Chapter 6 MOTION IN A CIRCLE

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- Summary

Learning Outcomes

Candidates should be able to:

(a) express angular displacement in radians

(b) show an understanding of and use the concept of angular velocity to solve problems

(c) recall and use $v = r\omega$ to solve problems

(d) describe qualitatively motion in a curved path due to a perpendicular force, and understand the centripetal acceleration in the case of uniform motion in a circle

(e) recall and use centripetal acceleration $a = r\omega^2$, and $a = v^2/r$ to solve problems

(f) recall and use centripetal force $F = mr\omega^2$, and $F = mv^2/r$ to solve problems

6.1 Introduction

Circular motion is a common occurrence in our daily lives. The second hand of a clock goes round in a circle in 60 seconds while a geostationary satellite revolves around the earth once every 24 hours. You probably have also experienced circular motion when you sat on a carousel or in the Singapore Flyer.

As physics students, we want to go beyond the physical experience. We want to understand what causes an object to move in a circular motion and the quantities associated with it. Let's begin by asking 2 questions which we will address in detail in **6.3**.

1. Why must an object moving with uniform speed in a circle experience a force?
2. Why is that force directed towards the centre of the circle?

At the end of this topic, you should be able to answer questions these confidently.

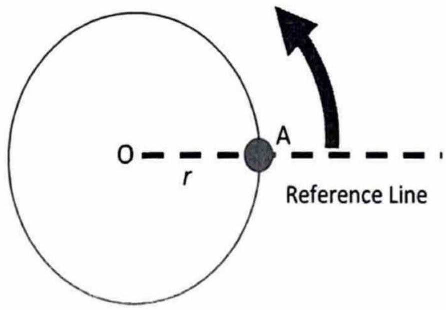
6.2 Kinematics of Uniform Circular Motion

It is not possible to apply the kinematics equations found in Chapter 2 to uniform circular motion because the direction of acceleration is constantly changing. Hence, we need a new set of equations. These new equations require the understanding of the following new terminologies:

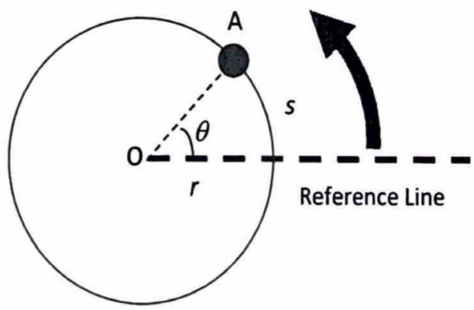
angular displacement, angular velocity, period and frequency.

Angular Displacement

Consider an object moving in a circle of radius r about a fixed point O as the centre.



(a) at $t = 0$



(b) at $t = \Delta t$

Fig 6.1

At time $t = 0$, the object A is on the reference line as shown in **Fig 6.1(a)**. After a time interval Δt has elapsed, the object has moved to a new position as shown in **Fig 6.1(b)**. In this time interval, the line OA has moved through the angle θ with respect to the reference line. The angle θ is said to be the **angular displacement** of the object.

Definition

Angular displacement is the angle an object makes with respect to a reference line.

The unit of angular displacement is **radian**.

Definition

The **radian** is the angle subtended by an arc length equal to the radius of the arc.

In general, any angle θ measured **in radians** is defined by the relation:

Formula

$$\theta = \frac{s}{r}$$

where s : arc length
r : the radius of the circle.

When the body moves one complete circle, the arc length s would be the circumference of the circle, which is equal to $2\pi r$. Therefore,

$$\theta = \frac{2\pi r}{r} = 2\pi$$

And we know that in one complete circle, the body moves through 360°. Therefore $2\pi \equiv 360°$

$$\Rightarrow \quad \pi \equiv 180°$$

In general,

$$x \text{ rad} \equiv x \times \frac{180°}{\pi}$$

Conversely,

$$y° \equiv y \times \frac{\pi}{180} \text{ rad}$$

Example 1 **Fill in the blanks.**

Angle in degrees	Angle in radians (in terms of π)
360	2π
90	π / 2
180	π
45	$\frac{\pi}{4}$
30	$\frac{\pi}{6}$
1	$\frac{\pi}{180}$

Example 2

The laser in a CD player is 5.0 cm from the central of the disc. What length of the disc is scanned by the laser when the disc turns through an angle of 0.45 radians?

Solution

$$\theta = \frac{s}{r}$$

$$0.45 = \frac{s}{5.0}$$

$$s = 2.25 \text{ or } 2.3 \text{ cm}$$

Angular Velocity

Definition

Angular velocity ω of a body is defined as the rate of change of its angular displacement with respect to time.

Formula

$$\omega = \frac{d\theta}{dt}$$

where θ : angular displacement
t : time
S.I. Unit : rad s^{-1}

Important Note

Angular velocity is a **vector** – it has both magnitude and direction. The direction is described as "clockwise" or "anti-clockwise" at A-level.

Period and Frequency

The **period** of an object in circular motion is the time taken for it to make one complete revolution.

S.I. Unit: s

The **frequency** of an object in circular motion is the number of complete revolutions made per unit time.

S.I. Unit: s^{-1} or Hz

Formula

Period T and frequency f are related by the equation:

$$f = \frac{1}{T}$$

Relationship between Angular Velocity and Linear Velocity

Consider an object P rotating in a horizontal circle of radius r about point O. Suppose the object has an angular velocity ω and a linear velocity v at any point in its path (**Fig 6.2**).

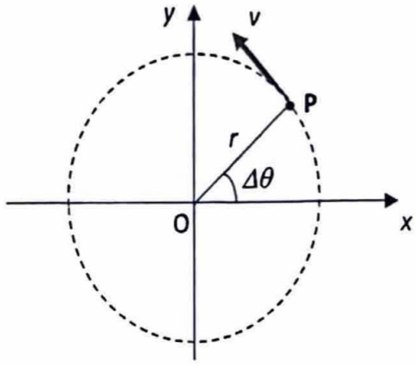


Fig 6.2

Differentiating $s = r\,\theta$ with respect to t and since r is a constant,

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

Formula

Therefore,

$$\boxed{v = r\omega}$$

where $v = \frac{ds}{dt}$ and $\omega = \frac{d\theta}{dt}$

Linear velocity v is also known as the **tangential velocity**.

Relationship between Period, Angular Velocity and Frequency

Consider an object moving in uniform circular motion, i.e. with a constant ω.

Time taken for one complete revolution = T.

Angular displacement for one complete revolution = 2π.

