2023 H2 Maths Prelim Paper 1 Marking Scheme

Qn	Solutions	
1	$\mathrm{f}(x)=x^3+ax^2+bx+c$ $-32=1+a+b+c$ $\mathrm{f}'(x)=3x^2+2ax+b$ $0=3+2a+b$ $y=\dfrac{1}{\mathrm{f}(x)}$ has a vertical asymptote at $x=5$ implies that $y=\mathrm{f}(x)$ has an x-intercept at $x=5$. $0=125+25a+5b+c$ $\begin{cases} a+b+c=-33 \\ 2a+b=-3 \\ 25a+5b+c=-125 \end{cases}$ By GC, $\mathrm{f}(x)=x^3-5x^2+7x-35$. [$a=-5, b=7, c=-35$]	
2	$w+z^*=-2+4\mathrm{i}$ ----(1) $z+2=3\mathrm{i}w$ ----(2) From (1): $w=-2+4\mathrm{i}-z^*$ Substituting into (2): $z+2=3\mathrm{i}\left(-2+4\mathrm{i}-z^*\right)$ $z+3\mathrm{i}z^*=-14-6\mathrm{i}$ Let $z=a+\mathrm{i}b$, $a+\mathrm{i}b+3\mathrm{i}a+3b=-14-6\mathrm{i}$ $(a+3b)+(3a+b)\mathrm{i}=-14-6\mathrm{i}$ Comparing real and imaginary parts: $a+3b=-14$ ---(3) $3a+b=-6$ ---(4) Solving simultaneously with $(4)-3(3)$, we have $-8b=36\Rightarrow b=-4.5$ Hence $a=-0.5$, and $z=-0.5-4.5\mathrm{i}$ $w=-2+4\mathrm{i}-z^*$: $w=-2+4\mathrm{i}+0.5-4.5\mathrm{i}=-1.5-0.5\mathrm{i}$	

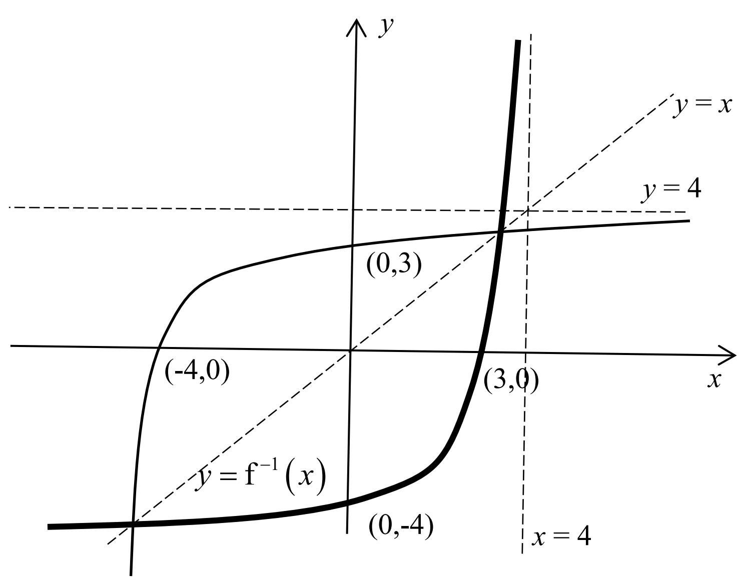
3(i)	**Method 1: Algebraic Manipulation** $\int_0^1 \frac{x}{\sqrt{x+1}}\,\mathrm{d}x$ $=\int_0^1 \frac{(x+1)}{\sqrt{x+1}}-\frac{1}{\sqrt{x+1}}\,\mathrm{d}x$ $=\int_0^1 \sqrt{x+1}\,\mathrm{d}x-\int_0^1 \frac{1}{\sqrt{x+1}}\,\mathrm{d}x$ $=\left[\frac{2}{3}(x+1)^{\frac{3}{2}}-2(x+1)^{\frac{1}{2}}\right]_0^1$ $=\left[\frac{2}{3}(2)^{\frac{3}{2}}-2(2)^{\frac{1}{2}}\right]-\left[\frac{2}{3}(1)^{\frac{3}{2}}-2(1)^{\frac{1}{2}}\right]$ $=\left[\frac{4}{3}\sqrt{2}-2\sqrt{2}\right]-\left[\frac{2}{3}-2\right]$ $=\frac{4}{3}-\frac{2}{3}\sqrt{2}$ $=\frac{2}{3}(2-\sqrt{2})$ **Method 2: Integration by Parts** Let $u=x$ $\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{1}{\sqrt{x+1}}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=1$ $v=2\sqrt{x+1}$ $\int_0^1 \frac{x}{\sqrt{x+1}}\,\mathrm{d}x$ $=\left[2x\sqrt{x+1}\right]_0^1-\int_0^1 2\sqrt{x+1}\,\mathrm{d}x$ $=\left[2x\sqrt{x+1}\right]_0^1-2\left[\frac{2(x+1)^{\frac{3}{2}}}{3}\right]_0^1$ $=\left[2\sqrt{2}-0\right]-\frac{4}{3}\left[(2)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right]$ $=\frac{4}{3}-\frac{2}{3}\sqrt{2}$ $=\frac{2}{3}\left(2-\sqrt{2}\right)$	

	Method 3: Using Substitution (Change of Variable) Let $u=x+1$. Then $\frac{\mathrm{d}u}{\mathrm{d}x}=1$. When $x=0,\ u=1$ When $x=1,\ u=2$ $\int_0^1 \frac{x}{\sqrt{x+1}}\,\mathrm{d}x$ $=\int_1^2 \frac{u-1}{\sqrt{u}}\,\mathrm{d}u$ $=\int_1^2 \frac{u}{\sqrt{u}}-\frac{1}{\sqrt{u}}\,\mathrm{d}u$ $=\int_1^2 u^{\frac{1}{2}}-u^{-\frac{1}{2}}\mathrm{d}u$ $=\left[\frac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right]_1^2$ $=\frac{2}{3}\left[2\sqrt{2}-1\right]-2\left[\sqrt{2}-1\right]$ $=\frac{4}{3}-\frac{2}{3}\sqrt{2}$ $=\frac{2}{3}(2-\sqrt{2})$	
3(ii)	Area of 1st rectangle: $\frac{1}{n}\mathrm{f}\left(\frac{1}{n}\right)=\frac{1}{n}\frac{\left(\frac{1}{n}\right)}{\sqrt{1+\frac{1}{n}}}=\frac{1}{n}\frac{\left(\frac{1}{n}\right)}{\sqrt{\frac{n+1}{n}}}=\left(\frac{1}{n^2}\right)\frac{1}{\sqrt{\frac{n+1}{n}}}=\frac{1}{n\sqrt{n}}\frac{1}{\sqrt{n+1}}$ Area of 2nd rectangle: $\frac{1}{n}\mathrm{f}\left(\frac{2}{n}\right)=\frac{1}{n}\frac{\left(\frac{2}{n}\right)}{\sqrt{1+\frac{2}{n}}}=\frac{1}{n}\frac{\left(\frac{2}{n}\right)}{\sqrt{\frac{n+2}{n}}}=\left(\frac{1}{n^2}\right)\frac{2}{\sqrt{\frac{n+2}{n}}}=\frac{1}{n\sqrt{n}}\frac{2}{\sqrt{n+2}}$ Area of *n*th rectangle: $\frac{1}{n}f\left(\frac{n}{n}\right)=\frac{1}{n}\frac{\left(\frac{n}{n}\right)}{\sqrt{1+\frac{n}{n}}}=\frac{1}{n}\frac{\left(\frac{n}{n}\right)}{\sqrt{\frac{n+n}{n}}}=\left(\frac{1}{n^2}\right)\frac{n}{\sqrt{\frac{n+n}{n}}}=\frac{1}{n\sqrt{n}}\frac{n}{\sqrt{2n}}$	

