and turn fingers inward toward the direction of $\mathbf{b}$. The thumb outstretched at $90°$ gives the direction of the vector product $\mathbf{a}\times\mathbf{b}$.

Fig. 4.20



The vector product of two vectors in *Cartesian form* is as follows:

$$\boxed{\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}\times\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}y_1z_2-z_1y_2\\-(x_1z_2-z_1x_2)\\x_1y_2-y_1x_2\end{pmatrix}}$$

Procedure to obtain the cross product of 2 vectors:

To obtain Row 1 on RHS,	To obtain Row 2 on RHS,	To find Row 3 on RHS,
$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}\times\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}y_1z_2-z_1y_2\\\square\\\square\end{pmatrix}$	$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}\times\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}\square\\-(x_1z_2-z_1x_2)\\\square\end{pmatrix}$	$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}\times\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}\square\\\square\\x_1y_2-y_1x_2\end{pmatrix}$

E.g. $\begin{pmatrix}2\\1\\-2\end{pmatrix}\times\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{pmatrix}1(3)-1(-2)\\-(2(3)-0(-2))\\2(1)-0(1)\end{pmatrix}=\begin{pmatrix}5\\-6\\2\end{pmatrix}.$

4.2.3.1 Properties of the Vector Product

1. $\mathbf{a}\times\mathbf{b}$ is a vector	E.g. $\begin{pmatrix}1\\1\\-1\end{pmatrix}\times\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{pmatrix}4\\-3\\1\end{pmatrix}$
2. $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$ (*not commutative*)	By definition $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ have the same magnitude but are in opposite directions. Why?
3. $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ $(\mathbf{b}+\mathbf{c})\times\mathbf{a}=\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}$ (*distributive over addition*)	E.g. $\begin{pmatrix}2\\-1\\4\end{pmatrix}\times\left(\begin{pmatrix}0\\3\\5\end{pmatrix}+\begin{pmatrix}1\\1\\2\end{pmatrix}\right)=\begin{pmatrix}2\\-1\\4\end{pmatrix}\times\begin{pmatrix}0\\3\\5\end{pmatrix}+\begin{pmatrix}2\\-1\\4\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}$
4. $(\lambda\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(\lambda\mathbf{b})=\lambda(\mathbf{a}\times\mathbf{b})$ where $\lambda\in\mathbb{R}$	E.g. $2\begin{pmatrix}2\\-1\\4\end{pmatrix}\times\begin{pmatrix}0\\3\\5\end{pmatrix}=\begin{pmatrix}2\\-1\\4\end{pmatrix}\times 2\begin{pmatrix}0\\3\\5\end{pmatrix}=2\left(\begin{pmatrix}2\\-1\\4\end{pmatrix}\times\begin{pmatrix}0\\3\\5\end{pmatrix}\right)$
5. $\mathbf{a}\times\mathbf{a}=\mathbf{0}$	By definition, $\lvert\mathbf{a}\times\mathbf{b}\rvert=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$ Hence $\lvert\mathbf{a}\times\mathbf{a}\rvert=\lvert\mathbf{a}\rvert\lvert\mathbf{a}\rvert\sin 0^\circ=0$ Thus $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ (Note: it is a zero vector)
6. If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors, $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ $\Leftrightarrow$ **a** and **b** are parallel	$(\Rightarrow)$ If $\mathbf{a}\times\mathbf{b}=\mathbf{0}\Rightarrow\lvert\mathbf{a}\times\mathbf{b}\rvert=0\Rightarrow\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta=0\Rightarrow\sin\theta=0$ (since $\lvert\mathbf{a}\rvert\neq 0$ and $\lvert\mathbf{b}\rvert\neq 0$) thus $\theta=0^\circ$ or 180° $(\Leftarrow)$ If **a** and **b** are parallel, $\theta=0^\circ$ or 180°, then $\sin\theta=0$, thus $\mathbf{a}\times\mathbf{b}=\mathbf{0}$.

Example 4.16 $\begin{pmatrix}4\\1\\-1\end{pmatrix}\times\begin{pmatrix}2\\2\\-1\end{pmatrix}=\begin{pmatrix}-1-(-2)\\-(-4-(-2))\\8-2\end{pmatrix}=\begin{pmatrix}1\\2\\6\end{pmatrix}$

Find the unit vector(s) which are perpendicular to both $\mathbf{a}=4\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

Solution:

A vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{a}\times\mathbf{b}=$ $\begin{pmatrix}4\\1\\-1\end{pmatrix}\times\begin{pmatrix}2\\2\\-1\end{pmatrix}=\begin{pmatrix}1\\2\\6\end{pmatrix}$

Hence a unit vector perpendicular to **a** and **b** is $\dfrac{\mathbf{a}\times\mathbf{b}}{\lvert\mathbf{a}\times\mathbf{b}\rvert}=$ $\dfrac{1}{\sqrt{1^2+2^2+6^2}}\begin{pmatrix}1\\2\\6\end{pmatrix}=\dfrac{1}{\sqrt{41}}\begin{pmatrix}1\\2\\6\end{pmatrix}$

The other unit vector perpendicular to **a** and **b** is	$\frac{-1}{\sqrt{41}}\begin{pmatrix}1\\2\\6\end{pmatrix}$

Example 4.17

Given that $\mathbf{b}$ is a non-zero vector and $2\mathbf{a}\times\mathbf{b}=3\mathbf{b}\times\mathbf{c}$, show that $2\mathbf{a}+3\mathbf{c}=\lambda\mathbf{b}$ for some $\lambda\in\mathbb{R}$.

Solution:

$2\mathbf{a}\times\mathbf{b}=3\mathbf{b}\times\mathbf{c}$

$2\mathbf{a}\times\mathbf{b}-3\mathbf{b}\times\mathbf{c}=\mathbf{0}$

$2\mathbf{a}\times\mathbf{b}+3\mathbf{c}\times\mathbf{b}=\mathbf{0}$

$(2\mathbf{a}+3\mathbf{c})\times\mathbf{b}=\mathbf{0}$

Thus $2\mathbf{a}+3\mathbf{c}$ is parallel to $\mathbf{b}$, ie. $2\mathbf{a}+3\mathbf{c}=\lambda\mathbf{b}$ for some $\lambda\in\mathbb{R}$.

