JURONG PIONEER JUNIOR COLLEGE
9749 H2 PHYSICS

MOTION IN A CIRCLE

Content

(I) **Kinematics of uniform circular motion**
(II) **Centripetal acceleration**
(III) **Centripetal force**

Learning Outcomes

Candidates should be able to:

(a) express angular displacement in radians.

(b) show an understanding of and use the concept of angular velocity to solve problems.

(c) recall and use $v = r\omega$ to solve problems.

(d) describe qualitatively motion in a curved path due to a perpendicular force, and understand the centripetal acceleration in the case of uniform motion in a circle.

(e) recall and use centripetal acceleration $a = r\omega^2$, and $a = \dfrac{v^2}{r}$ to solve problems.

(f) recall and use centripetal force $F = mr\omega^2$, and $F = \dfrac{mv^2}{r}$ to solve problems.

Introduction

- In the previous topic on Kinematics, we learnt about uniformly accelerated motion. An object accelerating along a straight line experiences a net force that acts along its direction of motion. However, if a constant net force acts at an angle to the direction of motion at any instant, the object moves in a curved path (e.g. projectile motion).

- In this topic, we will study the circular motion of objects in which the acceleration is not uniform. In particular, we will focus on uniform circular motion in which an object travels at a constant speed.

1 Kinematics of uniform circular motion

(a) Candidates should be able to express angular displacement in radians.

1.1 Angular displacement

- Consider an object moving from A to B in a circle with uniform speed *v* round a fixed point O as centre.

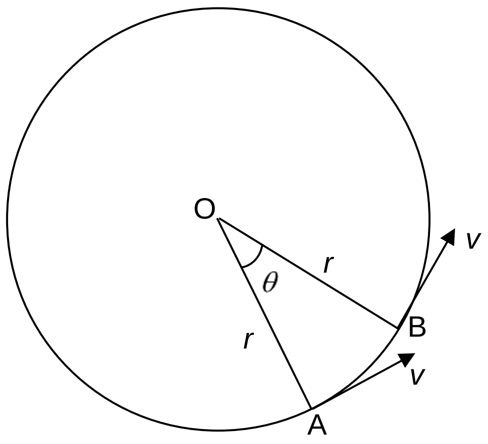


Fig. 1 Object moving in uniform circular motion from A to B

- The angle θ swept through by the radius is known as the *angular displacement*.

- It is defined by the equation

$$\boxed{\theta = \frac{s}{r}}$$

where *s* is the arc length AB and *r* is the radius of the circle.

- The angular displacement θ is measured in radians.

One radian (rad) is defined as the angle subtended at the centre of a circle by an arc equal in length to the radius.

(b) *Candidates should be able to show an understanding of and use the concept of angular velocity to solve problems.*

1.2 Angular velocity

- *Angular velocity* ω is the rate of change of angular displacement.

$$\boxed{\omega = \frac{d\theta}{dt}}$$

  It is measured in rad s^{-1}.

- For an object moving with constant angular velocity, $\omega = \frac{\theta}{t}$.

<u>Example 1</u>

A boy seated 2.0 m away from the centre of a merry-go-round completes one-sixth of a revolution in 3.0 s. Calculate

(a) his angular displacement θ from his starting position,
(b) the distance he travels from his starting position,
(c) his angular velocity.

Solution:

(a) For one revolution, $\theta = 2\pi$ rad; for one-sixth of a revolution,

$$\text{angular displacement } \theta = \frac{1}{6} \times 2\pi = 1.047$$
$$= 1.0 \text{ rad}$$

(b) The distance travelled is the arc length s of the circle.

$$s = r\theta$$
$$= 2.0 \times 1.047$$
$$= 2.1 \text{ m}$$

(c) Angular velocity $\omega = \frac{\theta}{t} = \frac{1.047}{3.0}$

$$= 0.35 \text{ rad s}^{-1}$$

(c) *Candidates should be able to recall and use $v = r\omega$ to solve problems.*

1.3 Tangential velocity

- The *tangential velocity v* is the instantaneous velocity of the object along its circular path. The direction is therefore tangential to the circular path.

