Name:__________________________________ Class:_____________

JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2024

FURTHER MATHEMATICS **9649/01**

Higher 2 **12 September 2024**

Paper 1 **3 hours**

Additional materials: 12-page Answer Booklet
(additional 4-page Answer Booklet(s), if applicable)
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 Show by induction that $\sum_{r=1}^{n} \frac{1}{r^2} < 2 - \frac{1}{n}$ for all integers $n \geq 2$. **[6]**

2 The linear transformation $\mathrm{T}: \mathbb{R}^4 \to \mathbb{R}^3$ is represented by the matrix **A** where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 6 \\ 1 & 1 & 2 & 4 \\ 3 & 7 & 6 & 20 \end{pmatrix}$$

(i) Let R_{T} and K_{T} be the range space and null space of T respectively. Find the dimension of R_{T} and deduce the dimension of K_{T}. **[3]**

(ii) By finding the bases for R_{T} and K_{T}, find the general solution of $\mathrm{T}(\mathbf{x}) = \begin{pmatrix} 3 \\ 1 \\ 11 \end{pmatrix}$. **[5]**

3 Marine scientists calculated that when the concentration of a particular chemical in the waters at a beach along the coast reaches 7 milligrams per litre (mg/l), the level of pollution endangers all marine life in the area.

A factory wishes to release waste containing the chemical into the coastal waters. It is claimed that the discharge will not endanger the marine life in the region.

The local authority is provided with the following information:

- The coastal waters contains none of this chemical at present.
- The factory manager has applied for a permit to discharge waste on a weekly basis into the coastal waters. The discharge, which will be done at the beginning of each week, will result in an increase in concentration of 2.5 mg/l of the chemical along the coast.
- The tidal streams will remove 7% of the chemical from the coastal waters every day.

(i) Based on the information, form a recurrence relation for the concentration level of chemical, u_n, at the beginning of week *n*. Hence, find the concentration at the beginning of week *n*. **[4]**

(ii) Based on the concentration level of the chemical, should the local authority allow the factory to go ahead with the discharge? Justify your answer. **[2]**

4 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & -3 & -3 \\ 2 & a-5 & -3 \\ -2 & 8 & a+6 \end{pmatrix},$$

where a is a real constant.

Given that $\mathbf{A}$ has an eigenvalue of 2, find the possible values of a exactly. **[3]**

For the negative value of a that you have obtained, determine all the eigenvalues and corresponding eigenvectors of

(i) $\mathbf{A}$, **[5]**

(ii) $2\mathbf{A} - 3\mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix. **[2]**

5 The motion of the tip of a tuning fork can be modelled by the differential equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + m\omega^2 x = 0$$

where x is the displacement of the tip from its equilibrium position at time t and m, k and ω are positive constants. It is known that k is so small that k^2 can be ignored as k models the slight damping due to the resistance of the air. It is given that the tip of the fork is initially in its equilibrium position and moving with speed v in the positive x-direction.

(i) Solve the differential equation. **[4]**

The amplitude of a vibration is the maximum displacement of the tip from its equilibrium position and one period of a vibration is the time interval between the occurrences of two consecutive amplitudes.

(ii) Consider the period of the vibrations over time and show that the amplitude of successive vibrations follows a geometric progression. **[3]**

(iii) Given that k is no longer small and $k^2 > 4m^2\omega^2$, describe the behaviour of x as time progresses and sketch a possible graph of x vs t. Justify your answer. **[3]**

6 The number of branches on a tree in a particular year is modelled as the number of branches that were on the tree in the previous year plus new growth of k times the number that were on the tree the year before that, where $0 < k < 1$.

(i) Let x_n be the number of branches on the tree n years after it was bought. Write down a recurrence relation for x_{n+2} in terms of x_{n+1}, x_n and k. **[1]**

(ii) The tree was bought with 20 branches. It had 25 branches after one year. Given that $k = 0.11$, solve your recurrence relation. **[5]**

To control the growth of the tree it is pruned each year after the new growth has taken place. New growth is not pruned, but a proportion, r, where $0 < r < 1$, of old branches is removed. Let y_n be the number of branches, after pruning, on the tree n years after it was bought.

(iii) Modify your answer to part **(i)** to produce a recurrence relation for y_{n+2} in terms of y_{n+1}, y_n, k and r. **[1]**

7 A particular solution of the differential equation

$$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} - 2y + (1+x)y^2 = 0$$

has $y = 1$ when $x = 0$.

(i) Use the Euler method with step size 0.5 to estimate y at $x = 1$. **[2]**

(ii) Show by means of the substitution $y = \frac{1}{z}$, that the differential equation reduces to

$$\frac{\mathrm{d}z}{\mathrm{d}x} + \left(\frac{2}{1+x}\right)z = 1.$$ **[2]**

Hence find y in terms of x. **[6]**

(iii) Find the percentage error of your estimation in part **(i)** and suggest a way to improve on your estimation in part **(i)**. **[2]**

8 A curve C is defined parametrically by

$$x = r(\theta - \sin\theta),\ y = r(1 - \cos\theta),$$

where r is a positive constant and $0 \le \theta \le 2\pi$.

(i) The curve C is rotated through one revolution about the x-axis. Given that the area of the surface generated is 432π units^2, find the exact value of r. **[7]**

(ii) Suppose instead that $r = 2$. Let L be the line that passes through the origin and the point $(3\pi + 2, 2)$ on C. The region R is enclosed by C and L. Calculate the volume of the solid generated when R is rotated through 2π radians about the y-axis. **[6]**

9 A logistic growth model for the population $P(t)$ of a certain species of elephants in a habitat is given by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 2P - \frac{1}{150}P^2 .$$

(i) State the carrying capacity of the habitat. **[1]**

(ii) Find the general solution of the above differential equation, expressing P explicitly in terms of t. **[3]**

The population of elephants is then subjected to a constant poaching rate h and the population is now modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 2P - \frac{1}{150}P^2 - h .$$

(iii) Determine the maximum sustainable poaching rate and show that it occurs when the population is at half its carrying capacity. **[3]**

For the rest of the question, assume $h = 100$.

(iv) Determine the eventual size of the population if there are initially 80 elephants and sketch the solution curve. **[4]**

(v) Given instead that the initial population of elephants is 10, determine the earliest time T, correct to 2 decimal places, that poaching can start to ensure the survival of the population. **[3]**

10 **(a)** The matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is such that $\mathbf{A}^2 = \mathbf{A}$, where a, b, c, d are constants, $a \neq 0$, $d \neq 0$.

(i) Prove that det $\mathbf{A}$ must be 1 or 0. **[2]**

(ii) Prove that if det $\mathbf{A} = 1$, then $\mathbf{A} = \mathbf{I}$. **[2]**

(iii) Prove that if det $\mathbf{A} = 0$, then $a + d = 1$. **[3]**

(b) The sets A and B are defined as follows:

$$A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y - z = 0 \right\}, \quad B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 2x + 3y - 5z = 1 \right\}.$$

(i) Show that A is a subspace of $\mathbb{R}^3$ but B is not a subspace of $\mathbb{R}^3$. **[3]**

(ii) Show that $A \cup B'$ and $A \cap B'$ are both not subspaces of $\mathbb{R}^3$. **[4]**

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