Self-Review 4.7
If $\mathbf{a}=\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{b}=\mathbf{i}-\mathbf{j}$ and $\mathbf{c}=2\mathbf{i}+\mathbf{k}$, find $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

[$-\mathbf{i}-3\mathbf{j}+2\mathbf{k}$; $\mathbf{i}-\mathbf{j}$]

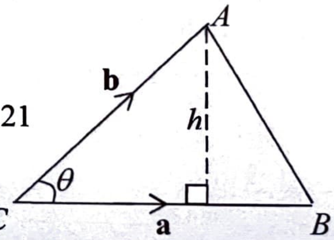
4.2.3.2 Finding Areas of Triangles and Parallelograms

Area of $\Delta ABC=\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

Proof: Area of $\Delta ABC=\frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta$

$=\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

Fig. 4.21



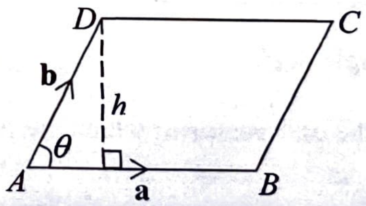
Area of parallelogram $ABCD=|\mathbf{a}\times\mathbf{b}|$

Proof: Area of $ABCD$ = base × height

$=|\mathbf{a}||\mathbf{b}|\sin\theta$

$=|\mathbf{a}\times\mathbf{b}|$

Fig. 4.22



Example 4.18

The points A, B, C, and D have coordinates $(1,2,0)$, $(5,4,0)$, $(7,5,3)$ and $(3,3,3)$ respectively.

(i) Find the area of triangle ABD.

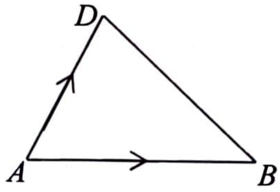
(ii) Show that $ABCD$ is a parallelogram and hence find the area of $ABCD$.

Solution:

The concept of vectors is one of the most useful ideas in applied mathematics.

(i) $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}5\\4\\0\end{pmatrix}-\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}4\\2\\0\end{pmatrix}$

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=\begin{pmatrix}3\\3\\3\end{pmatrix}-\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}$



Area of triangle $ABD=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|$

$=\frac{1}{2}\left|\begin{pmatrix}4\\2\\0\end{pmatrix}\times\begin{pmatrix}2\\1\\3\end{pmatrix}\right|$

$=\frac{1}{2}\left|\begin{pmatrix}6\\-12\\0\end{pmatrix}\right|=\frac{1}{2}(6)\left|\begin{pmatrix}1\\-2\\0\end{pmatrix}\right|=3\sqrt{5}$ sq. units.

(ii) Since $\overrightarrow{DC}=\overrightarrow{OC}-\overrightarrow{OD}=\begin{pmatrix}7\\5\\3\end{pmatrix}-\begin{pmatrix}3\\3\\3\end{pmatrix}=\begin{pmatrix}4\\2\\0\end{pmatrix}=\overrightarrow{AB}$.

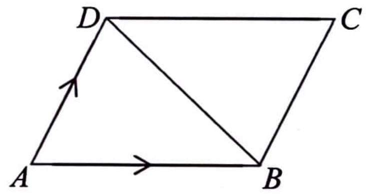
Thus, $ABCD$ is a parallelogram.

Area of parallelogram $ABCD$ $=2\times$ Area of triangle ABD

$=6\sqrt{5}$ sq. units

ThinkZone:

Can we find the area of triangle ABD using $\frac{1}{2}\left|\overrightarrow{BD}\times\overrightarrow{BA}\right|$?



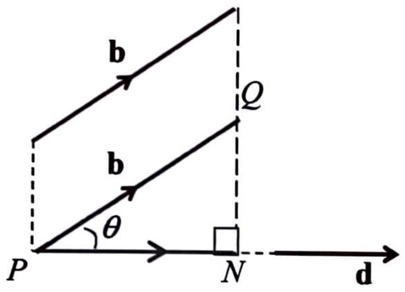
Do we also need to show that $\overrightarrow{AD}=\overrightarrow{BC}$?

Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

4.3 Vector Projections

4.3.1 Projection and Length of Projection of a Vector onto Another Vector

Fig. 4.23



Given 2 non-parallel vectors $\mathbf{b}$ and $\mathbf{d}$,

$\overrightarrow{PN}$ is called the **projection of b onto d**

PN is the **length of projection of b onto d**

$$PN = |\mathbf{b}|\cos\theta$$
$$= |\mathbf{b}||\hat{\mathbf{d}}|\cos\theta \quad (\because |\hat{\mathbf{d}}| = 1)$$
$$= \mathbf{b}\cdot\hat{\mathbf{d}}$$

Note: The projection of **b** onto **d** is different from the projection of **d** onto **b**.

Recall that if θ is obtuse, then the angle between **b** and **d** is obtuse and so $\mathbf{b}\cdot\hat{\mathbf{d}} < 0$. Since length is always positive, we have

> Length of projection of **b** onto **d** = $|\mathbf{b}\cdot\hat{\mathbf{d}}|$

> Projection of **b** onto **d**, $\overrightarrow{PN} = (\mathbf{b}\cdot\hat{\mathbf{d}})\hat{\mathbf{d}}$
>
> **Note:**
> If the angle θ between **b** and **d** is obtuse, then $\overrightarrow{PN}$ is in the opposite direction as **d**.

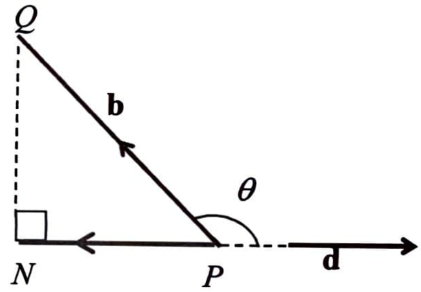


Fig. 4.24

On the other hand, the perpendicular distance from Q to PN is QN.

$$QN = |\mathbf{b}|\sin\theta$$
$$= |\mathbf{b}||\hat{\mathbf{d}}|\sin\theta = |\mathbf{b}\times\hat{\mathbf{d}}|$$

Hence

> the shortest (perpendicular) distance from point Q to PN is given by $|\overrightarrow{QN}| = |\mathbf{b}\times\hat{\mathbf{d}}|$

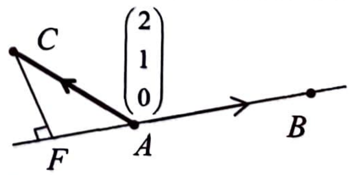
Example 4.19

The points A, B and C have coordinates $(1,0,-1)$, $(-1,2,1)$ and $(3,1,-1)$ respectively. Find the length of projection of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$. Hence, or otherwise, find the length of the perpendicular from C to $\overrightarrow{AB}$.

Solution

Let F be the foot of the perpendicular from C to $\overrightarrow{AB}$.

$$\overrightarrow{AC} = \begin{pmatrix}3\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}, \quad \overrightarrow{AB} = \begin{pmatrix}-1\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}-2\\2\\2\end{pmatrix}$$



The concept of vectors is one of the most useful ideas in applied mathematics.

Length of projection of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$, $\left|\overrightarrow{AF}\right| = \left|\overrightarrow{AC}\bullet\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}\right| = \left|\begin{pmatrix}2\\1\\0\end{pmatrix}\bullet\frac{1}{\sqrt{12}}\begin{pmatrix}-2\\2\\2\end{pmatrix}\right| = \frac{1}{\sqrt{3}}$

[Note that $\overrightarrow{AC}\bullet\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} < 0 \Rightarrow$ angle between AC and AB is obtuse.]