- Referring to Fig. 1, recall that if s is the length of the arc AB, then $s = r\theta$. Differentiating the equation with respect to time t, we have

$$\frac{ds}{dt} = \frac{d(r\theta)}{dt} = r\frac{d\theta}{dt}$$

Hence, the following relationship is obtained.

$$\boxed{v = r\omega}$$

- This relationship shows that the larger the radius r, the larger the tangential velocity v for a constant angular velocity ω.

1.4 Period and frequency

- The *period* T of a circular motion is defined as the time taken for an object to complete one revolution. In time T, the object moves through an angle of 2π rad. Therefore,

$$\omega = \frac{2\pi}{T}$$

$$\boxed{T = \frac{2\pi}{\omega}}$$

- The *frequency* f of a circular motion is defined as the number of revolutions completed per unit time. It is related to the period by

$$f = \frac{1}{T}$$

Also,
$$\omega = 2\pi f$$

Example 2

A toy car moves round a circular track of radius 0.30 m at 2.0 revolutions per second. Calculate its

(a) period T,

(b) angular velocity ω,

(c) tangential velocity v.

Solution:

(a) Period
$$T = \frac{1}{f} = \frac{1}{2.0} = 0.50 \text{ s}$$

(b) Angular velocity
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.50} = 12.566 = 13 \text{ rad s}^{-1}$$

(c) Tangential velocity
$$v = r\omega = 0.30 \times 12.566 = 3.8 \text{ m s}^{-1}$$

2 Centripetal acceleration

(d) Candidates should be able to describe qualitatively motion in a curved path due to a perpendicular force, and understand the centripetal acceleration in the case of uniform motion in a circle.

2.1 Centripetal acceleration

- The velocity of an object in circular motion is always changing even when the speed is constant, since the direction is always changing; velocity is a vector, so direction must be taken into account.
- This means that even when an object is moving with constant speed in a circular path, it is still accelerating.
- This acceleration cannot have a component in the direction of motion of the object, for if it had, it would either increase or decrease the speed of the object.
- Hence, this acceleration must be perpendicular to the direction of motion of the object, and is directed towards the centre of the circle.

(e) Candidates should be able to recall and use centripetal acceleration $a = r\omega^2$*, and* $a = \dfrac{v^2}{r}$ *to solve problems.*

2.2 Derivation of centripetal acceleration

- Consider an object moving from A to B in a circle with uniform speed *v*. It moves through an angle of $\Delta\theta$ in time Δt .

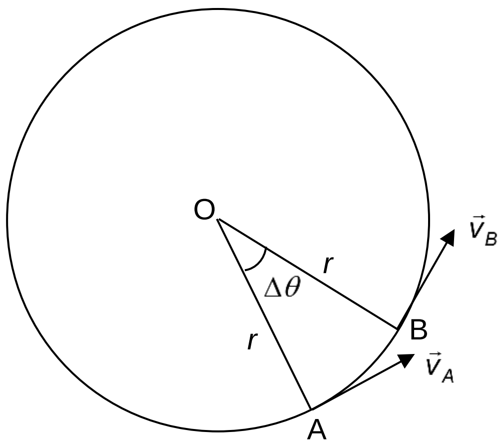
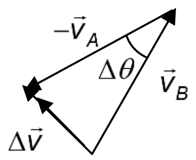


Fig. 2 Object moving in uniform circular motion from A to B

- The change in velocity of the object is given as $\Delta\vec{v} = \vec{v}_B - \vec{v}_A$, where $\vec{v}_A$ and $\vec{v}_B$ are the velocities of the object at A and B respectively.

$$\Delta\vec{v} = \vec{v}_B - \vec{v}_A$$
$$\Delta\vec{v} = \vec{v}_B + \left(-\vec{v}_A\right)$$

Note: $|\vec{v}| = |\vec{v}_A| = |\vec{v}_B|$



- Consider the angle $\Delta\theta$ to be very small (when Δt is very small). Therefore, $\Delta\vec{v} \approx \vec{v} \times \Delta\theta$, where $\Delta\vec{v}$ is directed towards the centre of the circle.