From the definition, $\omega = \frac{d\theta}{dt}$

Formulas

Therefore,

$$\boxed{\omega = \frac{2\pi}{T}}$$

Since $T = \frac{1}{f}$,

$$\boxed{\omega = 2\pi f}$$

Relationship between Period and Linear Speed

If the object is moving in uniform circular motion, its linear speed v can also be found by:

$$v = \frac{\text{Circumference of the circle}}{T}$$

Formula

Therefore,

$$\boxed{T = \frac{2\pi r}{v}}$$

Example 3

Alice and Bob are riding on a merry-go-round, which is rotating at a constant angular velocity. Alice stands on a point twice as far from the centre of the platform as Bob.

Which of the following statements is correct?

A. Alice's linear velocity is twice of Bob's.
B. Alice's linear velocity is the same as Bob's.
C. Alice's linear velocity is half of Bob's.
D. Alice's linear velocity is a quarter of Bob's.

6.3 Dynamics of Uniform Circular Motion

Consider an object moving in uniform circular motion.

NOTE

A. Why must the object experience a force?

Although the object is moving at a constant speed, its velocity is always changing because its direction of motion is constantly changing. Hence, this object experiences an acceleration. This leads to a force acting on the object since $F = ma$.

OR

According to Newton's First Law of motion, the object will continue to move at its constant speed in the same direction unless there is a net force acting on it. Hence, there must be a force acting on the object to constantly change its direction of motion.

B. Why is the force (acceleration) directed towards the centre of the circle?

Or Why is the force (acceleration) perpendicular to the motion of object?

As the object is moving at a constant speed, there must be no component of force (acceleration) in the direction of the motion of object. Otherwise, it will increase or decrease the speed of the object. Therefore, the direction of this force (acceleration) has to be perpendicular to the direction of motion.

Using a vector diagram, it can be found that the direction of the change in velocity (acceleration) is towards the centre of the circular motion.

An acceleration of this nature is called the ***centripetal acceleration****. And the force is called the* ***centripetal force****.*

Centripetal Acceleration and Centripetal Force

The magnitude of the **centripetal acceleration** is given by:

$a = v\omega$ (refer to Appendix A for derivation)

Formulas

Since $v = r\omega$,

$$a = v\omega$$
$$a = r\omega^2$$
$$a = \frac{v^2}{r}$$

By Newton's 2nd Law, $F = ma$, the **centripetal force** acting on the object is

Formula

$$F = ma = mv\omega = mr\omega^2 = \frac{mv^2}{r}$$

where F : centripetal force
m : mass of body
a : centripetal acceleration

Note

1. As the centripetal force acts at right angles to the motion, it only changes the direction of the velocity, but not the speed of the object.
2. Although a centripetal force is needed to keep an object moving in a circle, the force does **no work** on the object since there is no displacement of the object in the direction of the force.
3. The centripetal force is not a new force! It can be the tension in a string, gravitational force of attraction between 2 masses, frictional force between 2 surfaces or a combination of forces. (A common mistake made by students is to include the centripetal force in free body diagram on top of the other forces acting on the body.)
4. If the force that produces the centripetal acceleration is removed, the object does not continue to move in its circular path; instead, it will move tangentially to its original path.

Example 4

Assuming that the Earth orbits round the Sun at a constant speed of 29.9 × 10³ m s⁻¹, find the centripetal acceleration of the Earth.

Solution

T = 365 days

$a = r\omega^2 = v\omega$ (since $v = r\omega$)

$= v\left(\frac{2\pi}{T}\right)$

$= (29.9 \times 10^3)\left(\frac{2\pi}{365 \times 24 \times 60 \times 60}\right) = 5.96 \times 10^{-3}$ m s^{-2}

Problem Solving Strategy

1. Draw a free-body diagram of the body under consideration.
2. Identify the centre of the circular motion.
3. Resolve the forces along two perpendicular axes. One of the axes must be in the direction towards the centre of motion.
4. Determine and equate the resultant force towards the centre of circular path to *ma*, where *m* is the mass of the body and $a = \frac{v^2}{r} = r\omega^2$.
5. Equate the forces perpendicular to the plane of motion. There should be no acceleration in the plane perpendicular to the uniform circular motion.

Uniform Circular Motion in a Horizontal Table

A. A mass *m* moving in uniform circular motion

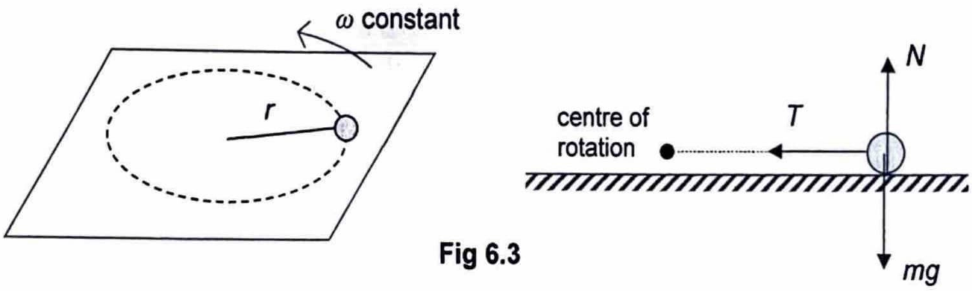


Fig 6.3

Resolving forces,

Vertically : $N = mg$

Horizontally : $T = \frac{mv^2}{r}$ (tension *T* provides the centripetal force)

B. Conical pendulum

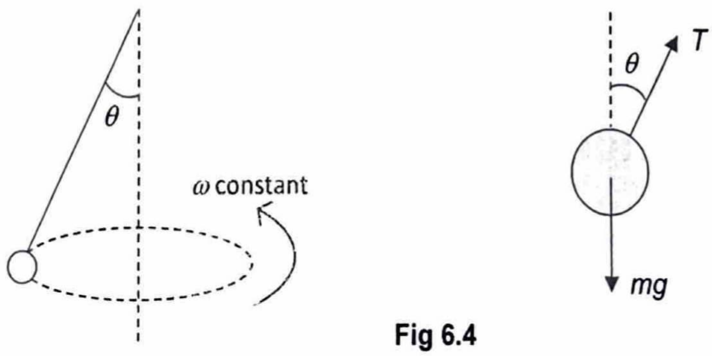


Fig 6.4

Resolving forces,

Vertically : $T\cos\theta = mg$ ..... (1)

Horizontally : $T\sin\theta = \frac{mv^2}{r}$ ..... (2)

$(2)/(1)$: $\tan\theta = \frac{v^2}{rg}$

C. An aircraft performing a horizontal circular turn

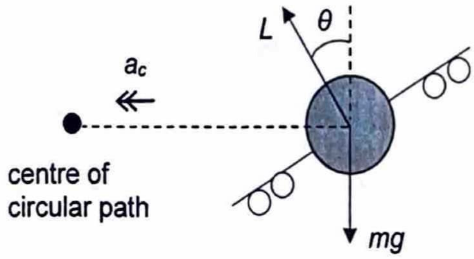


Fig 6.5

An aircraft, moving with speed v, describes a circular path of radius r in a horizontal plane. The forces acting on the aircraft are the lift L of the aircraft and the weight mg of the aircraft.

