AHMAD IBRAHIM SECONDARY SCHOOL
GCE O-LEVEL PRELIMINARY EXAMINATION 2023

SECONDARY 4 EXPRESS

Name: MARKING SCHEME	Class:	Register No.:

ADDITIONAL MATHEMATICS **4049/02**
Paper 2 **11 August 2023**

Candidates answer on the Question Paper. **2 hours 15 minutes**

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 90.

For Examiner's Use
/90

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}bc\sin A$$

1 **(a)** Find the range of values of x for which the expression $3-2x^2$ is negative. [2]

$$3-2x^2<0$$
$$2x^2-3>0$$
$$x<-\sqrt{\frac{3}{2}} \quad \text{or} \quad x>\sqrt{\frac{3}{2}} \quad \text{OR}$$
$$x<-\frac{\sqrt{6}}{2} \quad \text{or} \quad x>\frac{\sqrt{6}}{2}$$

M1: finding roots or factorisng

A1

(b) Find the set of values of the constant k for which the curve $y=x^2$ lies entirely above the line $y=k(x+1)$. [3]

$$x^2>k(x+1)$$
$$x^2-kx-k>0$$

For the quadratic expression to be always positive,

$$b^2-4ac<0$$
$$k^2+4k<0$$ M1
$$k(k+4)<0$$ M1
$$-4<k<0$$ A1

2 **(a)** Find the range of values of k such that the line $x+y=3$ intersects the curve $x^2-2x+2y^2=k$. [4]

$x+y=3$

$y=3-x$--------(1)

$x^2-2x+2(3-x)^2=k$

$x^2-2x+2(9-6x+x^2)=k$

$x^2-2x+18-12x+2x^2-k=0$

$3x^2-14x+18-k=0$

Since the line and curve intersects,

$b^2-4ac\geq 0$

$(-14)^2-4(3)(18-k)\geq 0$

$196-216+12k\geq 0$

$-20+12k\geq 0$

$k\geq \frac{5}{3}$

M1 manipulation to get quadratic equation in 1 unknown

M1

M1

A1

(b) State a possible value of k if if there is no intersection between the line and the curve. [1]

Any value that is < $\frac{5}{3}$.

B1

3 A polynomial, P, is $x^{2n}-(k+1)x^2+k$ where n and k are positive integers.

(a) Explain why $x-1$ is a factor of P for all values of k. [2]

let $f(x)=x^{2n}-(k+1)x^2+k$

$f(1)=1-k-1+k$

$=0$

$\therefore$ since remainder =0, $(x-1)$ is a factor.

M1

A1: must mention remainder = 0, or by factor theorem

(b) Given that k = 4, find the value of n for which $x-2$ is a factor of P. Hence factorise P completely. [4]

$f(x)=x^{2n}-5x^2+4$

$f(2)=2^{2n}-16$

M1

Since $x-2$ is a factor,

$2^{2n}-16=0$

$n=2$

A1

$f(x)=x^4-5x^2+4$

$=(x-1)(x-2)(x^2+3x+2)$

M1, must write factors $(x-1)(x-2)$ first since it's a hence question

$=(x-1)(x-2)(x+1)(x+2)$

A1

[Turn over

4 A projectile was launched from a catapult to hit a defence structure on a fort. The height, h metres, of the projectile above ground is given by the equation $h = -2x^2 + 3x + 1.5$, where x metres is the horizontal distance from the catapult.

(i) By expressing the function in the form $h = a(x-m)^2 + n$, where a, m and n are constants, explain whether the projectile can reach a height of 3 metres. [2]

$h = -2x^2 + 3x + 1.5$

$= -2(x^2 - 1.5x - 0.75)$

$= -2[(x-0.75)^2 - 0.75^2 - 0.75]$

$= -2\ (x-0.75)^2 - 2.625$

M1 for completing the square

maximum point is $(0.75,\ 2.625)$

Therefore the projectile cannot reach a height of 3m since the maximum height is 2.625m.

