National Junior College
2016 – 2017 H2 Further Mathematics
Topic F7: Further Complex Numbers (Assignment 1 Solutions)

1.

$$\begin{aligned} z^4 &= -8+8\left(\sqrt{3}\right)\mathrm{i} \\ &= 16\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)} \\ &= 16\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}+2k\pi\right)}, k \in \mathbb{Z} \end{aligned}$$

$$\therefore z = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}+\frac{k\pi}{2}\right)}, k = -2, -1, 0, 1.$$

When $k=-2,\ z = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}-\frac{2\pi}{2}\right)} = 2\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}$.

When $k=-1,\ z = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}-\frac{\pi}{2}\right)} = 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$.

When $k=0,\ z = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}+\frac{0\pi}{2}\right)} = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}$.

When $k=2,\ z = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}+\frac{\pi}{2}\right)} = 2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$.

$$\begin{aligned} z^8+16z^4+256 &= 0 \\ z^8+16z^4+64 &= -192 \\ \left(z^4+8\right)^2 &= -192 \\ z^4+8 &= \pm\sqrt{-192} = \pm\left(8\sqrt{3}\right)\mathrm{i} \end{aligned}$$

$$z^4 = -8+\left(8\sqrt{3}\right)\mathrm{i} \text{ or } z^4 = -8-\left(8\sqrt{3}\right)\mathrm{i} = \left(-8+\left(8\sqrt{3}\right)\mathrm{i}\right)^*$$

$$z^4 = -8+\left(8\sqrt{3}\right)\mathrm{i} \Rightarrow 2\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}, 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}, 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}, 2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$$

or

$$z^4 = \left(-8+\left(8\sqrt{3}\right)\mathrm{i}\right)^* \Rightarrow 2\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{6}\right)}, 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}\right)}, 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}, 2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$$

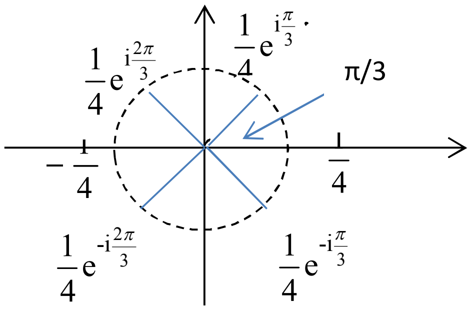
2. (i) $\left(1-\sqrt{3}\mathrm{i}\right)^{-12}=\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)^{-12}=2^{-12}\mathrm{e}^{\mathrm{i}4\pi}=2^{-12}\left[\cos(4\pi)+\mathrm{i}\sin(4\pi)\right]=\frac{1}{2^{12}}=\frac{1}{4096}$.

(ii) $z^6\left(1-\sqrt{3}\mathrm{i}\right)^{12}=1 \qquad \Rightarrow z^6=\left(1-\sqrt{3}\mathrm{i}\right)^{-12}=\frac{1}{4096}=\frac{1}{4096}\mathrm{e}^{\mathrm{i}0}$

$$\Rightarrow z^6=\frac{1}{4096}\mathrm{e}^{\mathrm{i}\pi(2n)/6}, n=0,\pm1,\pm2,3$$

$$\Rightarrow z=\frac{1}{4}\mathrm{e}^{\mathrm{i}\frac{(n)\pi}{3}}, \ n=0,\pm1,\pm2,3$$

$$\Rightarrow z=\frac{1}{4}\mathrm{e}^{\mathrm{i}\frac{\pi}{3}},\frac{1}{4}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}},\frac{1}{4}\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}},\frac{1}{4}\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}},\frac{1}{4},-\frac{1}{4}$$



(iii)

$$\left(z-re^{\mathrm{i}\theta}\right)\left(z-re^{-\mathrm{i}\theta}\right)=z^2-r(e^{\mathrm{i}\theta}+e^{-\mathrm{i}\theta})z+r^2$$

$$=z^2-r(2\cos\theta)z+r^2$$

(iv)

$$z^6\left(1-\sqrt{3}\mathrm{i}\right)^{12}-1$$

$$=4096\left(z-\frac{1}{4}\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)\left(z-\frac{1}{4}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)\left(z-\frac{1}{4}\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}\right)\left(z-\frac{1}{4}\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\right)\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)$$

$$=4096\left(z^2-\frac{1}{2}z\cos\frac{\pi}{3}+\frac{1}{16}\right)\left(z^2-\frac{1}{2}z\cos\frac{2\pi}{3}+\frac{1}{16}\right)\left(z^2-\frac{1}{16}\right)$$

$$=\left(16z^2-4z+1\right)\left(16z^2+4z+1\right)\left(16z^2-1\right)$$

3. (i)