Resolving forces,

Vertically : $L\cos\theta = mg$ ..... (1)

Horizontally : $L\sin\theta = \dfrac{mv^2}{r}$ ..... (2)

$^{(2)}/_{(1)}$: $\tan\theta = \dfrac{v^2}{rg}$

D. A cyclist moving round a circular track

A cyclist is moving with speed v around a bend of radius r on a horizontal road. The forces acting on the cyclist are the contact force R due to the road, and the weight mg of the cyclist.

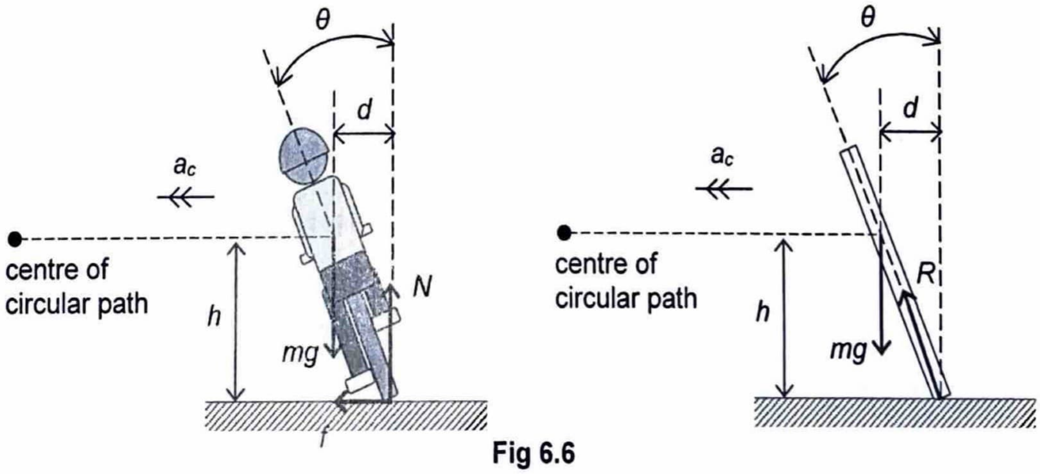


Fig 6.6

Resolving forces,

Vertically : $R\cos\theta = mg$ or $N = mg$ --- (1)

Horizontally : $R\sin\theta = \dfrac{mv^2}{r}$ or $f = \dfrac{mv^2}{r}$ --- (2)

$^{(2)}/_{(1)}$: $\tan\theta = \dfrac{v^2}{rg}$

Why does the cyclist have to lean inwards, towards the centre of the circular motion?

The centripetal force necessary for the cyclist to execute the turn is provided by the frictional force f acting on the bicycle. This force f produces a clockwise moment about the centre of gravity G which tends to turn the cyclist outwards. When the rider leans inwards, the normal contact force N produces a counterbalance moment about G so that no rotation would occur.

E. A racing car travelling round a circular bend along a flat horizontal track

Each of the 4 tyres experiences its own normal contact force and frictional force. By combining the forces on front and rear tyres so that we have just two normal contact forces, N_1 and N_2 and two sideward frictional forces f_1 and f_2 as shown.

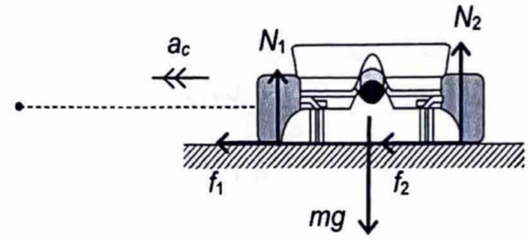


Fig 6.7

Resolving forces,

Vertically : $N_1 + N_2 = mg$

Horizontally : $f_1 + f_2 = ma = \dfrac{mv^2}{r}$

Note

As the racing car travels faster, the centripetal acceleration ($a = r\omega^2$) required to maintain the same radius increases. Therefore, the sideward friction needed to produce this acceleration must increase. At some particular speed, the force that is required to keep the car on the circular path will be greater than the maximum sideward frictional force that can be provided by the track. At this point the car will skid off the track.

To solve the above problem on a sharp bend (i.e. a small radius of curvature), the road surface is often tilted or sloped to minimize friction. The slope is called **banking**. It cuts down wear on the road surface and on the tyres. It also cuts down noise and aids safety. The banking helps to provide part of the centripetal force from the normal contact force, and thus reduce the sideward friction that is needed to produce the centripetal force.

F. A racing car travelling round a circular bend along a banked track

A racing car of mass *m* is travelling round a circular bend along a banked track so that there is no need for sideward frictional forces to provide for the centripetal acceleration. **(Ideal banking)**

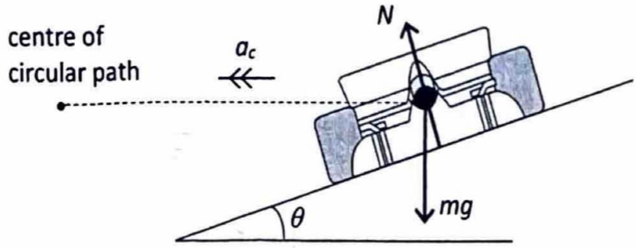


Fig 6.8

Resolving forces,

Vertically : $N\cos\theta = mg$ --- (1)

Horizontally : $N\sin\theta = \dfrac{mv^2}{r}$ --- (2)

$^{(2)}/_{(1)}$: $\tan\theta = \dfrac{v^2}{rg}$

The angle of banking θ is ideal as no frictional forces is providing for the centripetal acceleration. Only the horizontal component ($N\sin\theta$) of the normal contact force *N* provides the centripetal force required to keep the car moving round the bend.

For a particular angle of ideal banking θ,

$$\tan\theta = \frac{a_c}{g} = \frac{v_{\text{ideal}}^2}{rg}$$

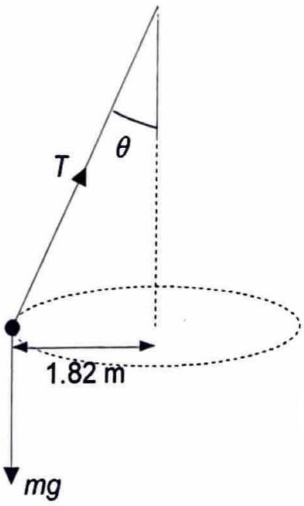
If $v_{actual} > v_{ideal}$, sideward frictional force on the tyres will act down the slope as the car tends to slide up the bank.

If $v_{actual} < v_{ideal}$, sideward frictional force on the tyres will act up the slope because the car tends to slide down the bank.

Example 5

A pendulum bob of mass 0.15 kg suspended from a fixed point moves in a horizontal circular motion of radius 1.82 m as shown. The bob moves at a steady speed, taking 18.0 s to make 10 complete revolutions.