A1 for comparing 2.625 and 3

(ii) Given that the defence structure is 1.4 metres horizontally from the catapult and 0.8 metres above the ground, justify if the projectile will hit the structure. [2]

Subs $(1.4, 0.8)$ into $h = -2x^2 + 3x + 1.5$

$h = -2(1.4)^2 + 3(1.4) + 1.5$

$= 1.78$

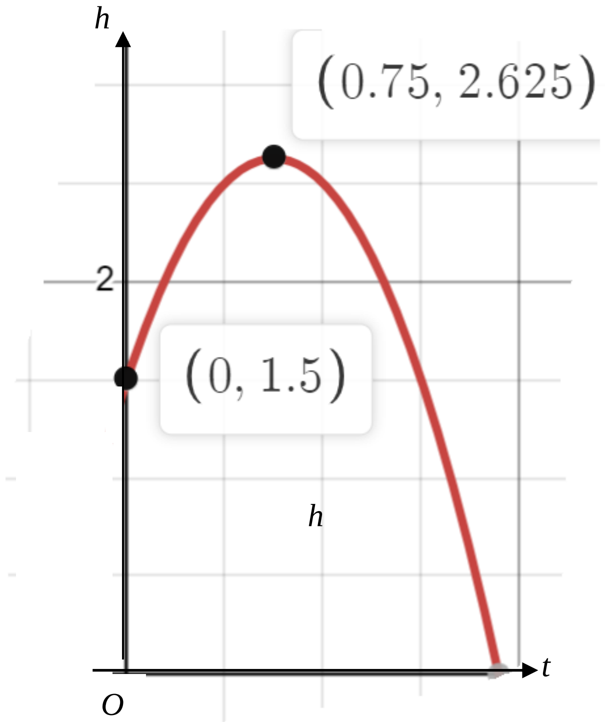
$\neq 0.8$

M1 for determining if the point lies on the equation

Since the point $(1.4, 0.8)$ does not lie on the curve $h = -2x^2 + 3x + 1.5$,
therefore the projectile will not hit the structure.

A1

(iii) Sketch the curve of $h = -2x^2 + 3x + 1.5$. [2]



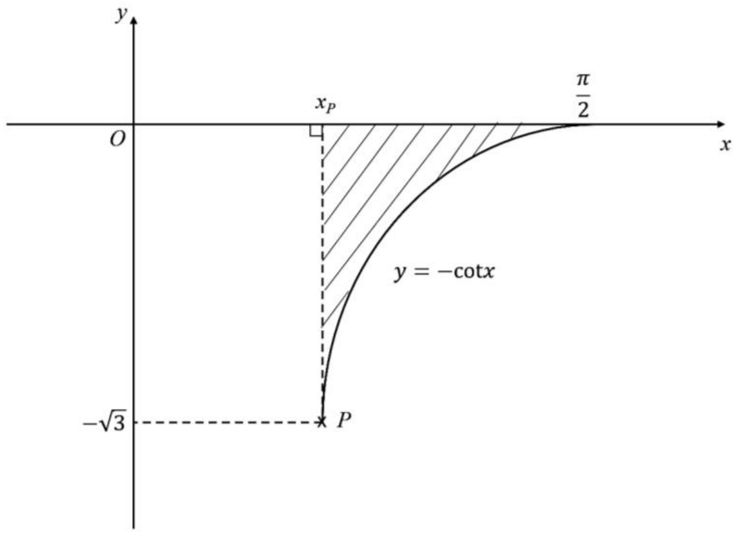
B! correct shape with turning point

B1 correct y-intercept

5 **(a)** Differentiate $\ln(\sin x)$ with respect to x. [2]

$$\frac{d}{dx}\ln(\sin x)$$
$$= \frac{1}{\sin x}\cos x \quad \text{M1}$$
$$= \cot x \quad \text{A1}$$

(b)



The diagram shows part of the curve $y = -\cot x$, cutting the x-axis at $\left(\frac{\pi}{2}, 0\right)$.
The line $y = -\sqrt{3}$ intersects the curve at P.

(i) State the value of x_p, the x-coordinate of P. [1]

$$\sqrt{3} = \frac{1}{\tan x}$$
$$\tan x = \frac{1}{\sqrt{3}}$$
$$x = \frac{\pi}{6} \quad \text{B1}$$

(ii) Explain why the expression $\int_{x_P}^{\frac{\pi}{2}} -\cot x \, dx$ does not give the area of the shaded region. [1]

The shaded area is below the *x*-axis. If we $\int_{Q_x}^{\frac{\pi}{2}} -\cot x \, dx$, we will get a **negative value** for the area. Thus $\int_{Q_x}^{\frac{\pi}{2}} -\cot x \, dx$ does not give area of the shaded region.