"Hence" method (By Pythagoras' Theorem)

Let $\left|\overrightarrow{CF}\right|$ be the perpendicular distance.

$$\left|\overrightarrow{CF}\right|^2 + \left|\overrightarrow{AF}\right|^2 = \left|\overrightarrow{AC}\right|^2$$

$$\left|\overrightarrow{CF}\right| = \sqrt{\left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AF}\right|^2}$$

$$= \sqrt{(2^2+1^2+0^2) - \left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \sqrt{5-\frac{1}{3}} = \sqrt{\frac{14}{3}}$$

"Otherwise" method (Use of cross product)

$$\left|\overrightarrow{CF}\right| = \left|\overrightarrow{AC}\times\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}\right| = \left|\begin{pmatrix}2\\1\\0\end{pmatrix}\times\frac{1}{\sqrt{12}}\begin{pmatrix}-2\\2\\2\end{pmatrix}\right|$$

$$= \frac{1}{\sqrt{12}}\left|\begin{pmatrix}2\\-4\\6\end{pmatrix}\right| = \frac{1}{\sqrt{3}}\left|\begin{pmatrix}1\\-2\\3\end{pmatrix}\right|$$

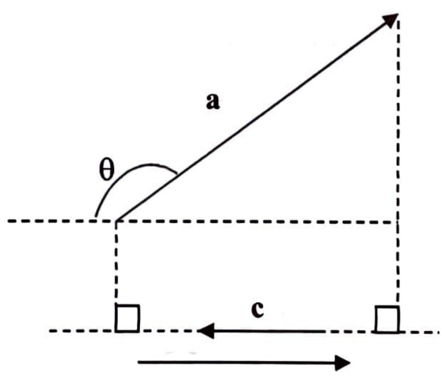
$$= \frac{\sqrt{14}}{\sqrt{3}} = \sqrt{\frac{14}{3}} \text{ units}$$

Example 4.20

Find $\mathbf{a}\bullet\hat{\mathbf{c}}$ if $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{j}$. Hence find the projection of **a** onto **c** and its length.

Solution:

$$\mathbf{a}\bullet\hat{\mathbf{c}} = \begin{pmatrix}4\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}0\\3\\0\end{pmatrix}\frac{1}{3} = -3$$



Projection of **a** onto **c**

$$= (\mathbf{a}\bullet\hat{\mathbf{c}})\hat{\mathbf{c}} = (-3)\begin{pmatrix}0\\3\\0\end{pmatrix}\frac{1}{3} = -3\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-3\\0\end{pmatrix}.$$

Length of projection of **a** onto **c** = 3

ThinkZone:

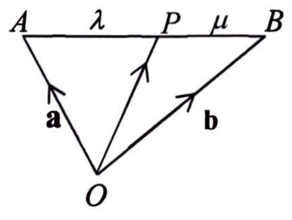
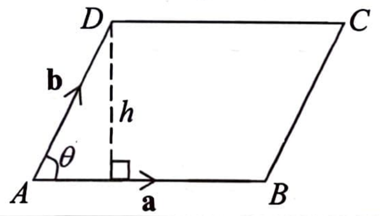
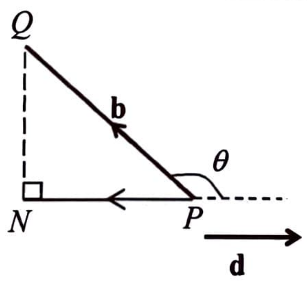
If $\mathbf{a}\bullet\hat{\mathbf{c}} < 0$, what does it mean for the angle between **a** and $\hat{\mathbf{c}}$?

Self-Review 4.8

Relative to an origin O, the points A, B and C have position vectors $-3\mathbf{i} + 8\mathbf{j} + \mathbf{k}$, $7\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ and $-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ respectively. Find

(i) the lengths of the projection of $\overrightarrow{AB}$ onto $\overrightarrow{BC}$ and $\overrightarrow{BC}$ onto $\overrightarrow{AB}$, [10 units; $5\sqrt{6}$ units;]

(ii) the perpendicular distance from C to the line AB. [$5\sqrt{3}$ units]

Summary of Main Concepts of Vectors	
1. Collinearity and non-parallel vectors	Three distinct points *A*, *B* and *C* are collinear if and only if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some real scalar *k*. If **a** and **b** are non-zero and non-parallel vectors, $\lambda\mathbf{a} = \mu\mathbf{b} \Rightarrow \lambda = \mu = 0$
2. Ratio Theorem	$\overrightarrow{OP} = \dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{\mu + \lambda}$ 
3. Scalar Product	$\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$ where θ is the angle between **a** and **b** ($0° \le \theta \le 180°$) • To find angle between two vectors • To show perpendicular vectors (if **a** and **b** are non-zero vectors and $\mathbf{a}\cdot\mathbf{b} = 0$, then $\mathbf{a} \perp \mathbf{b}$)
4. Vector Product	$\lvert\mathbf{a}\times\mathbf{b}\rvert = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$ where θ is the angle between **a** and **b** ($0° \le \theta \le 180°$) • To find a vector perpendicular to both **a** and **b** • To show parallel vectors (if **a** and **b** are non-zero vectors and $\mathbf{a}\times\mathbf{b} = \mathbf{0}$, then $\mathbf{a} \mathbin{//} \mathbf{b}$)
5. Area of a triangle and a parallelogram	• Area of triangle *ABD* $= \frac{1}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert$ • Area of parallelogram *ABCD* $= \lvert\mathbf{a}\times\mathbf{b}\rvert$ 
6. Projection of a vector onto another vector and length of projection	 • length of projection of **b** onto **d** = $\lvert\mathbf{b}\cdot\hat{\mathbf{d}}\rvert$ • Projection of **b** onto **d**, $\overrightarrow{PN} = (\mathbf{b}\cdot\hat{\mathbf{d}})\hat{\mathbf{d}}$ • Shortest (perpendicular) distance from point *Q* to *PN* is given by $\lvert\overrightarrow{QN}\rvert = \lvert\mathbf{b}\times\hat{\mathbf{d}}\rvert$.

4.4 Equation of a Line

In coordinate geometry, the cartesian equation of a line consists of points whose cartesian coordinates satisfy the cartesian equation of the line. Similarly in vector geometry, the vector equation of a line consists of infinitely many points whose position vectors satisfy a given vector equation.

4.4.1 Vector Equation of a Line

In this section, we are interested in finding the vector equation of a line in 3-D space.

Let **a** be the position vector of a point A on the line l and **d** be a vector parallel to the line l. If P is a point on the line l, then

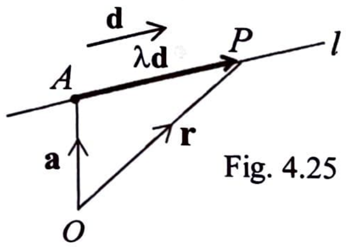


Fig. 4.25

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$

for some real scalar λ.