Calculate (a) the centripetal acceleration of the bob,
(b) the tension in the thread.



Solution

(a) $$a_c = r\omega^2 = r\left(\frac{2\pi}{T}\right)^2$$

$$= 1.82 \times \left(\frac{2\pi}{1.80}\right)^2$$

$$= 22.2 \text{ m s}^{-2}$$

(b) Horizontally,

$T\sin\theta = ma_c$ --- (1)

Vertically,

$T\cos\theta = mg$ --- (2)

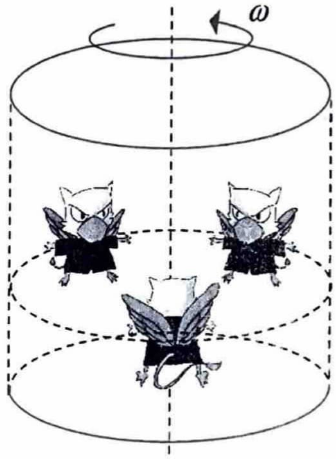
$\frac{(1)}{(2)}$: $\tan\theta = \frac{a_c}{g}$

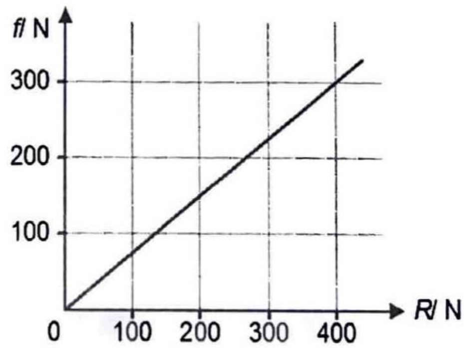
$$\theta = \tan^{-1}\left(\frac{22.18}{9.81}\right) = 66.1°$$

$$\therefore T = \frac{mg}{\cos\theta} = \frac{0.15\times 9.81}{\cos 66.14°} = 3.6 \text{ N}$$

Example 6

Three baby Griffles, each of mass 30 kg, are in a large vertical cage of radius 5.0 m, spinning about a vertical axis through its centre. When the cage is spinning sufficiently fast, the floor is dropped and the Griffles are stuck to the wall. The graph shows the variation in frictional force *f* with normal contact force *R* between the baby Griffles and the wall.





Determine the minimum angular speed at which the cage must be rotated for the Griffles to remain in position when the floor is dropped.

(Take g = 10 m s^{-2})

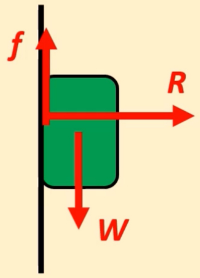
Example 6 (Cont'd)

Solution

For the Griffles to remain in position, $f = W = 30 \times 10 = 300$ N.

From the graph $R = 400$ N.

Since the centripetal force is provided by the normal contact force R,



$$R = mr\omega^2$$

$$\omega = \sqrt{\frac{R}{mr}} = \sqrt{\frac{400}{30 \times 5.0}}$$

$$\therefore \omega = 1.63 \text{ rad s}^{-1}$$

Uniform Circular Motion in a Vertical Circle

A ball moving in a vertical circle at uniform speed

A ball of mass m is attached to a light rod and is moving in a vertical circle of radius r at uniform speed v.

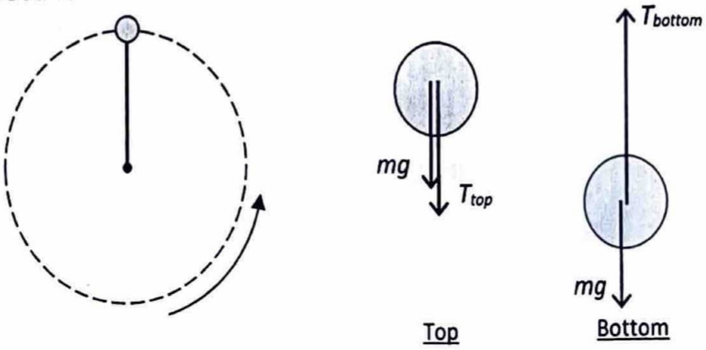


Fig 6.9

At the top,

$$T_{top} + mg = \frac{mv^2}{r}$$

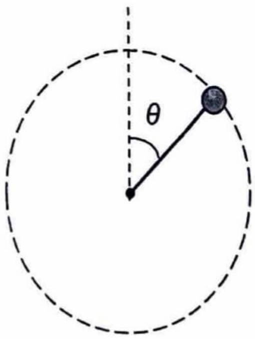
$$T_{top} = \frac{mv^2}{r} - mg$$

At the bottom,

$$T_{bottom} - mg = \frac{mv^2}{r}$$

$$T_{bottom} = \frac{mv^2}{r} + mg$$

Note that the tension in the rod is changing according to θ.



$$T + mg\cos\theta = \frac{mv^2}{r}$$

$$T = \frac{mv^2}{r} - mg\cos\theta$$

Non-uniform Circular Motion in a Vertical Circle

A body undergoing *non-uniform* circular motion has varying tangential speed due to non-zero tangential acceleration in addition to centripetal acceleration. In *uniform* circular motion, the net force is always directed towards the centre. In *non-uniform* circular motion, this is not true.

Roller coaster moving in a vertical loop

A roller coaster train of mass *m* moves around a vertical circular loop of radius *r*.

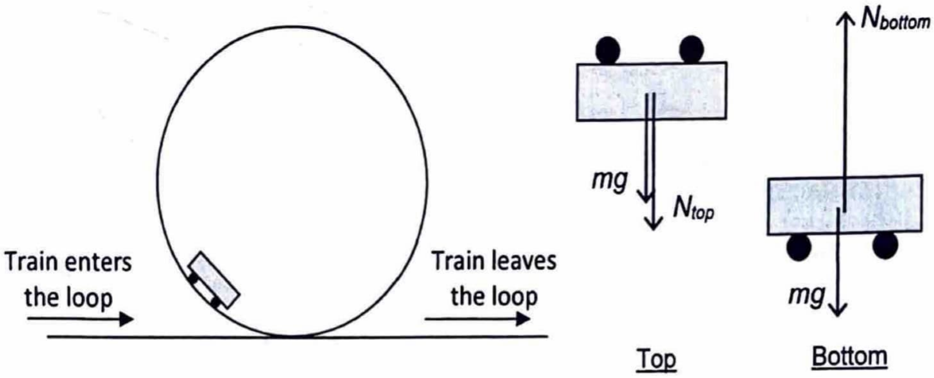


Fig 6.10

At the top,

$$N_{top} + mg = \frac{mv_{top}^2}{r}$$

$$N_{top} = \frac{mv_{top}^2}{r} - mg \qquad ---(1)$$

At the bottom,

$$N_{bottom} - mg = \frac{mv_{bottom}^2}{r}$$

$$N_{bottom} = \frac{mv_{bottom}^2}{r} + mg \qquad ---(2)$$

Comparing (1) and (2), we can conclude that $N_{bottom} > N_{top}$ (similar to the case of a ball swung in vertical circular motion where $T_{bottom} > T_{top}$).