- Since acceleration is defined as the rate of change of velocity,

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$
$$\vec{a} = \frac{\vec{v} \times \Delta\theta}{\Delta t}$$
$$\vec{a} = \vec{v}\omega$$

- The acceleration is in the same direction as the change of velocity, which is towards the centre of the circle. It is known as the *centripetal acceleration*, a_c and can be written as

$$\boxed{a_c = v\omega = r\omega^2 = \frac{v^2}{r}}$$

Example 3

The Moon orbits around the Earth with a speed of 1020 m s^{-1} and takes 27.3 days for a complete revolution. Calculate

(a) the acceleration of the Moon,

(b) the distance between the centres of the Moon and Earth.

Solution:

(a) Centripetal acceleration,

$$a_c = v\omega = v\left(\frac{2\pi}{T}\right)$$
$$= 1020\left(\frac{2\pi}{27.3 \times 24 \times 3600}\right)$$
$$= 2.72 \times 10^{-3} \text{ m s}^{-2}$$

(b) Let r be the distance between the centres of the Moon and Earth (which is the radius of the orbit of the Moon, assumed circular).

Since $v = r\omega$,

$$1020 = r \times \frac{2\pi}{27.3 \times 24 \times 3600}$$
$$r = 3.83 \times 10^8 \text{ m}$$

2.3 Summary of terms

- Fig. 3 gives a summary of the terms used for circular motion and the relationship between them.

	tangential	angular	relationship
displacement	s	θ	$s = r\theta$
velocity	v	ω	$v = r\omega$

Fig. 3 Summary of physical quantities used for circular motion

- Centripetal acceleration, $a_c = r\omega^2 = \dfrac{v^2}{r}$.

3 Centripetal force

(d) *Candidates should be able to describe qualitatively motion in a curved path due to a perpendicular force, and understand the centripetal acceleration in the case of uniform motion in a circle.*

3.1 Centripetal force

- Applying Newton's second law of motion, it can be deduced that if an object is moving along a circular path, there must be a resultant force acting on it since there is acceleration.

- The direction of this resultant force is in the same direction as the centripetal acceleration, and this resultant force is known as the *centripetal force*.

- The centripetal force causes no change in the speed of the object since its direction is perpendicular to the direction of motion of the object.

- The magnitude of the centripetal force F_c required to keep an object of mass m moving in a circle of radius r is given by

$$\boxed{F_c = ma_c = mr\omega^2 = \frac{mv^2}{r}}$$

- Note that centripetal force is <u>only a name</u> for the <u>resultant force</u> that causes an object to move in circular motion. As such, when drawing a free-body diagram of an object in circular motion, centripetal force <u>should not be included</u> as an independent force.

(f) *Candidates should be able to recall and use centripetal force* $F = mr\omega^2$, *and* $F = \dfrac{mv^2}{r}$ *to solve problems.*

3.2 Problem solving

- As in any question on forces and dynamics, it is necessary for us to know how to draw and interpret free-body diagrams in order to help us solve the problem.
- The following strategies are applied when solving problems involving circular motion using Newton's second law of motion.

  1. Draw a free-body diagram of the object in circular motion.
  2. Resolve the forces into 2 components; one in the direction parallel to the resultant force (centripetal force), and the other in the direction perpendicular to the direction of the resultant force.
  3. Apply Newton's second law of motion and solve for unknowns.

3.2.1 Horizontal circle

Horizontal circular motion requires horizontal resultant force.

<u>Example 4</u>

A 1000 kg car rounds a curve on a flat road of radius 50 m at a speed of 14 m s^{-1}.

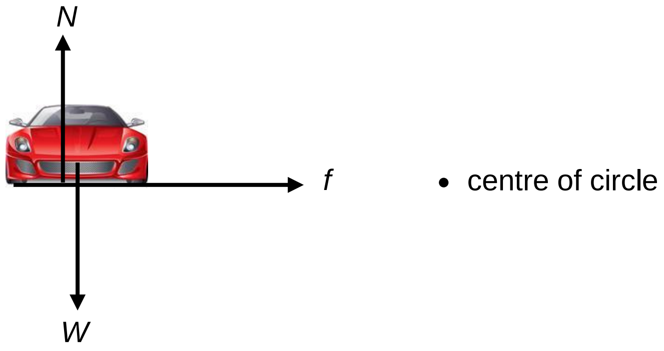
(a) Draw a free-body diagram of the car.