Thus, the line l passing through point A and parallel to **d** has vector equation

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \quad \lambda \in \mathbb{R}$$

Note:

1. λ is a real scalar.
2. **d** is called a **direction vector** of the line l.
3. **a** is the position vector of a particular point A on the line l.
4. **r** is the position vector of a point P on the line corresponding to a given value of λ. As λ varies over all real values, we obtain the position vector of every point on the line
5. In general, **r** is *not parallel* to **d** and hence not parallel to the line l.

Q: When is **r** parallel to l?

To find the vector equation of a line l passing two given points
Let the two given points be A and B with position vector **a** and **b** respectively.
Let position vector of a point P on line l be **r**.
Then

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}, \quad \lambda \in \mathbb{R}$$
$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}), \quad \lambda \in \mathbb{R}.$$

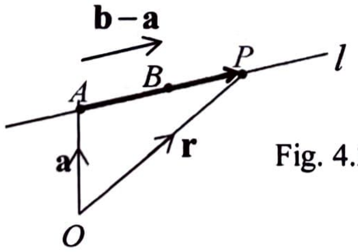


Fig. 4.26

Q: Is the vector equation of a line unique?

Example 4.21

The points A and B have coordinates $(1,0,-1)$ and $(-1,2,1)$ respectively. Find an equation of the line that passes through

(i) O and A,

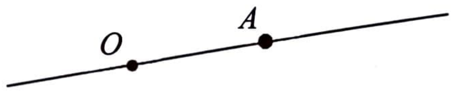
(ii) A and is parallel to the vector $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$,

(iii) A and B.

Solution:

(i) The line passing through O and is parallel to $\overrightarrow{OA}$ has vector equation

$$\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}=\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix},\quad \lambda\in\mathbb{R}.$$

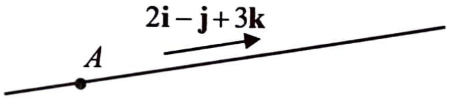


Alternatively, $\mathbf{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}, \lambda\in\mathbb{R}$ is also acceptable.

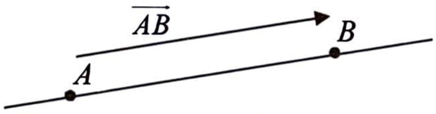
(ii)

The line passing through A and is parallel to $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ has vector equation

$$\mathbf{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix},\quad \lambda\in\mathbb{R}.$$



(iii)



The line passing through A and B is parallel $\overrightarrow{AB}=\begin{pmatrix}-1\\2\\1\end{pmatrix}-\begin{pmatrix}1\\0\\-1\end{pmatrix}$
$=\begin{pmatrix}-2\\2\\2\end{pmatrix}$ $\lambda\in\mathbb{R}$

another vector equation can be

$r=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\mu\begin{pmatrix}-1\\1\\1\end{pmatrix}, \mu\in\mathbb{R}$

ThinkZone:

Note that $\overrightarrow{AB}\,//\begin{pmatrix}-1\\1\\1\end{pmatrix}$

Is

$\mathbf{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}\ \lambda\in\mathbb{R}$

another possible equation of the line?

Note:
- The vector equation of a line is not unique.
- If a line passes through the origin, the equation of the line can be written as $\mathbf{r}=\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$.

4.4.2 Cartesian Equation of a Line

Given the vector equation of a line $\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \ \lambda \in \mathbb{R}$

By writing $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ and equating each row, we get

$$x = a_1 + \lambda d_1$$
$$y = a_2 + \lambda d_2$$
$$z = a_3 + \lambda d_3$$

where $\lambda \in \mathbb{R}$

If $d_i \neq 0$, $i = 1,2,3$, we have $\lambda = \dfrac{x-a_1}{d_1}$, $\lambda = \dfrac{y-a_2}{d_2}$ and $\lambda = \dfrac{z-a_3}{d_3}$. Hence we have

> Equation of a line in **Cartesian form** is
> $$\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$$

Example 4.22

Express the following vector equations in Cartesian form.

(a) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix}, \ \lambda \in \mathbb{R}$,

(b) $\mathbf{r} = (2+\mu)\mathbf{i} + (1-4\mu)\mathbf{j} - 3\mathbf{k}, \ \mu \in \mathbb{R}$.

Solution:

(a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix}$

$$x = 1 - 4\lambda \quad \Rightarrow \quad \lambda = \frac{x-1}{-4}$$

$$y = 2 + 5\lambda \quad \Rightarrow \quad \lambda = \frac{y-2}{5}$$

$$z = 6\lambda \quad \Rightarrow \quad \lambda = \frac{z}{6}$$

Cartesian equation: $\dfrac{x-1}{-4} = \dfrac{y-2}{5} = \dfrac{z}{6}$.

ThinkZone:
Do you think that $\dfrac{1-x}{4} = \dfrac{y-2}{5} = \dfrac{z}{6}$ is also an answer?

(b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2+\mu \\ 1-4\mu \\ -3+0\mu \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}, \ \mu \in \mathbb{R}$

$x = 2 + \mu \quad \Rightarrow \quad \mu = x - 2$

$y = 1 - 4\mu \quad \Rightarrow \quad \mu = \dfrac{y-1}{-4}$

$z = -3$

Cartesian equation: $\dfrac{x-2}{1} = \dfrac{y-1}{-4}, z = -3.$

ThinkZone:
Do you think that $\dfrac{x-2}{1} = \dfrac{1-y}{4}, \quad z = -3$ is also an answer?

Example 4.23
Express each of the following equations in vector form:

(a) $\dfrac{x-5}{3} = \dfrac{y+4}{7} = \dfrac{z-6}{2},$ (b) $x - 5 = \dfrac{2z+1}{3},\ y = -3.$

Solution:

(a) Let $\lambda = \dfrac{x-5}{3} = \dfrac{y+4}{7} = \dfrac{z-6}{2}$

Then $x = 5 + 3\lambda$
$y = -4 + 7\lambda$
$z = 6 + 2\lambda$

Hence $\mathbf{r} = \begin{pmatrix} 5 \\ -4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R}.$

(b) Hence $\underset{\sim}{r_1} = \begin{pmatrix} 5 \\ -3 \\ -\frac{1}{2} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ \frac{3}{2} \end{pmatrix}, \lambda \in \mathbb{R}$

Another equation could be

$\mathbf{r} = \begin{pmatrix} 5 \\ -3 \\ -\frac{1}{2} \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \ \mu \in \mathbb{R}$

Self-Review 4.9

Express the Cartesian equation $\dfrac{1-x}{3} = \dfrac{y}{5} = z$ in vector form. $\left[\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}\right]$

4.4.3 Problems Involving a Point and a Line

4.4.3.1 To determine if a given point lies on a line:

A point lies on a line if the position vector of the point satisfies the equation of the line.

Example 4.24

Find an equation of the line l that passes through the point with position vector $2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$ and is parallel to the vector $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. Determine whether l passes through the following points:

(i) A with coordinates $(3, 6, 11)$,

(ii) the origin O.

Solution:

Equation of line l, $\quad \mathbf{r}=\begin{pmatrix}2\\4\\8\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\4\end{pmatrix}, \lambda\in\mathbb{R}$.