The presence of a normal contact force acting on the train means that it remains in contact with the track. Since $N_{top} < N_{bottom}$, the train will more likely lose contact with the track at the top.

Hence for the train to remain on track throughout the motion around the loop, set N_{top} = 0 in (1). This will give you the minimum speed of the train at the top of the loop where it just remains in contact with the track. Therefore,

$$0 + mg = \frac{mv_{top,\min}^2}{r}$$

$$v_{top,\min} = \sqrt{gr}$$

Assuming that energy lost due to air resistance and friction are negligible, applying principle of conservation of energy,

Total initial energy = Total final energy

$$\frac{1}{2}mv_{bottom}^{2} = \frac{1}{2}mv_{top,min}^{2} + mg(2r)$$

$$\frac{1}{2}mv_{bottom}^{2} = \frac{1}{2}m(gr) + mg(2r)$$

$$v_{bottom} = \sqrt{5gr}$$

This is the minimum speed at which the train enters the loop so that it will remain in contact throughout the loop.

Example 7

A pail of water is whirled in a vertical circular path such that no water is spilled.

(a) Why does the water remain in the pail, even when the pail is upside down?

(b) Write an expression of the minimum linear speed v_{min} of the pail at the highest point in terms of the radius r of the motion.

Solution

(a) In a vertical circular motion, the water tends to move tangential to the path. With sufficient speed, this causes the water to press against the bottom of the pail which gives rise to the normal force exerted by the pail on the water. The presence of the normal force means that the water remains in contact with the pail.

(b)

$$N + W = F_C$$

$$N + mg = \frac{mv^2}{r}$$

$$N = \frac{mv^2}{r} - mg$$

The min. speed v_{min} required is when the water just stays in contact with the pail where $N = 0$.

$$N = 0$$

$$\frac{mv_{min}^{2}}{r} = mg$$

$$v_{min} = \sqrt{gr}$$

Example 8

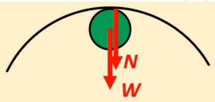
The figure below shows a roller coaster train of mass 50 kg moving in a circular loop of radius 3.0 m. Assume that frictional forces may be neglected.

(a) What minimum speed must the train have at the top for it to be *just in contact* with the loop?

(b) What speed must the train enter at the bottom of the track so that the train just makes it over the top of the loop?

Solution

(a)



At the top of the loop,

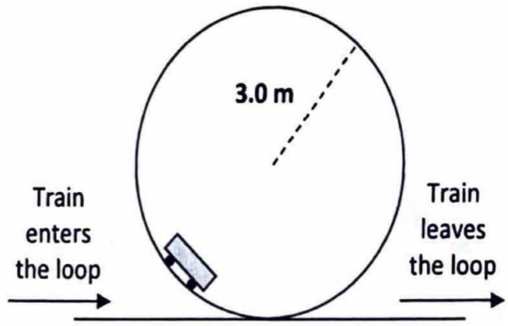
$$mg + N = \frac{mv_{top}^2}{R}$$

If the train is *just in contact*, *N* = 0

$$mg = \frac{mv_{top}^2}{R}$$

$$v_{top} = \sqrt{gR} = \sqrt{(9.81)(3.0)}$$

$$= 5.42 \text{ m s}^{-1}$$



(b)

Total initial energy = Total final energy

$$KE_{bot} + GPE_{bot} = KE_{top} + GPE_{top}$$

$$\frac{1}{2}mv_{\text{bot}}^2 + 0 = \frac{1}{2}mv_{\text{top}}^2 + mgh_{\text{top}}$$

$$\frac{1}{2}v_{\text{bot}}^2 = \frac{1}{2}gR + g(2R)$$

$$v_{\text{bot}}^2 = 5gR$$

$$v_{\text{bot}} = \sqrt{5gR} = \sqrt{5(9.81)(3.0)} = 12.1 \text{ m s}^{-1}$$

6.4 Summary

1. If a body moves in a circular path, its velocity is not constant since the direction changes continuously (remember velocity is a vector). Therefore there must be a resultant force acting on it according to Newton's First Law. If the resultant force is removed, the body would move off tangentially to the circular path.

2. For *uniform* circular motion, the body moves at constant speed. So the resultant force on the body cannot have a component along the direction of motion, otherwise the speed of the body will increase or decrease. The force is thus perpendicular to the direction of motion and is directed towards the centre of the circular path. This force is known as the centripetal force.

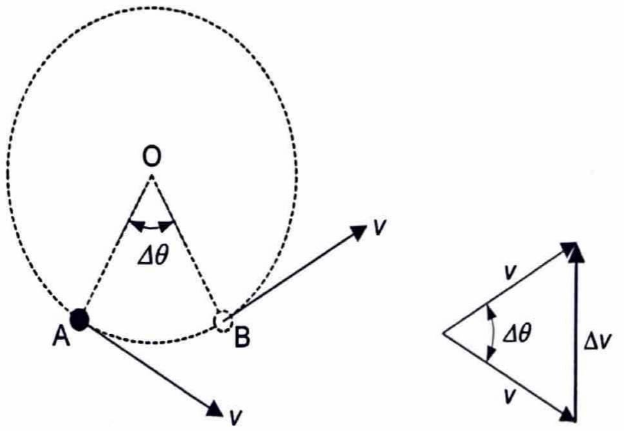
3. The centripetal force is **not** a new type of force. It is any force(s) that makes an object move in a circle e.g friction, gravity, electric force, etc. or a resultant of these forces.

4. The centripetal force results in centripetal acceleration which is in the direction of the centripetal force, i.e. towards the centre of the circle.

5. Uniform circular motion formulas:

Angular Velocity	$\omega = \frac{d\theta}{dt} = \frac{2\pi}{T} = 2\pi f$
Tangential Velocity	$v = r\omega$
Centripetal Acceleration	$a = \frac{v^2}{r} = v\omega = r\omega^2$
Centripetal Force	$F = \frac{mv^2}{r} = mv\omega = mr\omega^2$

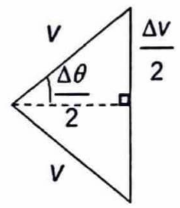
Annex A — Derivation of Centripetal Acceleration (not in Syllabus)



Consider a body moving from A to B with a constant speed v. In a time Δt, the body experiences an angular displacement of $\Delta\theta$. The change in velocity Δv is shown in the diagram on the right.

$$\sin\frac{\Delta\theta}{2} = \frac{\frac{\Delta v}{2}}{v}$$

$$\Delta v = 2 \times v \sin\frac{\Delta\theta}{2}$$



For small values of $\Delta\theta$, measured in radians,

$$\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2} \quad \text{(small angle approximation)}$$

$$\therefore \Delta v \approx v\Delta\theta$$

Dividing both sides by Δt,

$$\frac{\Delta v}{\Delta t} \approx v\frac{\Delta\theta}{\Delta t}$$

For infinitesimal small values of $\Delta\theta$ and Δt, the above approximation becomes equality

$$\frac{dv}{dt} = v\frac{d\theta}{dt}$$

$$\Rightarrow a = v\omega$$

Since $v = r\omega$,

$$a = v\omega = r\omega^2 = \frac{v^2}{r}$$

Tutorial 6 MOTION IN A CIRCLE

Self - Check Questions

S1. Define angular velocity.