B1

(iii) Find the exact area of the shaded region. [3]

$y = -\frac{1}{\tan x}$

when $y = -\sqrt{3}$

$-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -\cot x \, dx$ M1

$= [\ln(\sin x)]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ M1

$= \ln 1 - \ln \frac{1}{2}$

$= \ln 2 \text{ units}^2 \text{ or } -\ln \frac{1}{2} \text{ units}^2$

A1

6 **(a)** Without using a calculator, show that $\cos\left(\frac{7\pi}{12}\right)=\frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$. [3]

$$\cos\left(\frac{7\pi}{12}\right)=\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$$

M1: identifying special angles

$$=\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$$

M1: correct application of formula

$$=\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$=\frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$$

M1: recognising exact values and reach result given

(b) Evaluate $\int_0^{\frac{\pi}{12}} 3\cos^2 x-\sin^2 x \, dx$ exactly. [4]

$$\int_0^{\frac{\pi}{12}} 3\cos^2 x-\sin^2 x \, dx=\int_0^{\frac{\pi}{12}} \frac{3}{2}(2\cos^2 x-1+1)+\frac{1}{2}\left(1-2\sin^2 x-1\right) dx$$

$$=\int_0^{\frac{\pi}{12}} \frac{3}{2}\cos 2x+\frac{1}{2}\cos 2x+1 \, dx$$

M1: correct application of double angle formula

$$=\int_0^{\frac{\pi}{12}} 2\cos 2x+1 \, dx$$

M1

$$=\left[\sin 2x+x\right]_0^{\frac{\pi}{12}}$$

$$=\sin\frac{\pi}{6}+\frac{\pi}{12}$$

M1

$$=\frac{6+\pi}{12}$$

A1

7 (a) (i) Factorise $x^6 - 64$ completely. [2]

$$x^6 - 64$$
$$= (x^2)^3 - 4^3$$
$$= (x^2 - 4)(x^4 + 4x^2 + 16)$$
$$= (x-2)(x+2)(x^4 + 4x^2 + 16)$$

M1: either cubic factorisation or difference of squares factorisation

A1

OR

$$(x^3)^2 - (2^3)^2$$
$$= (x^3 - 8)(x^3 + 8)$$
$$= (x^3 - 2^3)(x^3 + 2^3)$$
$$= (x-2)(x^2 + 2x + 4)(x+2)(x^2 - 2x + 4)$$
$$= (x-2)(x+2)(x^2 + 2x + 4)(x^2 - 2x + 4)$$

M1

A1

(ii) Hence solve $x^6 - 64 = (x^2 + 4)^2 - (2x)^2$. [3]

$$x^6 - 64 = (x^2 + 4)^2 - (2x)^2$$
$$(x+2)(x-2)(x^4 + 4x^2 + 16) = x^4 + 8x^2 + 16 - 4x^2$$
$$(x+2)(x-2)(x^4 + 4x^2 + 16) = x^4 + 4x^2 + 16$$
$$(x+2)(x-2) = 1$$
$$x^2 - 5 = 0$$
$$x = \pm\sqrt{5}$$

M1 for expanding the RHS of the equation

M1

A1

(b) Find the values of the integers a and b for which $\dfrac{a+\sqrt{b}}{2}$ is the solution of the equation $2x\sqrt{3}+x\sqrt{125}=x\sqrt{45}+\sqrt{12}.$ [4]

$$2x\sqrt{3}+x\sqrt{125}=x\sqrt{45}+\sqrt{12}$$
$$2x\sqrt{3}+5x\sqrt{5}=3x\sqrt{5}+2\sqrt{3}$$
$$2x\sqrt{3}+5x\sqrt{5}-3x\sqrt{5}=2\sqrt{3}$$
$$x\left(2\sqrt{3}+2\sqrt{5}\right)=2\sqrt{3}$$