(i) If A lies in line l, then $\begin{pmatrix}3\\6\\11\end{pmatrix}=\begin{pmatrix}2\\4\\8\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\4\end{pmatrix}$ for some $\lambda\in\mathbb{R}$.

$$3=2+\lambda \quad ---(1)$$
$$6=4+2\lambda \quad ---(2)$$
$$11=8+4\lambda \quad ---(3)$$

From (1), $\lambda=1$. From (3), $\lambda=\dfrac{3}{4}$.

System of equations is inconsistent so A does not lie in l.

(ii) If origin lies in line l, then $\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}2\\4\\8\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\4\end{pmatrix}$ for some $\lambda\in\mathbb{R}$.

$0=2+\lambda \quad ---(1)$

$0=4+2\lambda \quad ---(2)$

$0=8+4\lambda \quad ---(3)$

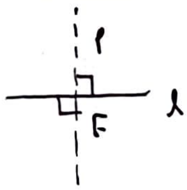
Since $\lambda=-2$ is consistent for all 3 equations, origin lies in line l.

Alternatively, by observation when $\lambda=-2$, $\mathbf{r}=\begin{pmatrix}2\\4\\8\end{pmatrix}-2\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$.

Hence l passes through the origin O.

4.4.3.2 Find

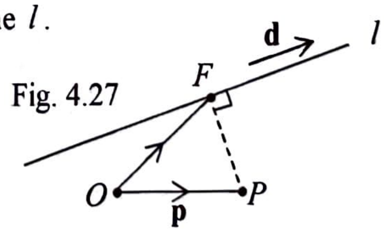
- **foot of perpendicular from a point to a line**
- **perpendicular (shortest) distance from a point to a line**
- **reflection of a point in a line**



Method to find position vector of foot of perpendicular:

Let P be a point with position vector $\mathbf{p}$ and l be the line with equation $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$.
Let F be the foot of the perpendicular from the point P to the line l.

Fig. 4.27



Step 1: Since F lies on l, $\overrightarrow{OF}=\mathbf{a}+\lambda\mathbf{d}$ for some $\lambda\in\mathbb{R}$.

Step 2: Find $\overrightarrow{PF}=\overrightarrow{OF}-\overrightarrow{OP}$ in terms of λ.

Step 3: $\overrightarrow{PF}\perp\mathbf{d} \Rightarrow \overrightarrow{PF}\cdot\mathbf{d}=0$ for some particular value of λ. Solve equation for λ.

Step 4: On knowing the value of λ, we can substitute this value of λ in $\overrightarrow{OF}$ to find the position vector of F.

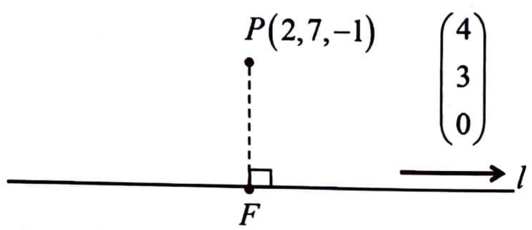
Example 4.25

The equation of a straight line l is $\mathbf{r}=(1+4\lambda)\mathbf{i}+3\lambda\mathbf{j}+2\mathbf{k}$, $\lambda\in\mathbb{R}$ and the point P has coordinates $(2,7,-1)$. Find the position vector of the foot of the perpendicular from P to l. Hence find

(i) the shortest distance from P to l.

(ii) the position vector of P', the reflection of P in the line l.

Solution:



l: $\mathbf{r}=\begin{pmatrix}1+4\lambda\\3\lambda\\2\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\0\end{pmatrix}, \lambda\in\mathbb{R}$

Step 1: Let F be the foot of the perpendicular from P to l.

Then $\overrightarrow{OF}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\0\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. -------- (*)

Step 2: $\overrightarrow{PF}=\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\0\end{pmatrix}-\begin{pmatrix}2\\7\\-1\end{pmatrix}$

$=\begin{pmatrix}-1\\-7\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\0\end{pmatrix}$ ------- (**)

Step 3: $\overrightarrow{PF}\perp\begin{pmatrix}4\\3\\0\end{pmatrix}\Rightarrow\overrightarrow{PF}\bullet\begin{pmatrix}4\\3\\0\end{pmatrix}=0$

$\left(\begin{pmatrix}-1\\-7\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\0\end{pmatrix}\right)\bullet\begin{pmatrix}4\\3\\0\end{pmatrix}=0$

$\begin{pmatrix}-1\\-7\\3\end{pmatrix}\bullet\begin{pmatrix}4\\3\\0\end{pmatrix}+\begin{pmatrix}4\\3\\0\end{pmatrix}\bullet\begin{pmatrix}4\\3\\0\end{pmatrix}\lambda=0$

$-25+25\lambda=0\Rightarrow\lambda=1$

Step 4: Hence $\overrightarrow{OF}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\begin{pmatrix}4\\3\\0\end{pmatrix}=\begin{pmatrix}5\\3\\2\end{pmatrix}$ from (*).

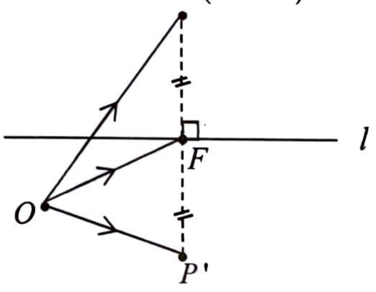
The concept of vectors is one of the most useful ideas in applied mathematics.

(i) $\overrightarrow{PF}=\begin{pmatrix}-1+4\\-7+3\\3+0\end{pmatrix}=\begin{pmatrix}3\\-4\\3\end{pmatrix}$ from (**)

$\left|\overrightarrow{PF}\right|=\sqrt{3^2+4^2+3^2}=\sqrt{34}$

Hence the shortest distance from P to l is $\sqrt{34}$ units.

(ii)



Using Ratio Theorem,

$$\overrightarrow{OF}=\frac{1}{2}\left(\overrightarrow{OP}+\overrightarrow{OP'}\right)\Rightarrow\overrightarrow{OP'}=2\overrightarrow{OF}-\overrightarrow{OP}$$

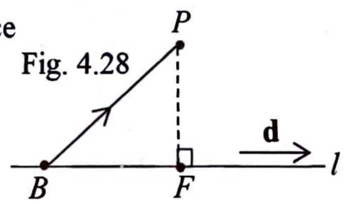
$$=2\begin{pmatrix}5\\3\\2\end{pmatrix}-\begin{pmatrix}2\\7\\-1\end{pmatrix}=\begin{pmatrix}8\\-1\\5\end{pmatrix}.$$

Alternative Approach to find the perpendicular distance from a point to a line:

Perpendicular distance from P to line l,

$PF=\left|\overrightarrow{BP}\times\hat{\mathbf{d}}\right|$

Fig. 4.28



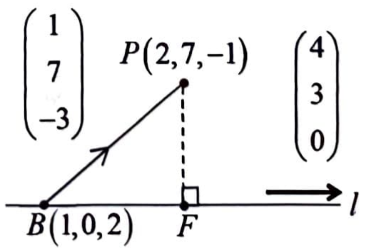
ThinkZone:
The formula is obtained from Section 4.3.1.
This result directly finds the perpendicular distance from a point to a line l without finding the position vector of foot of perpendicular.