S2. What is the relation between angular velocity ω and tangential velocity v for an object moving in circular motion?

S3. Why is an object moving in uniform circular motion accelerating? What is the direction of this acceleration?

S4. What are the formulas for centripetal acceleration and centripetal force?

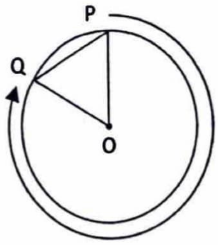
S5. Why does a car skid when it tries to negotiate a bend too fast?

S6. Why does the string of a pendulum swung in a circle break when the bob is turning too fast? How does the bob move immediately after the string breaks?

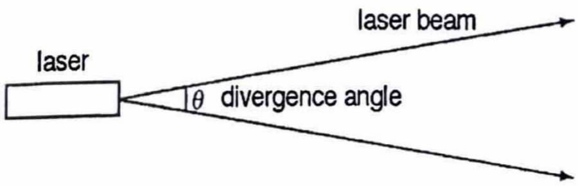
Self - Practice Questions

SP1. A disc rotates clockwise about its centre O until point P has moved to point Q, such that OP equals the length of the straight line PQ. What is the angular displacement of OQ relative to OP in terms of π?

[1]



SP2. An astronomer points a powerful laser beam at the Moon, a distance R away. The beam has a very small divergence angle θ, as shown in the diagram (which exaggerates greatly the size of the angle).



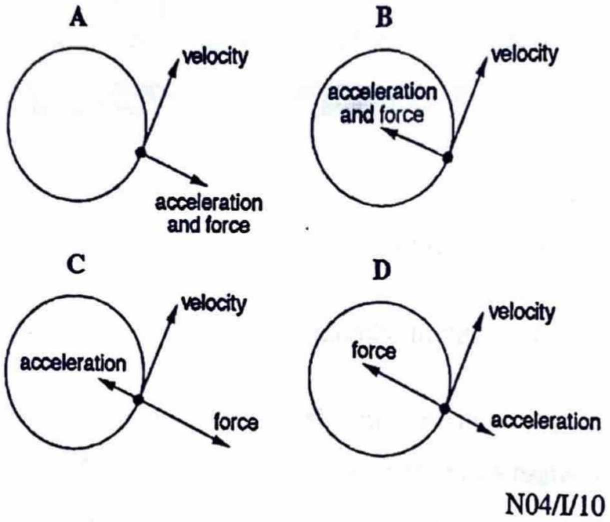
The astronomer looks up a value for the distance R in kilometres. He measures the angle θ in degrees.

Determine the diameter, in terms of R and θ in metres, of the circle of light the laser produces on the Moon. [2]

[N07/I/12]

SP3. A child on a roundabout is travelling in a horizontal circle with a constant speed. The child's velocity, acceleration and the resultant horizontal force on the child are all vectors.

Which diagram shows the correct directions for these vectors?



[N04/I/10]

SP4. A stone rotates in a horizontal circle with constant angular velocity ω.

What changes occur in the linear speed and in the centripetal acceleration of the stone as the radius of the circle increases?

	linear speed	centripetal acceleration
A	constant	decrease
B	constant	increase
C	increase	constant
D	increase	increase

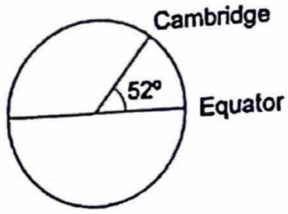
[N10/I/12]

SP5. A pendulum bob of mass 1.27 kg is supported by a string so that the radius of its path is 0.600 m. It is moving with velocity 0.575 m s^{-1} horizontally at the centre of its motion when the string is vertical.

Calculate the tension in the string at this instant. [2]

[N08/I/12]

SP6. Singapore is on the Equator. Cambridge is at a latitude of 52°N, as shown in the diagram.



A student at Singapore has a centripetal acceleration a_s because of the Earth's rotation about its axis. The centripetal acceleration of another student at Cambridge is a_c.

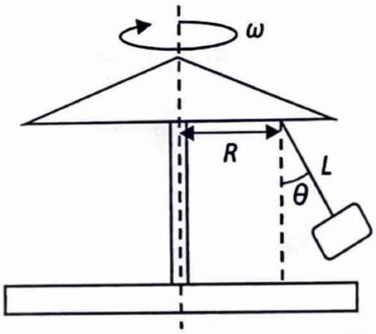
What are the magnitudes of the centripetal accelerations?

(radius of Earth = 6.4×10^6 m; angular velocity of Earth about axis = 7.3×10^{-5} rad s^{-1})

	a_s / m s^{-2}	a_c / m s^{-2}
A	3.4×10^{-2}	2.1×10^{-2}
B	3.4×10^{-2}	2.7×10^{-2}
C	3.4×10^{-2}	3.4×10^{-2}
D	4.7×10^{2}	4.7×10^{2}

[N06/I/9]

SP7. The vehicles of a fairground ride are supported by light cables with their upper ends at a radius R from the axis of rotation. The centre of mass of the vehicle is at a distance L from the upper point of support.



(a) Explain why, when the ride is rotating with angular velocity ω, the cables are inclined at an angle θ to the vertical, as shown in the diagram. [2]

(b) Show also that $\omega^2 = \dfrac{g\tan\theta}{R + L\sin\theta}$ [2]

(c) Calculate the rate of rotation, in rev s^{-1}, for which θ = 15°, given that R = 3.0 m and L = 2.5 m. [2]

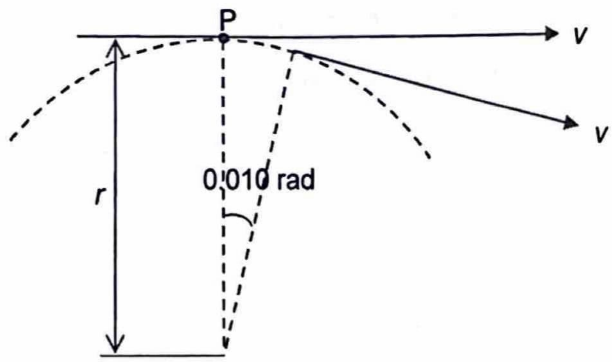
Discussion Questions

D1. **(a)** State the conditions for a particle to perform uniform circular motion about a fixed point. [2]

(b) Explain, from both the dynamical and work-energy points of view, why the speed of such a particle is constant. [2]

(c) Explain why the velocity and acceleration of this particle are both not constant. [1]

D2. An object P moves at a constant speed v through an arc of a circle of radius r. The arc subtends an angle 0.010 rad at the centre of the circle, as shown in figure.