(b) Calculate the required frictional force for the car to round the curve.

(c) If no frictional force is required, calculate the angle at which the road should be banked.

Solution:

(a)



(b) The static friction *f* provides the centripetal force. By Newton's second law of motion,

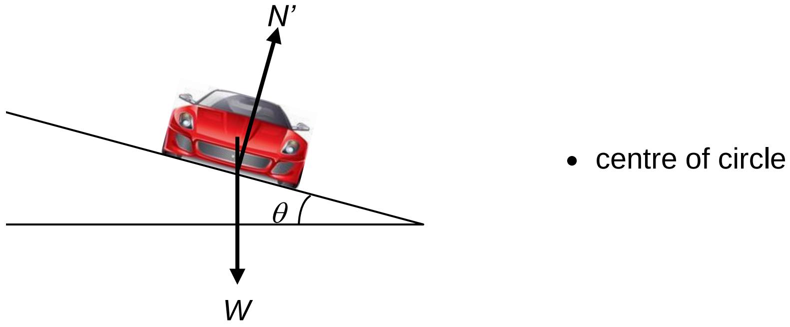
$$F_c = ma_c$$

$$f = m\frac{v^2}{r}$$

$$= 1000 \times \frac{14^2}{50}$$

$$= 3900 \text{ N}$$

(c)



Resolving N' horizontally and vertically, we have $N'_x = N'\sin\theta$ and $N'_y = N'\cos\theta$.

The horizontal component of N' provides the centripetal force necessary for the car to move in a horizontal circle.

By applying Newton's second law of motion,

$$N'\sin\theta = m\frac{v^2}{r} \qquad \text{- - - - (1)}$$
$$N'\cos\theta = mg \qquad \text{- - - - (2)}$$

Solving (1) and (2),

$$\tan\theta = \frac{v^2}{rg}$$
$$\tan\theta = \frac{14^2}{50 \times 9.81}$$
$$\theta = 22°$$

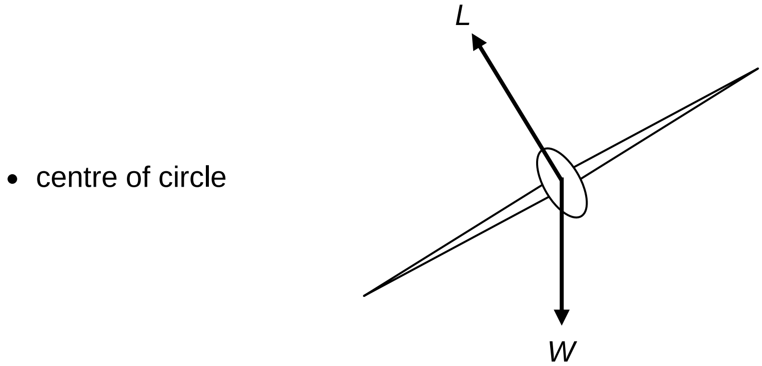
Example 5

An airplane moving at speed v is attempting to make a turn of radius r.

(a) Draw a free-body diagram of the airplane.

(b) Explain why the airplane has to be banked.

(a)



(b) In order for the airplane to turn, there must be a resultant force (centripetal force) that is directed towards the centre of the horizontal circle. This is provided by the horizontal component of the lift force, L_x. If the plane is not banked, then both lift L and weight W are in vertical directions and the airplane will not be able to make the turn.

Example 6

A particle of mass 300 g is attached to a fixed point on a smooth table by a string of length 2.0 m. It moves in a circular motion with speed 4.0 m s^{-1}.

(a) Calculate the tension in the string.

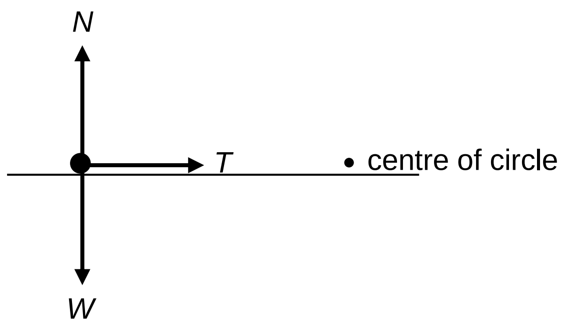
(b) If the maximum tension that the string can withstand is 12 N, determine the greatest number of revolutions per unit time which the particle can make without breaking the string.