In **Example 4.25** above, if the question did not ask for the position vector of the foot of perpendicular from P to l, then we can find the shortest distance from P to l by the above cross product approach.

$$\left|\overrightarrow{PF}\right|=\left|\overrightarrow{BP}\times\frac{1}{5}\begin{pmatrix}4\\3\\0\end{pmatrix}\right|$$

$$=\left|\begin{pmatrix}1\\7\\-3\end{pmatrix}\times\frac{1}{5}\begin{pmatrix}4\\3\\0\end{pmatrix}\right|=\frac{1}{5}\left|\begin{pmatrix}9\\-12\\-25\end{pmatrix}\right|=\sqrt{34}$$

Fig. 4.29



Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

Example 4.26

A space shuttle travels with a constant velocity of $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$ and passes through a point $(1,2,3)$ relative to a space station in space, assumed fixed. The time after which the space shuttle passes through the point $(1,2,3)$ is denoted by t.

(i) Write down the equation of the line of travel of the space shuttle in terms of t.

(ii) Regarding the space shuttle and the space station as points in space, find the least distance apart between the space shuttle and the space station.

Solution:

(i) The equation of the line of travel of the space shuttle is

$$\mathbf{r} = \underbrace{\begin{pmatrix}1\\2\\3\end{pmatrix}}_{\substack{\text{position}\\\text{vector of}\\\text{of initial}\\\text{point}}} + t\underbrace{\begin{pmatrix}2\\-1\\-2\end{pmatrix}}_{\substack{\text{velocity}\\\text{vector}}},\quad t \geq 0.$$

(ii) Let P be the point on the line of travel of the space shuttle which is closest to the space station.

Then $\overrightarrow{OP} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ for some $t \geq 0$.

Since $\overrightarrow{OP} \perp \begin{pmatrix}2\\-1\\-2\end{pmatrix}$ at closest approach,

$$\left[\begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right] \bullet \begin{pmatrix}2\\-1\\-2\end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \begin{pmatrix}2\\-1\\-2\end{pmatrix} + t\begin{pmatrix}2\\-1\\-2\end{pmatrix} \bullet \begin{pmatrix}2\\-1\\-2\end{pmatrix} = 0$$

$$\Rightarrow -6 + 9t = 0$$

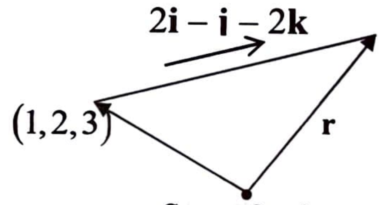
$$\Rightarrow t = \frac{2}{3}$$

Least distance apart

$$= OP = \left|\begin{pmatrix}1\\2\\3\end{pmatrix} + \frac{2}{3}\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right| = \frac{1}{3}\sqrt{90} = \sqrt{10} \text{ units}.$$

ThinkZone

Why is $\begin{pmatrix}1\\2\\3\end{pmatrix}$ the position vector of the initial point in the equation of the line of travel of the space shuttle?



Note: The space station is regarded as the fixed origin O.

P lies on the line of travel of the space shuttle.

Self-Review 4.10

Find the perpendicular distance from the origin, O, to the line $\mathbf{r} = (1-\lambda)\mathbf{i} + (2\lambda - 1)\mathbf{j} + \lambda\mathbf{k},\ \lambda \in \mathbb{R}$.

$[\sqrt{2}/2]$

4.4.4 Problems Involving Two Lines

4.4.4.1 Determine if two lines are parallel, skew or intersecting at a point

In 3-dimensional space, two straight lines may be

- coplanar (lying on the same plane) which means the lines are either parallel or intersecting
- non-coplanar which means the lines are non-parallel and non-intersecting, called **skew lines.**

Note: Two lines are parallel if their direction vectors are parallel. If two lines are parallel, then they are necessarily coplanar since it is always possible to create a plane that contains the two lines.

Example 4.27
Determine whether the following pairs of lines are parallel, intersecting, or skew. In the case where the two lines intersect, find the position vector of their point of intersection.

(a) l_1: $\mathbf{r}=\begin{pmatrix}1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\4\end{pmatrix}$, $\lambda\in\mathbb{R}$ and l_2: $\mathbf{r}=\begin{pmatrix}2\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}-6\\4\\-8\end{pmatrix}$, $\mu\in\mathbb{R}$,

(b) l_3: $\mathbf{r}=\begin{pmatrix}1\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$and l_4: $\mathbf{r}=\begin{pmatrix}2\\4\\6\end{pmatrix}+\mu\begin{pmatrix}2\\1\\3\end{pmatrix}$, $\mu\in\mathbb{R}$,

(c) l_5: $\mathbf{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\4\end{pmatrix}$, $\lambda\in\mathbb{R}$ and l_6: $\mathbf{r}=\begin{pmatrix}2\\3\\0\end{pmatrix}+\mu\begin{pmatrix}4\\-1\\1\end{pmatrix}$, $\mu\in\mathbb{R}$.

Solution:

Let $\underset{\sim}{d}_i$ be the direction vector of line l_i, where $i=1,2,3,\cdots,6$

(a) Since $\mathbf{d}_2=\begin{pmatrix}-6\\4\\-8\end{pmatrix}=-2\begin{pmatrix}3\\-2\\4\end{pmatrix}=-2\mathbf{d}_1$, the direction vectors of the two lines are parallel.

The two lines could be the same line.

Assuming the 2 lines are the same, $\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}2\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}-6\\4\\-8\end{pmatrix}$

$$1=2-6\mu\text{ --------(1)}$$
$$\Rightarrow\ 1=-1+4\mu\text{ -------(2)}$$
$$2=3-8\mu\text{ --------(3)}$$

Solving (1) will give $\mu=\frac{1}{6}$. However for (2), $\mu=\frac{1}{2}$.

Since μ is inconsistent for the 3 equations, the point (1, 1, 2) that lies on l_1 does not lie on l_2.

Hence the two lines are parallel but not intersecting.

(b) The two lines are non-parallel since $\mathbf{d}_3=\begin{pmatrix}1\\-1\\1\end{pmatrix}\neq k\begin{pmatrix}2\\1\\3\end{pmatrix}=k\mathbf{d}_4$, $k\in\mathbb{R}$

Assume that the two lines intersect. Suppose the two lines intersect at X. Then

$\overrightarrow{OX} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{OX} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ for some $\lambda, \mu \in \mathbb{R}$

$\therefore \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ for some $\lambda, \mu \in \mathbb{R}$.

$$\lambda - 2\mu = 1 \quad \text{\_\_\_(1)}$$
$$\Rightarrow \quad -\lambda - \mu = 5 \quad \text{\_\_\_(2)}$$
$$\lambda - 3\mu = 3 \quad \text{\_\_\_(3)}$$

Solving (1) and (2) simultaneously, we get $\lambda = -3$ and $\mu = -2$.

