1	(a) $$\mathrm{f}(r-1)-\mathrm{f}(r)=\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}$$ $$=\frac{(r+2)-r}{r(r+1)(r+2)}$$ $$=\frac{2}{r(r+1)(r+2)}$$ $$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\sum_{r=1}^{n}\frac{1}{2}\left(\mathrm{f}(r-1)-\mathrm{f}(r)\right)$$ $$=\frac{1}{2}\begin{bmatrix} \mathrm{f}(0)-\mathrm{f}(1) \\ +\ \mathrm{f}(1)-\mathrm{f}(2) \\ +\ \mathrm{f}(2)-\mathrm{f}(3) \\ \vdots \\ +\ \mathrm{f}(n-2)-\mathrm{f}(n-1) \\ +\ \mathrm{f}(n-1)-\mathrm{f}(n) \end{bmatrix}$$ $$=\frac{1}{2}\left[\mathrm{f}(0)-\mathrm{f}(n)\right]$$ $$=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$ **(b)(i)** $$\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$$ $$=\lim_{n\to\infty}\left[\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\right]$$ $$=\frac{1}{4}$$ **(ii)** $$\sum_{r=3}^{N-1}\frac{1}{r(r-1)(r-2)}=\sum_{i+2=3}^{i+2=N-1}\frac{1}{i(i+1)(i+2)}$$ $$=\sum_{i=1}^{N-3}\frac{1}{i(i+1)(i+2)}$$ $$=\frac{1}{4}-\frac{1}{2(N-1)(N-2)}$$

2

(a) **(i)** $S = a + ar + ar^2 + \ldots = \dfrac{a}{1-r}$

$-\tfrac{1}{2}S = ar + ar^3 + ar^5 + \ldots = \dfrac{ar}{1-r^2}$

Therefore $-\tfrac{1}{2}\left(\dfrac{a}{1-r}\right) = \dfrac{ar}{1-r^2}$

$\Rightarrow -\tfrac{1}{2} = \dfrac{r}{1+r}$ (since S exists, $r \neq 1$)

$\Rightarrow -1 - r = 2r$

$\Rightarrow 3r = -1$

$\Rightarrow r = -\tfrac{1}{3}$

(ii) $ar^2 = 2 \Rightarrow a = \dfrac{2}{r^2} = \dfrac{2}{\left(-\tfrac{1}{3}\right)^2} = \dfrac{2}{\tfrac{1}{9}} = 18$

$H : |a|, |ar|, |ar^2|, \ldots \Leftrightarrow a, a|r|, a|r|^2, \ldots$

Sum to infinity of $H = \dfrac{a}{1-|r|} = \dfrac{18}{1-\tfrac{1}{3}} = \dfrac{18}{\tfrac{2}{3}} = 27$

(b) $1000 + (n-1)(-1.4) < 0$

$\Rightarrow n > 715.3 \Rightarrow n \geq 716$

First negative term of series $= 1000 + 715(-1.4) = -1$

Sum of all positive terms of series

$= \dfrac{715}{2}\left(2 \times 1000 + 714 \times (-1.4)\right)$

$= 357643$

3

(a) Let F be the focus of the parabola. By the geometrical definition of parabola, $AF = AR$ and $BF = BS$.

Let $AR = k_1$ and $BS = k_2$. Then $n = m + 2(k_1 + k_2)$.

Therefore, $k_1 + k_2 = \dfrac{n-m}{2}$.

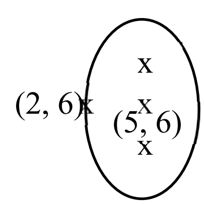
Area of $ABSR = \dfrac{1}{2}m(k_1 + k_2) = \dfrac{m(n-m)}{4}$

(b)(i) Foci: (5, 2) and (5, 10). So centre: (5, 6).

Conic is an ellipse with length of semi-minor axis, $b = 3$.

$c^2 = a^2 - b^2 \quad \Rightarrow \quad 4^2 = a^2 - 3^2$

$a = 5$



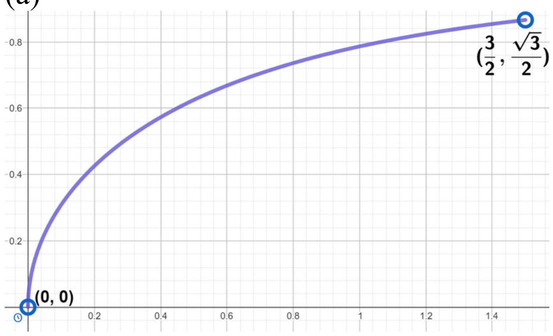
Cartesian equation in standard form: $\dfrac{(x-5)^2}{3^2} + \dfrac{(y-6)^2}{5^2} = 1$.