M1 , isolating x terms and simplifying the surds

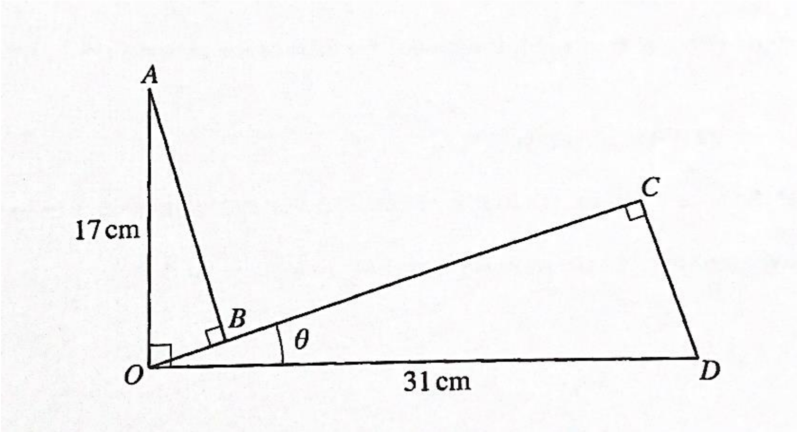
$$x=\frac{2\sqrt{3}}{\left(2\sqrt{3}+2\sqrt{5}\right)}\times\frac{\left(2\sqrt{3}-2\sqrt{5}\right)}{\left(2\sqrt{3}-2\sqrt{5}\right)}$$

M1 for rationalising

$$=\frac{12-4\sqrt{15}}{12-20}$$
$$=\frac{12-4\sqrt{15}}{-8}$$
$$=\frac{-3+\sqrt{15}}{2}$$
$$a=-3,\ b=15$$

A1 , A1

8



The diagram shows three fixed points *O*, *A* and *D* such that *OA* = 17 cm, *OD* = 31 cm and angle *AOD* = 90°.
The lines *AB* and *DC* are perpendicular to the line *OC* which makes an angle θ with the line *OD*.
The angle θ can vary in such a way that the point B lies between the points *O* and *C*.

(i) Show that $AB + BC + CD = (48\cos\theta + 14\sin\theta)$ cm. [3]

$\sin\theta = \dfrac{CD}{31}$

$CD = 31\sin\theta$

$\sin\theta = \dfrac{OB}{17}$

$OB = 17\sin\theta$

$\cos\theta = \dfrac{AB}{17}$ M2 for any 2 correct

$AB = 17\cos\theta$

$\cos\theta = \dfrac{OC}{31}$

$OC = 31\cos\theta$

$AB + BC + CD$

$= 17\cos\theta + 31\cos\theta - 17\sin\theta + 31\sin\theta$

$= (48\cos\theta + 14\sin\theta)$ cm A1

(ii) Find the values of θ for which $AB+BC+CD=49$ cm. [6]

$48\cos\theta+14\sin\theta=49$

$R\cos(\theta-\alpha)=49$ M1

$R=\sqrt{48^2+14^2}$

$=50$ M1

$\alpha=\tan^{-1}\left(\frac{14}{48}\right)$

$=16.26°$ M1

$50\cos(\theta-16.26°)=49$

$\cos(\theta-16.26°)=\frac{49}{50}$ M1

Reference angle $=11.48°$

$\theta=27.7°, 4.8°$ A1, A1

(iii) Find the maximum value of $AB+BC+CD$ and the corresponding value of θ. [2]

max $50\cos(\theta-16.26°)$

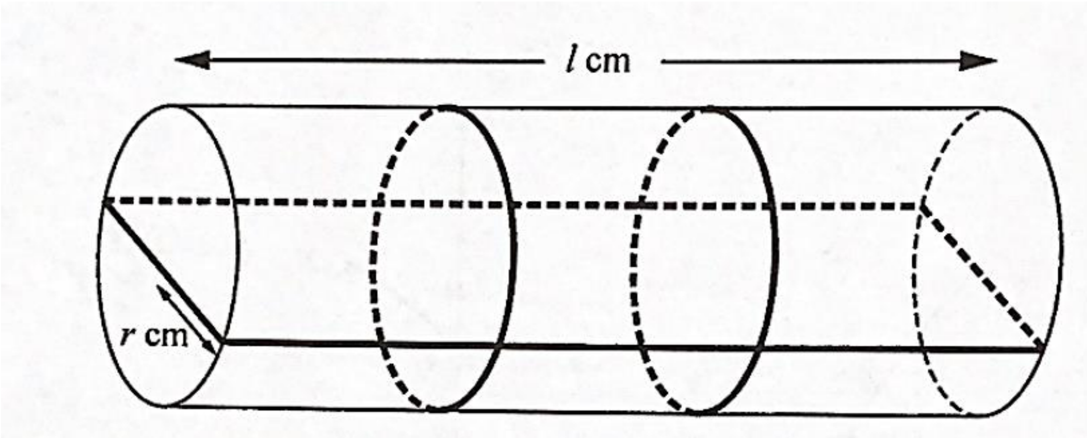
$=50$ B1

occurs when $\cos(\theta-16.26°)=1$

$\theta-16.26°=0$

$\theta=16.3°$ (1 d.p) B1

9 The diagram shows a roll of material in the shape of a cylinder of radius r cm and length l cm.
The roll is held together by three pieces of adhesive tape whose width and thickness may be ignored.
One piece of tape is in the shape of a rectangle, the other two pieces are in the shape of circles.
The total length of tape is 600 cm.