$$\sum_{k=1}^{n}(z+z^2+\ldots+z^k)=\sum_{k=1}^{n}\frac{z\left(1-z^k\right)}{1-z}$$

$$=\frac{z}{1-z}\sum_{k=1}^{n}\left(1-z^k\right)$$

$$=\frac{z}{1-z}\left(n-\sum_{k=1}^{n}z^k\right)$$

$$=\frac{z}{1-z}\left[n-\frac{z\left(1-z^n\right)}{1-z}\right]$$

$$=\frac{nz}{1-z}-\frac{z^2}{(1-z)^2}(1-z^n)$$

(ii)

$$\frac{z}{1-z}=\frac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}}$$

$$=\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}-\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\right)}$$

$$=\frac{\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\mathrm{i}\sin\frac{\theta}{2}-\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)}$$

$$=\frac{\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}}{-2\mathrm{i}\sin\frac{\theta}{2}}\cdot\frac{\mathrm{i}}{\mathrm{i}}$$

$$=\frac{\mathrm{i}}{2\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)$$

(iii)

$$\sum_{k=1}^{n}(\sin\theta+\sin 2\theta+\ldots+\sin k\theta)$$

$$=\operatorname{Im}\sum_{k=1}^{n}(z+z^2+\ldots+z^k)$$

$$=\operatorname{Im}\left[\frac{nz}{1-z}-\frac{z^2}{(1-z)^2}(1-z^n)\right]$$

$$=\operatorname{Im}\left[n\left(\frac{\mathrm{i}}{2\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)\right)-\left(\frac{\mathrm{i}}{2\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)\right)^2\left(1-(\cos\theta+\mathrm{i}\sin\theta)^n\right)\right]$$

$$=\operatorname{Im}\left[n\left(\frac{\mathrm{i}}{2\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)\right)-\left(\frac{-(\cos\theta+\mathrm{i}\sin\theta)}{4\sin^2\frac{\theta}{2}}\right)(1-(\cos n\theta+\mathrm{i}\sin n\theta))\right]$$

$$=\frac{n\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}-\frac{\cos\theta\sin n\theta-\sin\theta+\sin\theta\cos n\theta}{4\sin^2\frac{\theta}{2}}$$

$$=\frac{2n\sin\frac{\theta}{2}\cos\frac{\theta}{2}-\cos\theta\sin n\theta+\sin\theta-\sin\theta\cos n\theta}{4\sin^2\frac{\theta}{2}}$$

$$=\frac{n\sin\theta-\cos\theta\sin n\theta+\sin\theta-\sin\theta\cos n\theta}{4\sin^2\frac{\theta}{2}}$$

$$=\frac{(n+1)\sin\theta-\sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}$$

4.

$$\begin{aligned}\cos 5\theta&=\operatorname{Re}(\cos 5\theta+\mathrm{i}\sin 5\theta)\\&=\operatorname{Re}(\cos\theta+\mathrm{i}\sin\theta)^5\\&=\operatorname{Re}[\cos^5\theta+5\cos^4\theta(\mathrm{i}\sin\theta)+10\cos^3\theta(\mathrm{i}\sin\theta)^2+10\cos^2\theta(\mathrm{i}\sin\theta)^3\\&\qquad+5\cos\theta(\mathrm{i}\sin\theta)^4+(\mathrm{i}\sin\theta)^5]\\&=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\\&=\cos^5\theta-10\cos^3\theta(1-\cos^2\theta)+5\cos\theta(1-2\cos^2\theta+\cos^4\theta)\\&=16\cos^5\theta-20\cos^3\theta+5\cos\theta\end{aligned}$$

Let $x=\cos\theta$ where $\cos 5\theta=0\Rightarrow 5\theta=\dfrac{(2n+1)\pi}{2}$

$$\Rightarrow\theta=\frac{(2n+1)\pi}{10},n=0,1,3,4$$

Hence $x=\cos\dfrac{(2n+1)\pi}{10},n=0,1,3,4$.

Since $\cos\frac{\pi}{10} = -\cos\frac{9\pi}{10}$ and $\cos\frac{3\pi}{10} = -\cos\frac{7\pi}{10}$,

$16\left(x^2 - \cos^2\frac{\pi}{10}\right)\left(x^2 - \cos^2\frac{3\pi}{10}\right) = 0$.

$\therefore 16x^2 - 20x + 5$ has roots $\cos^2\frac{\pi}{10}$ and $\cos^2\frac{3\pi}{10}$.

Using product of roots,

$\cos^2\frac{\pi}{10}\cos^2\frac{3\pi}{10} = \frac{5}{16} \Rightarrow \cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \frac{\sqrt{5}}{4}$.