We need to check if the above values of λ and μ satisfy all 3 equations.
Sub. $\lambda = -3$ and $\mu = -2$ in (3):

$\text{LHS} = (-3) - 3(-2) = 3 = \text{RHS}$

Hence the two lines intersect.

Sub $\lambda = -3$: The point of intersection has position vector $\overrightarrow{OX} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + (-3) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$.

Alternatively, we can also substitute $\mu = -2$ to find $\overrightarrow{OX} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + (-2) \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$.

(c)

$d_5 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \neq k \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = k d_6 \quad k \in \mathbb{R}$

observation, the two lines are not parallel

Suppose the two lines intersect. Then

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ for some $\lambda, \mu \in \mathbb{R}$

$$\lambda - 4\mu = 1 \text{ --- (1)}$$
$$\Rightarrow 3\lambda + \mu = 3 \text{ --- (2)}$$
$$4\lambda - \mu = -1 \text{ --- (3)}$$

solving (2) and (3) simultaneously, we get $\lambda = \frac{2}{7}$ and $\mu = \frac{15}{7}$

Sub $\lambda = \frac{2}{7}$ and $\mu = \frac{15}{7}$ in (1)

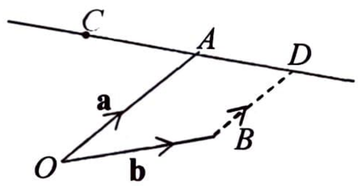
$\text{LHS} = \frac{2}{7} - 4\left(\frac{15}{7}\right) = -\frac{58}{7} \neq \text{RHS}$

Since LHS ≠ RHS, the two lines do not intersect. Since the 2 lines are non intersecting and non parallel, they are skew lines.

Example 4.28

Three non-collinear points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\frac{4}{3}\mathbf{a}-\frac{2}{3}\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. Find the position vector of the point D on CA produced such that BD is parallel to OA.

Solution:



$\overrightarrow{CA}=\mathbf{a}-\mathbf{c}=\mathbf{a}-(\frac{4}{3}\mathbf{a}-\frac{2}{3}\mathbf{b})=-\frac{1}{3}(\mathbf{a}-2\mathbf{b})$

Equation of line CA is $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{a}-2\mathbf{b})$, $\lambda\in\mathbb{R}$

Equation of line BD is $\mathbf{r}=\mathbf{b}+\mu\mathbf{a}$, $\mu\in\mathbb{R}$

At the intersection point D of lines CA and BD,

$\mathbf{a}+\lambda(\mathbf{a}-2\mathbf{b})=\mathbf{b}+\mu\mathbf{a}$ for some λ, $\mu\in\mathbb{R}$.

$\Rightarrow (1+\lambda-\mu)\mathbf{a}=(1+2\lambda)\mathbf{b}$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel,

$1+\lambda-\mu=0$ and $1+2\lambda=0$

Solving, we obtain $\lambda=-\frac{1}{2}$, $\mu=\frac{1}{2}$.

Hence $\overrightarrow{OD}=\mathbf{b}+\frac{1}{2}\mathbf{a}$.

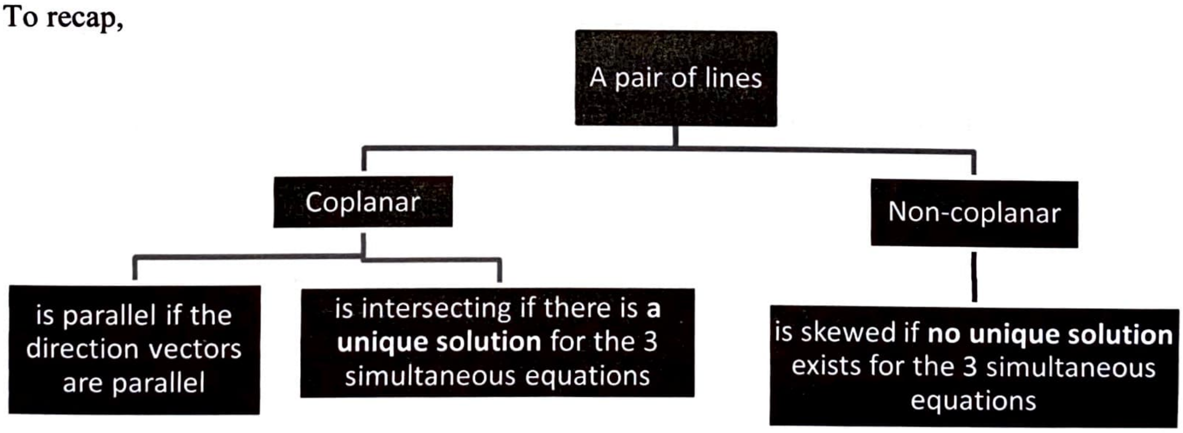
ThinkZone:

What do you understand by the words "non-collinear points" and "produced"?

See section 4.1.10.1. Alternatively, use the concepts of parallel vectors: $\overrightarrow{CA}\,//\,\overrightarrow{CD}$ & $\overrightarrow{BD}\,//\,\mathbf{a}$.

Try it on your own.

To recap,



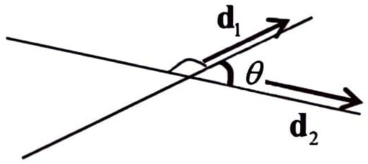
Self-Review 4.11

Determine whether the two lines $\frac{x-1}{2}=\frac{y+1}{3}=z$ and $\frac{x+1}{5}=y-2, z=2$ intersect. [No]

4.4.4.2 Finding the angle between two lines

The acute angle θ between two non-parallel lines (not necessarily intersecting) is given by $|\mathbf{d}_1 \cdot \mathbf{d}_2| = |\mathbf{d}_1||\mathbf{d}_2|\cos\theta$ where $\mathbf{d}_1$ and $\mathbf{d}_2$ are direction vectors of the two lines.

Fig. 4.30



Note: To find the angle between two skew lines, we do a parallel shift of one line until the two lines intersect. The angle between the two skew lines is defined to be the angle between the two intersecting lines.

Example 4.29

Find the acute angle between the lines:

$$p: \mathbf{r} = \begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R} \quad \text{and} \quad q: \mathbf{r} = \begin{pmatrix} -7 \\ 19 \\ 14 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix}, \mu \in \mathbb{R}$$

Solution:

$$\left| \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} \right| = \left| \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} \right| \cos\theta$$

$$|-9| = \sqrt{9}\sqrt{54}\cos\theta$$

$$\cos\theta = \frac{9}{3\sqrt{54}}$$

$$\theta = 65.9^\circ$$

ThinkZone:
What if you are asked to find the obtuse angle between lines?

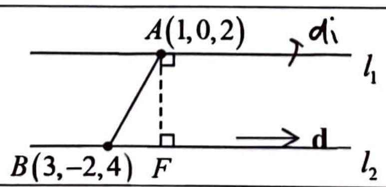
4.4.4.3 Determining the distance between two parallel lines

To find the (perpendicular) distance between two parallel lines, we first take an arbitrary point on one of the lines and then find the distance from this point to the other line as in Example 4.25.