(i) Show that the volume, V cm^3, of the cylinder is given by
$V = \pi r^2 \left(300 - 2r - 2\pi r\right)$. [3]

$2(2r)+2(l)+2(2\pi r)=600$ M1

$2r+l+2\pi r=300$

$l=300-2r-2\pi r$ M1

$V=\pi r^2 l$

$=\pi r^2 (300-2r-2\pi r)$ (shown) A1

(ii) Given that r can vary, show that V has a stationary value when $r = \frac{k}{1+\pi}$, where k is a constant to be found, and find the corresponding value of l. [5]

$$\frac{dV}{dr} = 600\pi r - 6\pi r^2 - 6\pi^2 r^2$$
$$= 6\pi r(100 - r - \pi r) \quad \text{M1}$$

For stationary value, $\frac{dV}{dr} = 0$.

$6\pi r(100 - r - \pi r) = 0$ M1

$r = 0$ rejected because $r > 0$ or

$100 - r - \pi r = 0$

$(1+\pi)r = 100$

$r = \frac{100}{1+\pi}$ A1

$\therefore$ V has a stationary value when $r = \frac{100}{1+\pi}$, where $k = 100$. (shown)

When $r = \frac{100}{1+\pi}$,

$l = 300 - 2\left(\frac{100}{1+\pi}\right) - 2\pi\left(\frac{100}{1+\pi}\right)$ M1

$= 300 - 2\left(\frac{100}{1+\pi}\right)(1+\pi)$

$= 300 - 200$

$= 100$ A1

(iii) Determine if the volume is a minimum or maximum. [3]

$$\frac{dV}{dr} = 6\pi r(100 - r - \pi r)$$
$$\frac{d^2V}{dr^2} = 6\pi r(-1-\pi) + (100 - r - \pi r)(6\pi)$$
$$= 6\pi r(-1-\pi) + (100 - r - \pi r)(6\pi)$$
$$= 6\pi(-r - \pi r + 100 - r - \pi r)$$
$$= 6\pi(-2r - 2\pi r + 100)$$

M1

when $r = \frac{100}{1+\pi}$,

$$\frac{d^2V}{dr^2} = 6\pi\left(-\frac{200}{1+\pi} - \frac{200\pi}{1+\pi} + 100\right)$$
$$= -600\pi$$

M1

Since $\frac{d^2V}{dr^2} < 0$, V is a maximum.

A1

10 A particle travelling in a straight line passes through a fixed point *O* with a speed of 8 m/s.
The acceleration, a m/s^2, of the particle t s after passing through *O*, is given by $a = -e^{-0.1t}$.
The particle comes to instantaneous rest at the point *P*.

(i) Show that the particle reaches *P* when $t = 10\ln 5$. [5]

$a = -e^{-0.1t}$

$v = \int a = -e^{-0.1t} \ \mathrm{d}t$

$= \dfrac{-e^{-0.1t}}{-0.1} + c$ M1

When $t = 0,\ v = 8$

$8 = \dfrac{-1}{-0.1} + c$

$8 = 10 + c$

$c = -2$ M1

$v = 10e^{-0.1t} - 2$

when $v = 0$ M1

$10e^{-0.1t} - 2 = 0$

$e^{-0.1t} = \dfrac{1}{5}$ M1

$-0.1t = -\ln 5$ A1

$t = 10\ln 5$ (shown)

(ii) Calculate the distance *OP*. [3]