(a) Determine, in terms of v, the magnitude of the change in velocity. [1]

(b) State the direction of the acceleration of P. [1]

(c) Deduce, in terms of r and v, the time taken for P to travel 0.010 . [2]

(d) Hence show that the magnitude of the acceleration of P is $\frac{v^2}{r}$. [1]

[N06/III/3 (Pa.

D3. A student builds and calibrates an accelerometer, which she uses to determine the speed of her car around a certain highway curve. The accelerometer consists of a simple pendulum with a protractor which she attaches to the roof of her car. Her friend observes that the pendulum hangs at an angle of 15° from the vertical when the car has a speed of 23.0 m s^{-1}.

(a) Calculate the centripetal acceleration of the car rounding the curve. [2]

(b) Determine the radius of the curve. [2]

(c) Determine the speed of the car if the deflection is 9.0°, while rounding the same curve. [2]

D4. An aircraft is travelling at a constant speed of 180 m s^{-1} in a horizontal circle of radius 20 km. A plumbline, attached to the roof of the cabin, settles at an angle ϕ to the vertical during the turn.

(a) Calculate the centripetal acceleration of the aircraft. [2]

(b) Sketch and label a diagram to show the forces acting on the bob of the plumbline. [2]

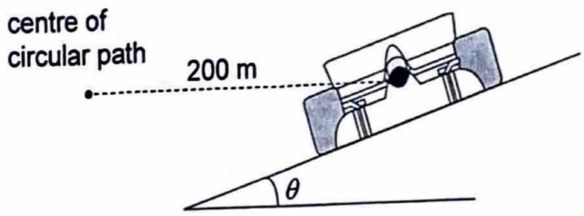
(c) State the direction of the resultant force acting on the plumbline in **(b)**. [1]

(d) Determine the angle ϕ. [2]

(e) Show by means of a simple sketch of the cross-section of the aircraft how the plumbline is orientated with respect to the aircraft. [1]

[J85/II/8 modified]

D5. When a car is rounding a bend of radius 200m on a flat road, the centripetal force is provided by the friction between the road and the tires. During a wet day, the friction is greatly reduced and it becomes dangerous to negotiate such a bend. This can be overcome by banking the road, as shown below.



(a) Determine the angle θ such that a 1500 kg car moving with speed 80 km h^{-1} can negotiate the bend without friction providing for the centripetal force. [2]

(b) A 2000 kg car is now moving on the same wet bend at 80 km h^{-1}. State with a reason if the car is able to negotiate the bend without skidding. [2]

(c) Sketch the free body diagram of the car moving round the bend on a dry day at a speed

(i) greater than 80 km h^{-1}

(ii) lesser than 80 km h^{-1} [4]

Write down the equations relating the horizontal forces acting on the car in both situations.

D6. **(a)** A pilot in a jet aircraft executes a "loop-the-loop" manoeuvre at a constant speed. State and explain at which points the pilot feels the lightest and the heaviest. [2]

(b) Some amusement park rides will swing you upside-down in a vertical circle. Suppose that you are wearing a hat while riding in one of these rides. As the ride swings you over the top, your head is nearer to the ground than your feet. If the ride goes over the top of the circle quickly, your hat stays on and you can hardly tell that you are upside down. But if the ride goes over the top slowly, your hat falls off.

Explain why your hat stays on in the first scenario, but falls off in the second scenario. [3]

(c) On one such ride, the loop has a radius of 7.0 m and a passenger of mass 60 kg is travelling at 12 m s^{-1} at the highest point of the loop. Assume that frictional forces may be neglected, calculate, for the passenger at the highest point,

(i) the centripetal acceleration, [2]

(ii) the force which the seat exerts on the passenger. [2]

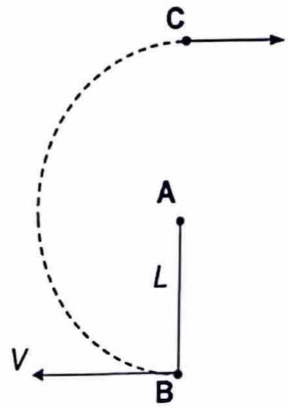
(d) The passenger now moves round and descends to the bottom of the loop. Calculate

(i) the change in potential energy of the passenger in moving from the top of the loop to the bottom, [2]

(ii) the speed of the passenger on leaving the loop. [2]

(e) Operators of this ride must ensure that the speed at which the passengers enter the loop is above a certain minimum value. Suggest a reason for this and calculate this minimum value. [4]

D7. A particle is suspended from a point A by an inextensible string of length L. It is projected from B with a velocity V, perpendicular to AB, which is just sufficient for it to reach C.



(a) Show that, if the string is just to be taut when the particle reaches C, its speed at C is $\sqrt{gL}$. [2]

(b) Determine the speed V with which the particle should be projected from B. [2]

(c) The particle continues to swing in a vertical circle, as its angular velocity is gradually increased. State and explain at which point of its circular path is the string most likely to break.

[2]

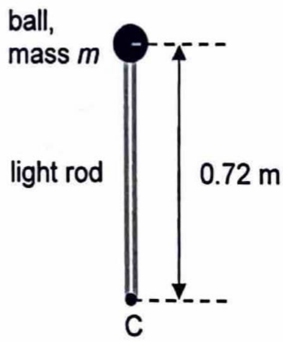
[N81/I/1 modified]

D8. A small ball of mass m is fixed to one end of a light rigid rod. The ball is made to move at constant speed around the circumference of a vertical circle with centre at C.

When the rod is vertical with the ball above C, the tension in the rod is given by

$$T = 2mg$$

where g is the acceleration of free fall.