(ii) New equation: $\dfrac{(x-5)^2}{3^2}+\dfrac{(5y-6)^2}{5^2}=1 \quad \Rightarrow \quad \dfrac{(x-5)^2}{3^2}+(y-6/5)^2=1$

Centre: $\left(5,\ \dfrac{6}{5}\right)$. $\qquad (c')^2=3^2-1^2=8 \quad \Rightarrow c'=\sqrt{8}=2\sqrt{2}$

Foci: $\left(5\pm 2\sqrt{2},\ \dfrac{6}{5}\right)$

4 (a)



(b) $x=r\cos\theta=\dfrac{\cos^2\theta}{\sin\theta}\Rightarrow\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\dfrac{\sin\theta\left(-2\cos\theta\sin\theta\right)-\cos^3\theta}{\sin^2\theta}$

$$=\frac{-\cos\theta\left(2\sin^2\theta+\cos^2\theta\right)}{\sin^2\theta}$$

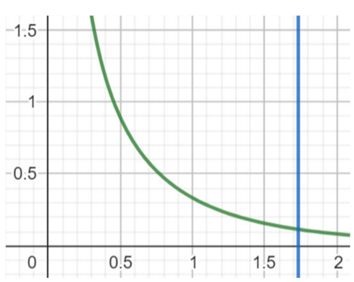
$y=r\sin\theta=\cos\theta\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=-\sin\theta$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sin^3\theta}{-\cos\theta\left(2\sin^2\theta+\cos^2\theta\right)}$$

$$=\frac{1}{\cot\theta\left(2+\cot^2\theta\right)}$$

$$=\frac{1}{r(r^2+2)} \quad \text{(shown)}$$

Since $\cot\theta$ is a decreasing function for $\dfrac{\pi}{6}<\theta<\dfrac{\pi}{2}$, $r\in\left(0,\ \cot\dfrac{\pi}{6}\right)=\left(0,\ \sqrt{3}\right)$.



From the sketch of $\mathrm{f}(r)=\dfrac{1}{r(r^2+2)}$,

$\mathrm{f}(r)>\mathrm{f}(\sqrt{3})=\dfrac{1}{5\sqrt{3}}$.

Therefore, $\dfrac{\mathrm{d}y}{\mathrm{d}x}>\dfrac{1}{5\sqrt{3}}$.

(c) $r=\cot\theta$

	$r^2 = \cot^2\theta = \dfrac{\cos^2\theta}{\sin^2\theta}$ $r^2\sin^2\theta = \cos^2\theta$ $y^2 = \dfrac{x^2}{r^2}$ $(x^2+y^2)y^2 = x^2$ $y^4 + x^2y^2 - x^2 = 0$ $y^2 = \dfrac{-x^2 \pm \sqrt{x^4+4x^2}}{2}$ Since $y^2 > 0$, choose $y^2 = \dfrac{-x^2+\sqrt{x^4+4x^2}}{2}$. Also, since $y > 0$, $y = \sqrt{\dfrac{-x^2+\sqrt{x^4+4x^2}}{2}}$ for $0 < x < \dfrac{3}{2}$.
5	(a) A point on the plane is $B(1,0,0)$. $\left\lvert \overrightarrow{BP}.\hat{\mathbf{n}} \right\rvert = 3\sqrt{3}$ $\dfrac{\left\lvert \begin{pmatrix}-5\\c\\c\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix} \right\rvert}{\sqrt{3}} = 3\sqrt{3}$ $\lvert 2c-5 \rvert = 9$ $\Rightarrow 2c-5=9$ or $2c-5=-9$ $c=7$ or $c=-2$ (rejected, since $c>0$) (b) A line perpendicular to the mirror and passing through A is: $\mathbf{r} = \begin{pmatrix}-15\\17\\5\end{pmatrix} + s\begin{pmatrix}1\\1\\1\end{pmatrix},\ s \in \mathbb{R}$ Let F be the intersection between this line and the mirror. $\begin{pmatrix}-15+s\\17+s\\5+s\end{pmatrix} \bullet \begin{pmatrix}1\\1\\1\end{pmatrix} = 1$ $-15+s+17+s+5+s=1$ $7+3s=1$ $s=-2$

