2012 J2 FYE P2

1 (i) Since θ is small, $\mathrm{h}(\theta)=\dfrac{1-\sqrt{2(1-\cos 3\theta)}}{2+\sin\theta}\approx\dfrac{1-\sqrt{2\left(1-1+\dfrac{9\theta^2}{2}\right)}}{2+\theta}=\dfrac{1-3\theta}{2+\theta}$

(ii)
$$\left(\frac{1-3\theta}{2+\theta}\right)^n=(1-3\theta)^n(2+\theta)^{-n}$$
$$=\frac{1}{2^n}\left(1-\frac{n\theta}{2}+....\right)(1-3n\theta+...)$$
$$\approx\frac{1}{2^n}\left(1-3n\theta-\frac{n\theta}{2}\right)$$
$$=\frac{1}{2^n}-\frac{7n\theta}{2^{n+1}}\qquad\Rightarrow a=\frac{1}{2^n},b=-\frac{7n}{2^{n+1}}$$

(iii) $-2<\theta<2$

(iv)

Let $y=\left(\dfrac{1-3\theta}{2+\theta}\right)^n$ $\Rightarrow \ln y=n\ln(1-3\theta)-n\ln(2+\theta)$

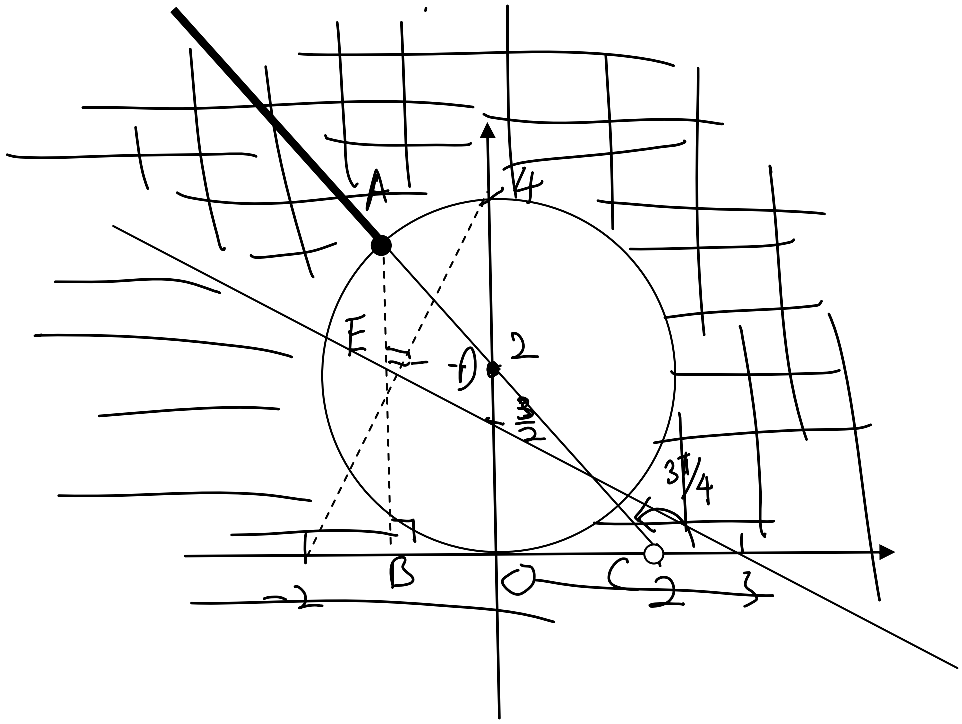
$$\Rightarrow\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}\theta}=-\frac{3n}{1+3\theta}-\frac{n}{2+\theta}$$

$\therefore$ when $\theta=0$, $y=\dfrac{1}{2^n}$ $\Rightarrow$ $\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\dfrac{1}{2^n}\left(-3n-\dfrac{n}{2}\right)=-\dfrac{7n}{2^{n+1}}$

$\therefore y\approx f(0)+f'(0)=\dfrac{1}{2^n}-\dfrac{7n\theta}{2^{n+1}}$ (Verified)

(i) $\dfrac{|z-4i|}{|2+z|}\le 1 \Rightarrow |z-4i| \le |z-(-2)|$

$$\arg(z-2)^{1/3}=\frac{\pi}{4}\Rightarrow\frac{1}{3}\arg(z-2)=\frac{\pi}{4}\Rightarrow\arg(z-2)=\frac{3\pi}{4}$$



(ii) z nearest to origin is complex represented by A.