Example 4.30

Find the distance between the lines $l_1: \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and $l_2: \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\mu \in \mathbb{R}$.

Solution:



Think Zone:

Why were A and B used?

The concept of vectors is one of the most useful ideas in applied mathematics.

Let F be the foot of the perpendicular from A with position vector $\begin{pmatrix}1\\0\\2\end{pmatrix}$ to l_2.

Can we use $|\overrightarrow{BA}\times\widehat{\mathbf{d}_2}|$?

$$\overrightarrow{AB}=\begin{pmatrix}3\\-2\\4\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}2\\-2\\2\end{pmatrix}.$$

Distance between l_1 and l_2,

$$\left|\overrightarrow{AF}\right|=|\overrightarrow{AB}\times\widehat{\mathbf{d}_2}|=\left|\begin{pmatrix}2\\-2\\2\end{pmatrix}\times\frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}\right|=\frac{1}{3}\left|\begin{pmatrix}-6\\0\\6\end{pmatrix}\right|=2\sqrt{2}\text{ units.}$$

4.4.4.4 Finding the reflection of a line in another coplanar line

To find the reflection of line m in line l

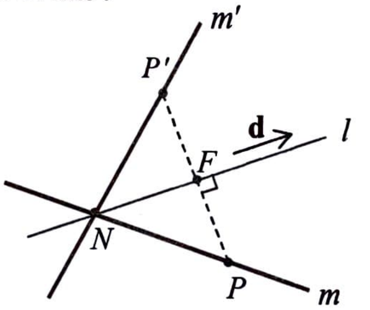


Fig 4.31

Step 1: Take a point P on line m. Find the foot of perpendicular, F from P to line l.

Step 2: Find the reflection P' of P in the line l (by using the Ratio Theorem).

Step 3: Find the point of intersection N of the lines l and m

Step 4: Find the equation of the reflected line m' which passes through points P' and N.

Example 4.31

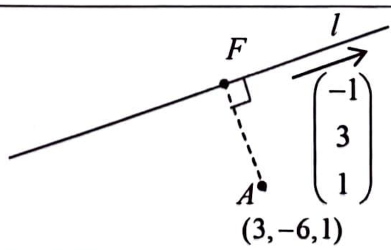
The point A has position vector $3\mathbf{i}-6\mathbf{j}+\mathbf{k}$ and the line l has equation $\mathbf{r}=7\mathbf{i}+\mathbf{j}-5\mathbf{k}+\lambda(-\mathbf{i}+3\mathbf{j}+\mathbf{k})$, $\lambda\in\mathbb{R}$.

(i) Find the position vector of the foot of perpendicular, F, from A to l, and hence find the position vector of the reflection of A in l.

(ii) Let B be a point on l with position vector $6\mathbf{i}+4\mathbf{j}-4\mathbf{k}$. Find the equation of the reflection of line AB in l.

Solution:

(i)



Mathematics is not about numbers, equations, computations or algorithms; it is about understanding.

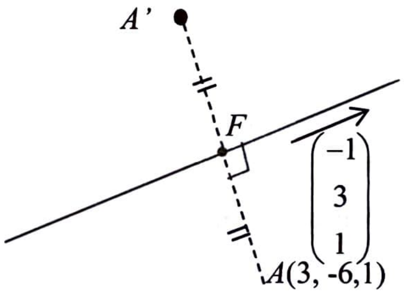
Since F lies on l, $\overrightarrow{OF} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

$$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \left(\begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right) - \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

Since $\overrightarrow{AF} \perp l$, $\overrightarrow{AF} \bullet \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = 0$

$$\left(\begin{pmatrix} 4 \\ 7 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right) \bullet \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = 0 \Rightarrow 11 + 11\lambda = 0 \Rightarrow \lambda = -1$$

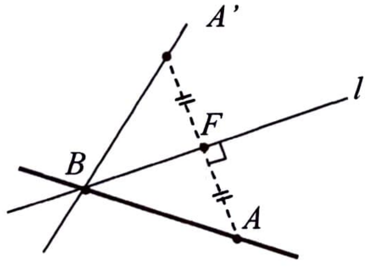
$$\overrightarrow{OF} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} + (-1) \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ -6 \end{pmatrix}.$$



By Ratio Theorem,

$$\overrightarrow{OF} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OA'} \right) \Rightarrow \overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA} = 2\begin{pmatrix} 8 \\ -2 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 2 \\ -13 \end{pmatrix}$$

(ii)



The reflection of line AB in the line l is the line $A'B$.

$$\overrightarrow{A'B} = \overrightarrow{OB} - \overrightarrow{OA'} = \begin{pmatrix} 6 \\ 4 \\ -4 \end{pmatrix} - \begin{pmatrix} 13 \\ 2 \\ -13 \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \\ 9 \end{pmatrix}$$

Equation of line $A'B$: $\mathbf{r}=\begin{pmatrix}6\\4\\-4\end{pmatrix}+\mu\begin{pmatrix}-7\\2\\9\end{pmatrix}$, $\mu\in\mathbb{R}$	

Main Concepts of Vectors (Lines)

Point and Line	Line and Line
1. Determine if a point lies on a line (E.g. 24) 2. Find the foot of perpendicular from a point to a line (E.g. 25) 3. Find the perpendicular (shortest) distance from a point to a line (E.g. 25) 4. Find the reflection of a point in a line (E.g. 25)	1. Determine if two lines are parallel (E.g. 27) 2. Determine if two non-parallel lines are intersecting or skew, and find the common point if they intersect (E.g. 27) 3. Find the (acute) angle between two lines (E.g. 29) 4. Find the distance between two parallel lines (E.g. 30) 5. Find the reflection of a line in another *coplanar* line (E.g. 31)

Self-Review 4.12 (HJC Prelim 1996/I/19)

A, *B*, *C* and *D* have coordinates $(0,-1,1)$, $(2,4,8)$, $(1,1,2)$ and $(3,3,6)$ respectively.

(i) Find the vector equations of the straight lines *AC* and *BD*.

(ii) Find the acute angle between lines *AC* and *BD*.

(iii) Determine whether the lines *AC* and *BD* are skew lines.

(iv) Find the foot of the perpendicular from *B* to the line *AC*.

(v) Hence find the shortest distance from *B* to the line *AC* and the area of triangle *ABC*.

(vi) Find the position vector of *B**, the image of *B* when reflected in the line *AC*.

(vii) Hence find the equation of the line obtained by reflecting the line *AB* in the line *AC*.

[(ii) 60^o (iv) $\left(\frac{19}{6},\frac{16}{3},\frac{25}{6}\right)$ (v) $\sqrt{\frac{107}{6}}$; $\frac{1}{2}\sqrt{107}$ (vi) $\left(\frac{13}{3},\frac{20}{3},\frac{1}{3}\right)$]