(a) **(i)** Explain why the centripetal force on the ball is greater than $2mg$. [1]

(ii) State, in terms of mg, the magnitude of the centripetal force. [1]

(iii) Determine the magnitude of the tension, in terms of mg, in the rod when the rod is vertical, with the ball below point C. [1]

(b) The distance from the centre of the ball to point C is 0.72 m.
Use your answer in **(a)(ii)** to determine, for the ball,

(i) the angular speed, [3]

(ii) the linear speed. [2]

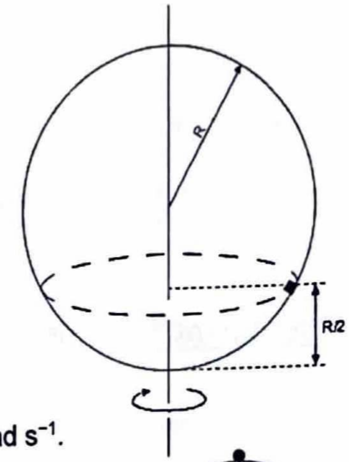
(c) The ball has a constant angular speed. [2]

(i) Explain why work has to be done for the ball to move from the position where it is vertically above point C to the position where it is vertically below C. [2]

(ii) Calculate the work done in **(c)(i)** for a ball of mass 240 g. [2]

[N10/III/2]

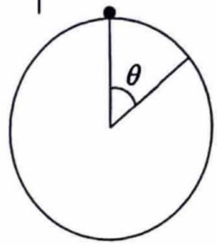
Challenging Questions



C1. A hollow sphere of radius $R = 0.5$ m rotates about a vertical axis through its centre with an angular velocity of $\omega = 5$ rad s^{-1}. Inside the sphere a small block (small enough to be considered lying flat on the inner surface of the sphere) is moving together with the sphere at the height of $R/2$. The maximum friction between an object and a surface is given by μN where N is the normal reaction force between the object and the surface and μ is the coefficient of friction.

(a) What is the least coefficient of friction to achieve this condition?

(b) Find the least coefficient of friction for the case of ω = 8.0 rad s^{-1}.



C2. A small bead is released from rest and allowed to slide down a smooth sphere. Find the angle θ at which the bead breaks contact from the surface of the sphere.

SUGGESTED SOLUTIONS

SELF-CHECK QUESTIONS

S1 Angular velocity is defined as the rate of change of angular displacement with respect to time.

S2 $v = r\omega$

The tangential velocity v is the product of angular velocity ω and the radius of circular motion r.

S3 Although the object is moving at a constant speed, its velocity is always changing because its direction of motion is constantly changing. Hence, this object experiences an acceleration.

As the object is moving at a constant speed, there must be no component of acceleration in the direction of the motion of object. Otherwise, it will increase or decrease the speed of the object. Therefore, the direction of this acceleration has to be perpendicular to the direction of motion which means that it will always act towards the centre of the circular motion.

S4

$$a = r\omega^2 = \frac{v^2}{r} = v\omega$$

$$F = mr\omega^2 = m\frac{v^2}{r} = mv\omega$$

S5 The force that enables the car to travel in a circular path is the sideward frictional force. If it travels too fast, the required centripetal force to keep it moving in circular path is larger than the maximum frictional force so the car is not able to continue its circular motion and skids.

S6 The tension in the string and the radial component of gravitational force provides the centripetal force required for circular motion. If the bob's speed is too high, the tension in the string will exceed its breaking strength and the string will break. The tension in the string goes to zero and the bob becomes a projectile, with its initial velocity tangent to the circle.

SELF-PRACTICE QUESTIONS

SP1. Since ∠POQ inside equilateral triangle POQ = π/3 rad, angular displacement of OQ relative to OP (in clockwise direction as shown in diagram) = 2π - π/3 = 5π/3 rad A1

SP2. Distance from laser to Moon = 1000*R* (in metres)

Angle = (π/180) × θ = 0.0175θ (in radians)

Diameter of circle of light ≈ arc length M1
= radius × angular displacement
= 1000*R* × 0.0175θ = 17.5$R\theta$ A1

SP3. **B**

In a uniform circular motion, the resultant force and thus acceleration is directed towards the centre of the circle. Linear velocity is tangential to the circular path.

SP4. **D**

Since linear speed $v = r\omega$, and centripetal acceleration $a = r\omega^2$, both will be proportional to *r* if angular velocity ω is constant.

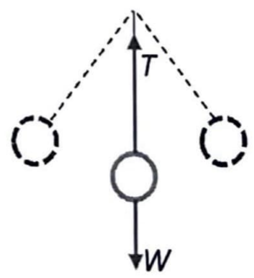
SP5. At the centre of its circular path, the pendulum is in a vertical position.
Resultant force provides for the centripetal force,

$$T - W = m\frac{v^2}{r}$$

$$T = m\frac{v^2}{r} + mg$$

$$= 1.27\left(\frac{0.575^2}{0.600} + 9.81\right)$$ M1

$$= 13.2 \text{ N}$$ A1

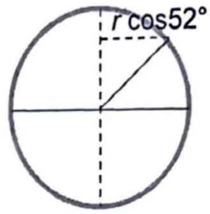


SP6. **A**

$a_s = r\omega^2 = 6.4 \times 10^6 \times (7.3 \times 10^{-5})^2 = 3.4 \times 10^{-2}$ m s^{-2}

The distance from Cambridge to the axis of rotation = $r\cos52°$

$a_c = r\cos52°\omega^2 = 6.4 \times 10^6 \times \cos 52° \times (7.3 \times 10^{-5})^2 = 2.1 \times 10^{-2}$ m s^{-2}



SP7. **(a)** The vehicles are moving in circular motion, hence there is a centripetal force that is directed towards the axis of rotation. B1

There are only weight and tension in the cable acting on each vehicle. As weight is always pointing downwards, the cables must incline at an angle to the vertical away from the axis of rotation such that the horizontal component of the tension in the cable provides the centripetal force. B1

(b) The distance from the vehicle to the centre of the circular path is $R + L\sin\theta$

Vertically $T\cos\theta = mg$ ... (1) M1

Horizontally $T\sin\theta = m(R + L\sin\theta)\omega^2$ ... (2) M1

$^{(2)}/_{(1)}$,

$$\tan\theta = \frac{(R + L\sin\theta)\omega^2}{g}$$

$$\therefore \omega^2 = \frac{g\tan\theta}{(R + L\sin\theta)}$$

(c)

$$\omega = \sqrt{\frac{9.81 \times \tan 15^0}{(3.0 + 2.5\sin 15^0)}}$$ C1

$= 0.848$ rad s^{-1}

$= (\frac{0.848}{2\pi})$ A1

$= 0.135$ rev s^{-1}

DISCUSSION QUESTIONS

D2. (a) 2.63 m s^{-2}, (b) 201 m, (c) 17.7 m s^{-1}
D3. (a) 0.010v, (c) 0.010r / v, (d) v^2 / r
D4. (a) 1.62 m s^{-2}, (d) 9.38°
D5. (a) 14.1°
D6. (c)(i) 21 m s^{-2}, (ii) 646 N, (d)(i) –8240 J, (ii) 20.5 m s^{-1}, (e) 18.5 m s^{-1}
D7. (b) $\sqrt{5gL}$
D8. (b)(i) 6.39 rad s^{-1}, (b)(ii) 4.60 m s^{-1}, (c)(ii) –3.39 J