Method 1

Using triangle ADE,

$$\cos\frac{\pi}{4}=\frac{DE}{AD}\Rightarrow DE=\frac{1}{\sqrt{2}}\times 2=\sqrt{2}\,,\quad \sin\frac{\pi}{4}=\frac{AE}{AD}\Rightarrow AE=\frac{1}{\sqrt{2}}\times 2=\sqrt{2}$$

Therefore $z=-\sqrt{2}+\mathrm{i}\left(2+\sqrt{2}\right)$

Method 2

$DC=\sqrt{2^2+2^2}=2\sqrt{2}$ =>AC= $2+2\sqrt{2}$

Using triangle ABC,

$$\sin\frac{\pi}{4}=\frac{AB}{AC}\Rightarrow AB=\frac{1}{\sqrt{2}}\left(2+2\sqrt{2}\right)=\sqrt{2}+2$$
$$\cos\frac{\pi}{4}=\frac{BC}{AC}\Rightarrow BC=\frac{1}{\sqrt{2}}\left(2+2\sqrt{2}\right)=\sqrt{2}+2\Rightarrow OB=BC-OC=\sqrt{2}$$

Therefore $z=-\sqrt{2}+\mathrm{i}\left(2+\sqrt{2}\right)$

Total volume of revolution of four rectangles =

$$\frac{\pi}{4}\left[\left(\frac{1}{1+5/4}\right)^2+\left(\frac{1}{1+6/4}\right)^2+\left(\frac{1}{1+7/4}\right)^2+\left(\frac{1}{1+2}\right)^2\right]=\frac{\pi}{4}\left[\left(\frac{4}{9}\right)^2+\left(\frac{4}{10}\right)^2+\left(\frac{4}{11}\right)^2+\left(\frac{4}{12}\right)^2\right]=\sum_{r=1}^{4}\frac{4\pi}{(8+r)^2}$$

Total volume of revolution of n rectangles =

$$\frac{\pi}{n}\left[\left(\frac{1}{1+\left(1+1/n\right)}\right)^2+\left(\frac{1}{1+\left(1+2/n\right)}\right)^2+\ldots+\left(\frac{1}{1+2}\right)^2\right]=\frac{\pi}{n}\left[\left(\frac{n}{2n+1}\right)^2+\left(\frac{n}{2n+2}\right)^2+\ldots+\left(\frac{n}{2n+n}\right)^2\right]=\sum_{r=1}^{n}\frac{n\pi}{(2n+r)^2}$$

(i)

Exact volume of solid formed $=\pi\int_1^2\frac{1}{(1+x)^2}\,\mathrm{d}x=-\pi\left[(1+x)^{-1}\right]_1^2=\frac{\pi}{6}$

Since estimated volume $<$ exact volume of solid formed $\Rightarrow\sum_{r=1}^{n}\frac{n\pi}{(2n+r)^2}<\frac{\pi}{6}\quad\Rightarrow\sum_{r=1}^{n}\frac{1}{(2n+r)^2}<\frac{1}{6n}$

(ii)

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n\pi}{(2n+r)^2}=\frac{\pi}{6}\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{(2n+r)^2}=\frac{1}{6}$$

4(a) By long division,

$$\frac{x^2}{x^2+4}=1-\frac{4}{x^2+4}=1-\frac{1}{\left(\frac{x}{2}\right)^2+1}$$ Hence, $A=1,\ B=-1,\ C=\frac{1}{2}$.

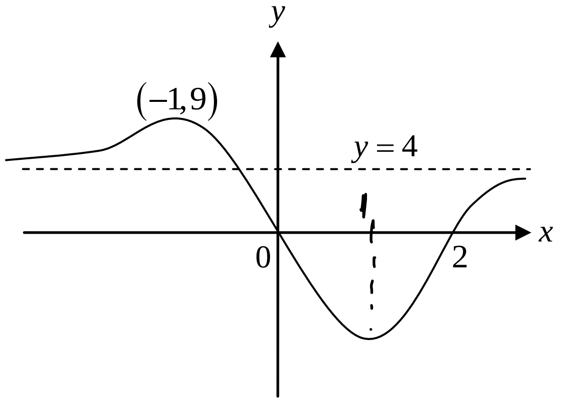
The sequence of transformations is:

(i) Scaling parallel to x-axis by a factor of 2

(ii) Reflection about the x-axis

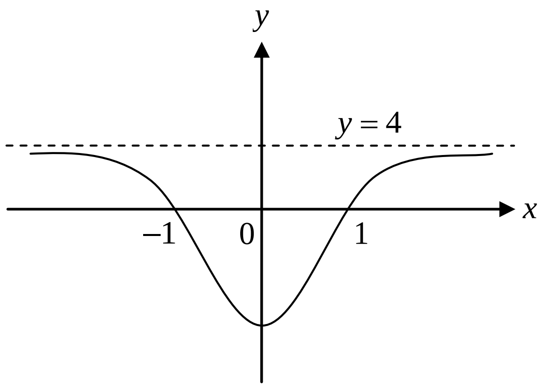
(iii) Translation of 1 unit in the direction of the y-axis

(b) (i)



(ii)

$y = \mathrm{f}(x)$



$y = \mathrm{f}(|x|+1)$

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(i) $\mathrm{f}\left(x\right) = \dfrac{x-\alpha}{x-\beta} = 1 + \dfrac{\beta-\alpha}{x-\beta}$

From sketch, any line, *y* = *k* will cut the graph at most once.

$\Rightarrow$ f is 1-1

$\Rightarrow \mathrm{f}^{-1}$ exists for any α,β

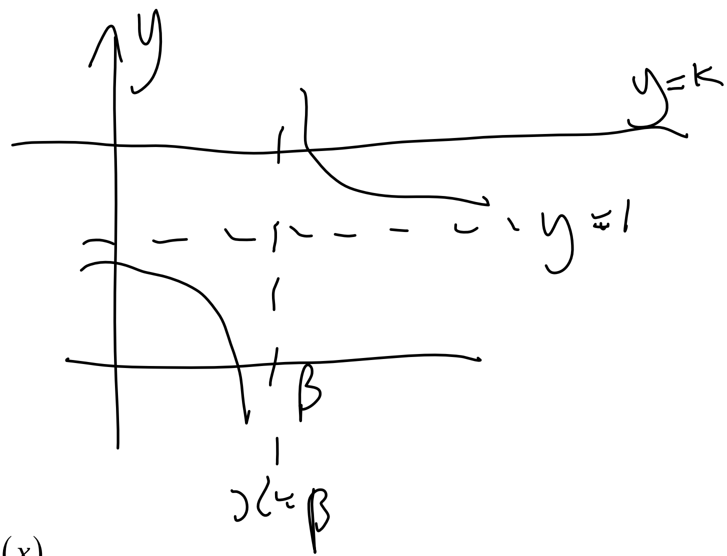
(ii) Since $\beta = 1$,

Let $y = \dfrac{x-\alpha}{x-1},\ x \neq 1$

$\Rightarrow xy - y = x - \alpha$

$\Rightarrow x = \dfrac{y-\alpha}{y-1}$

$\Rightarrow \mathrm{f}^{-1}(x) = \dfrac{x-\alpha}{x-1} = \mathrm{f}\left(x\right)$



$\mathrm{f}^{-1}(x) = \mathrm{f}\left(x\right)$

$\Rightarrow x = \mathrm{f}^{(2)}\left(x\right) \ \Rightarrow x = \mathrm{f}^{(2012)}\left(x\right) \Rightarrow \mathrm{f}\left(x\right) = \mathrm{f}^{(2013)}\left(x\right)$

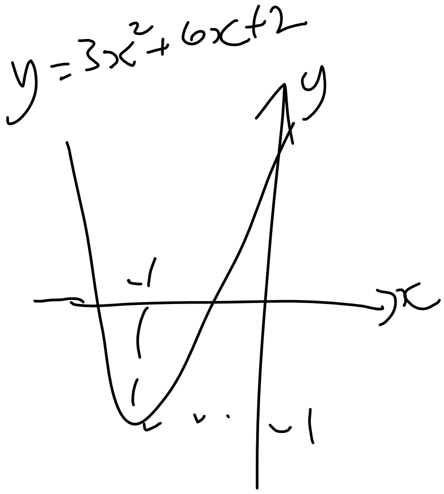
$\therefore \mathrm{f}^{(2013)}\left(\alpha\right) = \mathrm{f}\left(\alpha\right) = 0$