$s = \int_0^{10\ln 5} 10e^{-0.1t} - 2 \ \mathrm{d}t$

$= \left[\dfrac{10e^{-0.1t}}{-0.1} - 2t\right]_0^{10\ln 5}$ M1

$= -100e^{-0.1(10\ln 5)} - 20\ln 5 + 100$ M1

$= -100e^{-\ln 5} - 20\ln 5 + 100$

$= -20 - 20\ln 5 + 100$ A1

$= 47.8\text{m}$

(iii) Explain why the particle is again at *O* at some instant during the fiftieth second after first passing through *O*. [3]

when $t = 49$

$s = -100e^{-0.1(49)} - 2(49) + 100$

$= 1.255m$ M1

when $t = 50$

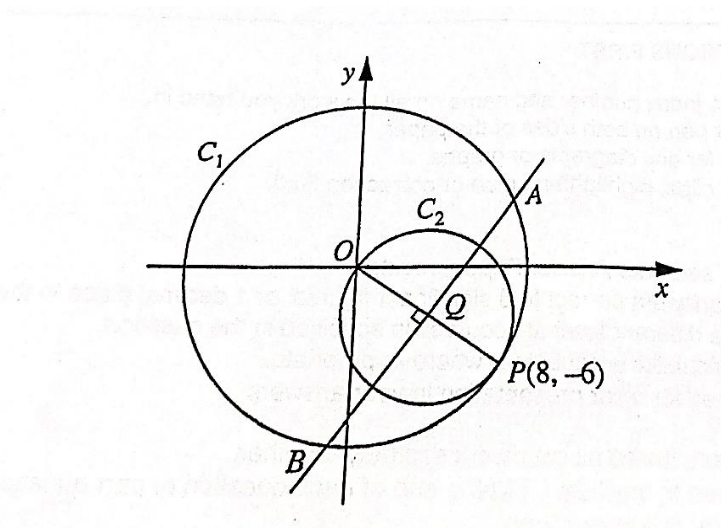
$s = -100e^{-0.1(50)} - 2(50) + 100$

$= -0.674\text{m}$ M1

Since displacement of the particle is positive when $t = 49$ and negative when $t = 50$, this shows that the particle must have passed through O at some point in the fiftieth second.

Thus the particle is again at O at some instant during the fifitieth second after passing through O. A1

11



The diagram shows two circles C_1 and C_2.
Circle C_1 has its centre at the origin *O*.
Circle C_2 passes through *O* and has its centre at *Q*.
The point $P(8,-6)$ lies on both circles and *OP* is a diameter of C_2.

(i) Find the equation of C_1. [2]

$$|OP| = \sqrt{(8-0)^2 + (6-0)^2}$$
$$= 10$$ M1

Equation of $C_1 : x^2 + y^2 = 100$ A1

(ii) Explain why the equation of C_2 is $x^2 + y^2 - 8x + 6y = 0$. [3]

$$x^2 + y^2 - 8x + 6y = 0$$
$$x^2 - 8x + y^2 + 6y = 0$$
$$(x-4)^2 - 16 + (y+3)^2 - 9 = 0$$
$$(x-4)^2 + (y+3)^2 = 5^2$$ M1

Centre $= (4,-3)$ because it is the mid point of *OP* A1

Radius is 5 units because it is $\frac{1}{2}|OP|$ A1

(iii) The line through Q perpendicular to OP meets the circle C_1 at the point A and Show that the x-coordinates of A and B are $a+b\sqrt{3}$ and $a-b\sqrt{3}$ respectively, where a and b are integers to be found. [7]

gradient $OP = -\frac{6}{8}$

$= -\frac{3}{4}$

gradient $AB = \frac{4}{3}$

$\frac{4}{3}(x-4) = y+3$

$4x-16 = 3y+9$

$y = \frac{4}{3}x - \frac{25}{3}$..........(1) M1

$x^2 + y^2 = 100$

$y^2 = 100 - x^2$.......(2)

$(1)^2 \quad y^2 = \frac{16}{9}x^2 - \frac{200}{9}x + \frac{625}{9}$..........(3)

$(2)=(3)$ M1 Reasonable attempt at manipulating the equations to obtain the quadratic equation

$\frac{16}{9}x^2 - \frac{200}{9}x + \frac{625}{9} = 100 - x^2$

$\frac{25}{9}x^2 - \frac{200}{9}x - \frac{275}{9} = 0$

$x^2 - 8x - 11 = 0$ M1

$x = \frac{8 \pm \sqrt{64+44}}{2}$ M1

$= \frac{8 \pm \sqrt{108}}{2}$

$= \frac{8 \pm 6\sqrt{3}}{2}$ M1

$= 4 \pm 3\sqrt{3}$

x-coordinate of A is $4+3\sqrt{3}$ A1, A1

x-coordinate of B is $4-3\sqrt{3}$

END OF PAPER