(c) State the direction in which the particle will move at the instant when the string breaks.

Solution:

(a) The forces acting on the particle are weight *W*, normal contact force *N* and the tension *T*.

The tension *T* provides the centripetal force necessary for circular motion.



By Newton's second law of motion,

$$T = m\frac{v^2}{r}$$

$$T = 0.300 \times \frac{4.0^2}{2.0}$$

$$= 2.4 \text{ N}$$

(b) By Newton's second law of motion,

$$T_{max.} = mr\omega_{max}^2$$

$$12 = 0.300 \times 2.0 \times \omega_{max}^2$$

$$\omega_{max} = 4.472 \text{ rad s}^{-1}$$

The greatest number of revolutions per unit time is maximum frequency f_{max}, where

$$f_{max} = \frac{\omega_{max}}{2\pi}$$

$$= \frac{4.472}{2\pi}$$

$$= 0.71 \text{ rev s}^{-1}$$

(c) The particle will move off tangentially at the instant when the string breaks. This is in accordance to Newton's first law of motion.

Example 7

The bob of a conical pendulum has a mass of 50 g and is attached to a fixed support by an inextensible string of length 1.0 m. The bob moves with constant angular speed in a horizontal circle about a vertical axis. The angle maintained by the string to the vertical is 30°.

Calculate

(a) the speed of the bob,

(b) the tension in the string.

Solution:

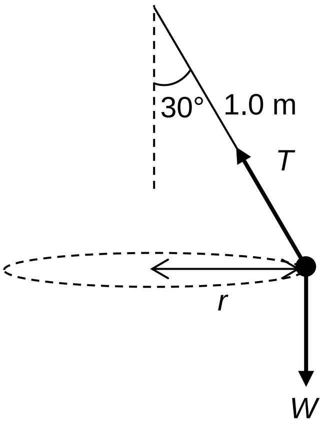
(a) For the conical pendulum, the forces acting on the bob are its weight W and tension T.

The horizontal component of T which is directed towards the centre of the circle provides the centripetal force. The vertical component of T is equal to the weight.

Applying Newton's second law of motion,

$$T\sin\theta = \frac{mv^2}{r} \qquad \text{- - - - (1)}$$

$$T\cos\theta = mg \qquad \text{- - - - (2)}$$



$$\frac{(1)}{(2)}: \quad \tan\theta = \frac{v^2}{rg}$$

$$\tan 30° = \frac{v^2}{1.0\sin 30° \times 9.81}$$

$$v = 1.7 \text{ m s}^{-1}$$

(b) From equation (2),

$$T\cos\theta = mg$$

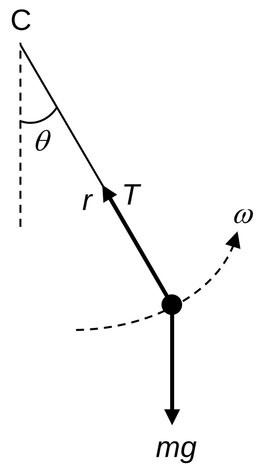
$$T\cos 30° = 0.050 \times 9.81$$

$$T = 0.57\text{ N}$$

3.2.2 Vertical circle

Vertical circular motion requires vertical resultant force.

Consider an object of mass m attached to a light inelastic string, moving in a circular path of radius r in a vertical plane, where C is the centre of the circle, as shown.
The object moves with a <u>constant</u> angular velocity ω.



At the position shown, the string is at an angle θ to the downward vertical.

The resultant force F towards C provides the centripetal force, i.e. $F = mr\omega^2 = T - mg\cos\theta$,

→ $T = mr\omega^2 + mg\cos\theta$

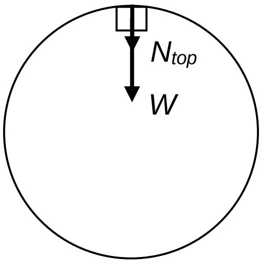
At the <u>bottom</u>, i.e. a point directly below C, $\theta = 0$ and $\cos\theta = 1$.
The tension is <u>maximum</u>, i.e. $T_{max} = mr\omega^2 + mg$

At the <u>top</u>, i.e. a point directly above C, $\theta = 180°$ and $\cos\theta = -1$.
The tension is <u>minimum</u>, i.e. $T_{min} = mr\omega^2 - mg$

<u>Example 8</u>

An amusement park ride rotates a person of mass 63.0 kg in a vertical circle of radius 6.60 m. The time taken for one revolution is 3.20 s. Calculate the force which is exerted by the seat on the person when the person is at
(a) the top of the circle,
(b) the bottom of the circle,
(c) half-way up the circle.