(iii) $g(x) = 3x^2 + 6x + 2 = 3(x+1)^2 - 1 \Rightarrow \text{min. value} = -1$

$\Rightarrow R_g = [-1, \infty)$

Since $D_f = \mathbb{R} \setminus \{\beta\}$ and fg does NOT exists

=> $R_g \not\subset D_f \Rightarrow \beta \in R_g \Rightarrow \beta \ge -1$



(iv) $fg(x) = \dfrac{3x^2 + 6x + 2 - \alpha}{3x^2 + 6x + 2 - \beta}$

$\Rightarrow fg(0) = 2 \Rightarrow \dfrac{2-\alpha}{2-\beta} = 2 \Rightarrow \alpha = -2 + 2\beta < -4 \quad (\text{from (iii) } \beta < -1)$

6

(i) Stratified Sampling (since the service provider should have complete list and relevant details of his subscribers)

(ii) (1) Divide the subscribers into ***mutually exclusive strata*** like age, gender, locations etc

(2) ***Simple random samples*** are drawn separately from each stratum with appropriate proportion.

Eg. If there are 60% male subscribers and 40% female subscribers => out of n subscribers to be selected, simple random sample of 0.6n male subscribers and 0.4n female subscribers need to be formed.

(3) Simple random samples ***put together to form a complete sample*** of subscribers to obtain feedback

(i) $9^6 = 531,441$

(ii) $^9C_3 \times \dfrac{6!}{2! \times 2! \times 2!} = 7560$

(iii) <u>Case1 : 3 odd 3 evens</u>

$^4C_3 \times 5^3 \times 4^3 = 32000$

<u>Case 2 : 2 odd 4 evens</u>

$^5C_2 \times 5^2 \times 4^4 = 64,000$

Required answer = 32,000 + 64,000 = 96,000

8(a)

Let X represent the number of times A occurs per day.

$X \sim \mathrm{Po}(8) \Rightarrow \mathrm{E}(X) = \mathrm{Var}(X) = 8$

$\Rightarrow \overline{X} \sim \mathrm{N}\left(8, \frac{8}{50}\right)$ approx (by Central limit theorem,since sample size =50 is large)

$\Rightarrow \mathrm{P}\left(\overline{X} \le 7\right) \approx 0.0062096799 \approx 0.006$

(b)

Let Y represent the number of times A occurs per day.

$Y \sim \mathrm{Po}(m+0.5)$

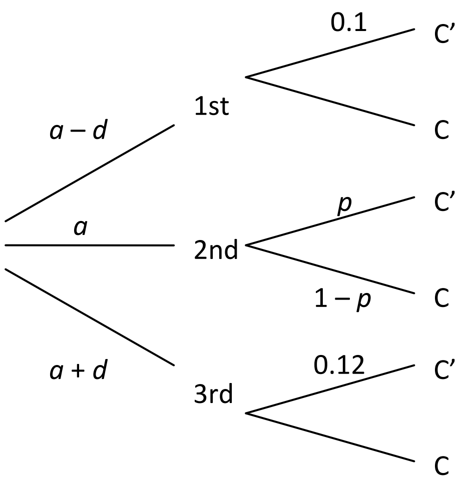
$\Rightarrow Y \sim \mathrm{N}(m+0.5, m+0.5)$ approx (since m is large, hence is $m+0.5$)

$\mathrm{P}(Y \le m) \approx \mathrm{P}(Y < m+0.5)$ (continuity correction)

$= 0.5$

9

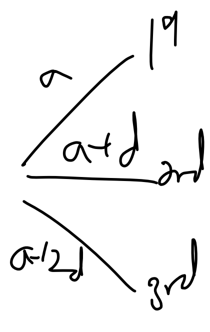
(i)



<u>Method 1</u>

$(a-d)+a+(a+d)=1 \Rightarrow a = \frac{1}{3}$

$\mathrm{P}(2nd) = a = \frac{1}{3}$ (Shown)



<u>Method 2</u>

$a+(a+d)+(a+2d)=1 \Rightarrow 3(a+d)=1 \Rightarrow a+d = \frac{1}{3}$

$\mathrm{P}(2nd) = a+d = \frac{1}{3}$ (Shown)