	The intersection point has position vector $\overrightarrow{OF}=\begin{pmatrix}-17\\15\\3\end{pmatrix}$ and $\overrightarrow{AF}=\begin{pmatrix}-2\\-2\\-2\end{pmatrix}$. Therefore, $\overrightarrow{OA'}=\begin{pmatrix}-17\\15\\3\end{pmatrix}+\begin{pmatrix}-2\\-2\\-2\end{pmatrix}=\begin{pmatrix}-19\\13\\1\end{pmatrix}$. Coordinates of A' is $(-19, 13, 1)$. (c) $\overrightarrow{A'P}=\begin{pmatrix}15\\-6\\6\end{pmatrix}$ Let θ be the acute angle between the line and the plane. $\theta=\sin^{-1}\dfrac{\left\|\begin{pmatrix}15\\-6\\6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|}{\sqrt{297}\sqrt{3}}=\sin^{-1}\dfrac{15}{\sqrt{891}}$ $\approx 30.1668°=30.2°$ (1 d.p.)
6	(a)(i) $w^2+\mathrm{i}w^*=(-2+\mathrm{i})^2+\mathrm{i}(-2-\mathrm{i})$ $=4-4\mathrm{i}-1-2\mathrm{i}+1$ $=4-6\mathrm{i}$ $z=\dfrac{\mathrm{i}}{4-6\mathrm{i}}\times\dfrac{4+6\mathrm{i}}{4+6\mathrm{i}}$ $=\dfrac{-6+4\mathrm{i}}{4^2+6^2}$ $=-\dfrac{3}{26}+\dfrac{1}{13}\mathrm{i}$ (ii) $u^3-i\|u\|^2+\dfrac{\mathrm{i}u}{z^*}=0$ $u^3-iuu^*=-\dfrac{\mathrm{i}u}{z^*}$ So $u=0$ or $u^2-iu^*=-\dfrac{\mathrm{i}}{z^*}$ $\Rightarrow \left(u^*\right)^2+iu=\dfrac{\mathrm{i}}{z}$ (considering conjugate) From (i), $u^*=-2+\mathrm{i} \Rightarrow u=-2-\mathrm{i}$.

So the two possible values of u are 0 and $-2-\mathrm{i}$.

(b) $p=\sin\frac{2\pi}{5}-\mathrm{i}\cos\frac{2\pi}{5}=-\mathrm{i}\left(\cos\frac{2\pi}{5}+\mathrm{i}\sin\frac{2\pi}{5}\right)$

$=\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}\right)}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{5}\right)}$

$=\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{10}\right)}$

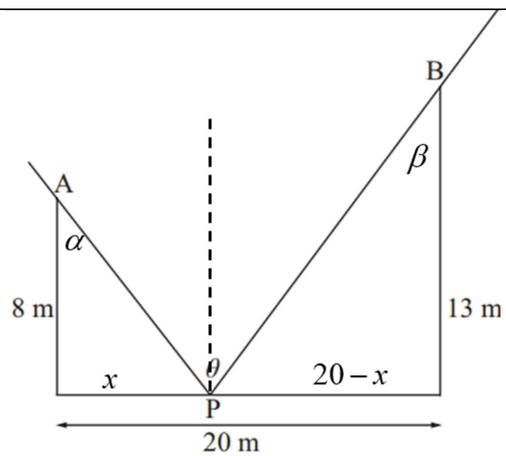
$\therefore\quad p^n=\mathrm{e}^{\mathrm{i}\left(-\frac{n\pi}{10}\right)}$

Equating arguments, $\quad -\frac{n\pi}{10}=\pi+k(2\pi),\ k\in\mathbb{Z}$

$n=-10(2k+1)$

When $k=-1,\ n=10$.
When $k=-2,\ n=30$.
When $k=-3,\ n=50$. The 3 smallest positive integers n are 10, 30, 50.

7



(a) $\theta=\alpha+\beta$

$\tan\alpha=\frac{x}{8}\Rightarrow\alpha=\tan^{-1}\left(\frac{x}{8}\right)$; $\tan\beta=\frac{20-x}{13}\Rightarrow\alpha=\tan^{-1}\left(\frac{20-x}{13}\right)$

Therefore, $\quad \theta=\tan^{-1}\frac{x}{8}+\tan^{-1}\frac{20-x}{13}$

(b) Either $\quad \theta=\tan^{-1}\frac{x}{8}+\tan^{-1}\frac{20-x}{13}\Rightarrow\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{\frac{1}{8}}{1+\left(\frac{x}{8}\right)^2}+\frac{-\frac{1}{13}}{1+\left(\frac{20-x}{13}\right)^2}$

$$\frac{d\theta}{dx} = \frac{8}{x^2+64} - \frac{13}{569-40x+x^2}$$
$$= \frac{8\left(569-40x+x^2\right)-13\left(x^2+64\right)}{\left(x^2+64\right)\left(569-40x+x^2\right)}$$
$$= \frac{5\left(744-64x-x^2\right)}{\left(x^2+64\right)\left(x^2-40x+569\right)}$$

(c) Maximum light intensity at P occurs when $\frac{d\theta}{dx} = 0$.

$$744-64x-x^2 = 0$$
$$x = 10.05$$

(d) Using the graph of θ against x, the minimum θ is $\theta = 0.994$ when $x = 0$.

(e) When $x = 10$, $\frac{d\theta}{dx} = \frac{20}{44116}$

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} = \frac{20}{44116} \times 0.5 = 0.000227 \quad \left(\text{rads}^{-1}\right)$$