Solution:
(a) At the top of the circle,

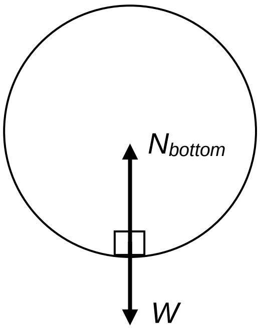


$$N_{top} + W = mr\omega^2 = mr\left(\frac{2\pi}{T}\right)^2$$

$$N_{top} + (63.0 \times 9.81) = 63.0 \times 6.60 \times \left(\frac{2\pi}{3.20}\right)^2$$

$$N_{top} = 985 \text{ N}$$

(b) At the bottom of the circle,

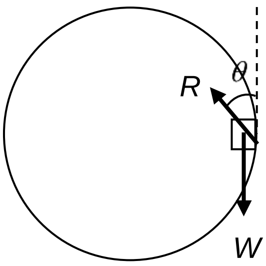


$$N_{bottom} - W = mr\omega^2 = mr\left(\frac{2\pi}{T}\right)^2$$

$$N_{bottom} - (63.0 \times 9.81) = 63.0 \times 6.60 \times \left(\frac{2\pi}{3.20}\right)^2$$

$$N_{bottom} = 2220 \text{ N}$$

(c) At half-way up the circle,



$$R\sin\theta = mr\omega^2 \qquad \text{- - - - (1)}$$
$$R\cos\theta = mg \qquad \text{- - - - (2)}$$

$$\frac{(1)}{(2)}: \quad \tan\theta = \frac{r\omega^2}{g}$$

$$\tan\theta = \frac{6.60 \times \left(\frac{2\pi}{3.20}\right)^2}{9.81}$$

$$\theta = 68.9^\circ$$

From equation (2),

$$R\cos 68.9^\circ = 63.0 \times 9.81$$
$$R = 1720 \text{ N}$$

Example 9

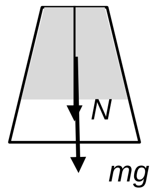
A pail of water can be swung round in a vertical circle of radius *r* without spilling.
Explain how this is possible, and hence derive an expression for the minimum speed of the pail at the top of the circle so that water stays within the pail.

Solution:
The water remains in contact with the pail throughout the circular motion and hence it does not spill. This is possible because the speed of the pail is large enough to ensure that the contact force is non-zero.

At the top of the circle,

$$N + W = \frac{mv^2}{r}$$

$$N = \frac{mv^2}{r} - mg$$

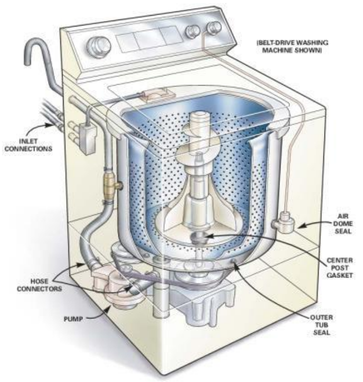


If the water does not spill, $N > 0$. Hence,

$$\frac{mv^2}{r} - mg > 0$$

$$v > \sqrt{rg}$$

Extra reading: Washing machine



Inner tub

In a washing machine, when the drum rotates, the sidewall provides the necessary centripetal force (reaction of the drum acting on the clothes) to keep the clothes in circular motion. When the drum is rotated with higher angular velocity, water particles acquire higher linear speed and they move in a spiral path, i.e. the radius of the circular path increases with the speed.

The holes in the sidewall cannot provide the necessary centripetal force, so the water particles move away from the centre through the holes along a tangential path, and this helps to dry the clothes.

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