(ii) $\frac{1}{3}(1-p) = \frac{14}{45}$

$p = \frac{1}{15}$

(iii) P(3rd yr | C') = 0.6

$$\frac{P(\text{3rd yr} \cap C')}{P(C')} = 0.6$$

$$\frac{\left(\frac{1}{3}+d\right)(0.12)}{\left(\frac{1}{3}-d\right)(0.1)+\frac{1}{3}\times\frac{1}{15}+\left(\frac{1}{3}+d\right)(0.12)} = 0.6 \Rightarrow \quad ..... \quad \Rightarrow d = \frac{13}{81}$$

Proportion of 3rd year students $= a + d = \frac{1}{3} + \frac{13}{81} = \frac{40}{81}$

(iv) P(3rd year | did not complete) = 0.6

P(3rd year) = $\frac{40}{81} \neq 0.6$

$\Rightarrow$ The two events are not independent.

10.

(i) Each phone is selected to purchase (by customers) independently from one another.
(**OR** the probability that a camera phone is purchased is constant for each phone purchased).

(ii) Let C' denote the number of non-camera phones purchased => $C' \sim \mathrm{B}(N, 0.05)$

$$P(C' > 5) < 0.1 \quad \Rightarrow \quad 1 - \mathrm{P}(C' \le 5) < 0.1 \Rightarrow \mathrm{P}(C' \le 5) > 0.9$$

From GC, When N = 63, $P(C' \le 5) = 0.90551 > 0.9$

When N = 64, $P(C' \le 5) = 0.89990 < 0.9$

Therefore, largest number is 63.

(iii) Let W denote the number of non-camera phones purchased out of 180 phones monitored.

=> W ~ B(180,0.02)

Since $n = 180$ is large, $np = 3.6 < 5 \Rightarrow W \sim \text{Po}(3.6)$ approx..

P(less than 171 camera phones) =P(at least 10 non-camera phones)

$= \mathrm{P}(W \ge 10)$

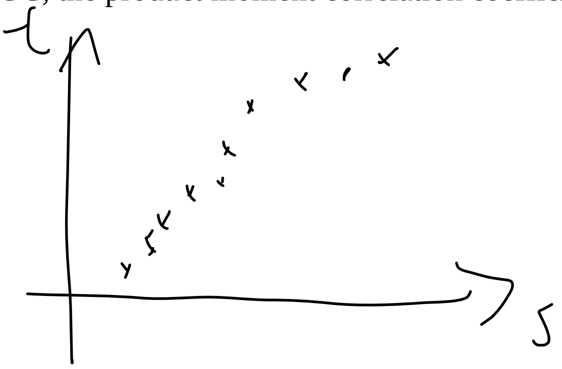
$= 1 - \mathrm{P}(W \le 9)$

$\approx 0.004024267 \approx 0.004$

11

(i) From GC, the product moment correlation coefficient for the data is 0.950. (3 s.f.)

(ii)



The product moment correlation figure may suggest a strong linear relationship, but the scatter plot suggests otherwise.

(iii) C : $t = a + b\ln s$, since the scatter plot has a shape similar to curve y=ln$x$.

(iv) Model C should have a higher positive value of product moment correlation coefficient than (ii)'s model

(v) Use regression line of Y=lns on X=t.
From GC, Y= 0.4837235603+0.019805927X

When t=5, $Y=0.5827532$

$=> s = \mathrm{e}^{Y} \approx \mathrm{e}^{0.5827532} \approx 1.790962527 \approx 1.79$

Not reliable as this is an extrapolation since t=5 is out of the data range for TV.

12

(i) Unbiased estimate for the mean $= \bar{x} = \dfrac{\sum x}{10} = 1997.6$

Unbiased estimate for the variance = $s^2 = 67.37777778 \approx 67.4$

(ii)

Test $H_0 : \mu = 2000$ *vs* $H_1 : \mu < 2000$
Level of significance : 0.5%

Since n is small and σ^2 is unknown, we use the t-test.

Test statistic: $t = \dfrac{\bar{x} - \text{"claimed value"}}{\frac{s}{\sqrt{n}}} = \dfrac{1997.6 - 2000}{\frac{8.208396785}{\sqrt{10}}} \approx -0.9246$

From GC, p-value is 0.18965 > 0.005

Since the p-value is _more___ than the level of significance, we do not reject H_0. There is __insufficient_______ evidence, at the 0.5% level, to conclude that the battery's lifespan less than 2000 hours.

Assumption : The lifespan of the battery is normally distributed.

Therefore Tom should recommend the battery to his classmates.

(iii) The 0.5% level of significance is the probability of rejecting the fact that the mean life span of the battery is 2000 hrs when it is true.

(iv) To have a different conclusion => to reject H_o

Use *z*-test

$$\Rightarrow z - test = \frac{1997.6 - 2000}{\frac{5}{\sqrt{n}}} < invnorm(0.005)$$

$\Rightarrow$ . . . .

$$\Rightarrow n > 28.779$$

13(i)

$$D_p \sim N(0.9, 0.005^2), \quad D_s \sim N(0.91, 0.005^2)$$

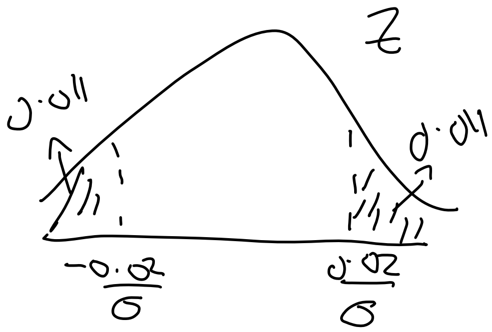
$$P(D_p > D_s) = P(D_p - D_s > 0)$$

$$D_p - D_s \sim N(-0.01, 2 \times 0.005^2)$$

Req. prob = $P(D_p - D_s > 0) = 0.0786496525 \approx 0.0786$

(ii) The pencil's and sharperner hole's diameters are independent of each other.

(iii)



$$D_p \sim N(0.9, \sigma^2) \Rightarrow Z = \frac{D_p - 0.9}{\sigma} \sim N(0,1)$$

$$\text{P}(|Dp - 0.9| > 0.02) = 0.022$$

$$\Rightarrow \text{P}\left(Z < \frac{-0.02}{\sigma}\right) + \text{P}\left(Z > \frac{0.02}{\sigma}\right) = 0.022$$

$$\Rightarrow \text{P}\left(Z < \frac{-0.02}{\sigma}\right) = 0.011$$

$$=> \frac{-0.02}{\sigma} = -2.290367878 \Rightarrow \sigma = 0.0087322217 \approx 0.00873$$

(iv)

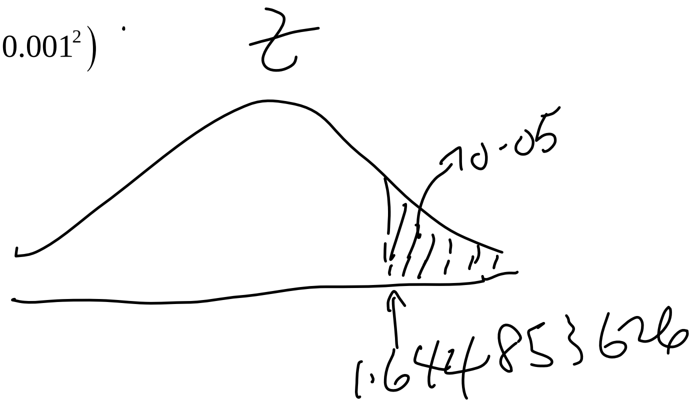
$P_i \sim N(0.9, 0.003^2), \quad S_i \sim N\left(0.91, 0.001^2\right), \ i = 1, 2, ..., n$

$P(\text{total pencil circumference+total sharperner's circumference}>30) \leq 0.05$

$\Rightarrow P\left(\pi\left(P_1 + P_2 + ... + P_n + S_1 + S_2 + ... + S_n\right) > 30\right) \leq 0.05$

$\Rightarrow P\left(P_1 + P_2 + ... + P_n + S_1 + S_2 + ... + S_n > \frac{30}{\pi}\right) \leq 0.05$

$P_1 + P_2 + ... + P_n + S_1 + S_2 + ... + S_n \sim N\left(n1.81, n0.003^2 + n0.001^2\right)$



$\Rightarrow P\left(Z > \frac{\frac{30}{\pi} - n1.81}{\sqrt{n0.003^2 + n0.001^2}}\right) \leq 0.05$

$\Rightarrow \frac{\frac{30}{\pi} - n1.81}{\sqrt{n0.003^2 + n0.001^2}} \geq 1.644853626$

$\Rightarrow n \leq 5.2692\,9 \quad \text{(by GC)}$

$\Rightarrow \max n = 5$