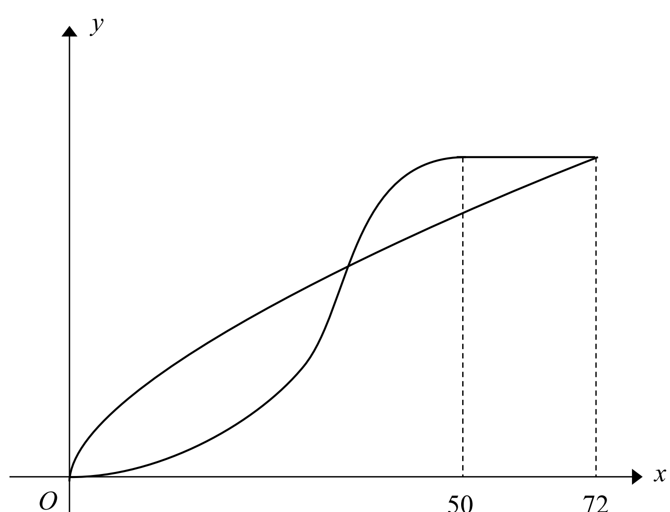
	Total area of all the rectangles: $=\frac{1}{n}\left[\mathrm{f}\left(\frac{1}{n}\right)+\mathrm{f}\left(\frac{2}{n}\right)+\mathrm{f}\left(\frac{3}{n}\right)+\cdots+\mathrm{f}\left(\frac{n}{n}\right)\right]$ $=\frac{1}{n\sqrt{n}}\left[\frac{1}{\sqrt{n+1}}+\frac{2}{\sqrt{n+2}}+\frac{3}{\sqrt{n+3}}+\cdots+\frac{n}{\sqrt{2n}}\right]$ All the rectangles above the curve enclosed the region bounded by the curve and the x-axis and include a small portion above the curve. Thus, the total area of all the rectangles is more than the area bounded by the curve and x-axis. Hence, $\frac{1}{n\sqrt{n}}\left[\frac{1}{\sqrt{n+1}}+\frac{2}{\sqrt{n+2}}+\frac{3}{\sqrt{n+3}}+\cdots+\frac{n}{\sqrt{2n}}\right]>\frac{2}{3}\left(2-\sqrt{2}\right)$.	
4(i)	$\lvert\mathbf{p}\cdot\mathbf{q}\rvert$ is the length of projection of **q** onto **p**. **<u>OR</u>** $\lvert\mathbf{p}\cdot\mathbf{q}\rvert$ is the length of projection of **q** onto the line with direction vector **p**.	
4(ii)	$\overrightarrow{OM}=(\mathbf{p}\cdot\mathbf{q})\mathbf{p}$ (vector projection, since M lies on OP.) $\therefore \overrightarrow{OM}=2\mathbf{p}$ (shown) **<u>OR</u>** Since M lies on line OP, $\overrightarrow{OM}=\lambda\mathbf{p}$ for some λ. Since M is the foot of the perpendicular from Q to the line OP, $\overrightarrow{QM}\cdot\overrightarrow{OP}=0$ $\left(\overrightarrow{OM}-\overrightarrow{OQ}\right)\cdot\overrightarrow{OP}=0$ $(\lambda\mathbf{p}-\mathbf{q})\cdot\mathbf{p}=0$ $\lambda\mathbf{p}\cdot\mathbf{p}=\mathbf{q}\cdot\mathbf{p}$ $\lambda\lvert\mathbf{p}\rvert^2=2$ $\lambda=2$ (as **p** is a unit vector) $\therefore \overrightarrow{OM}=2\mathbf{p}$ (shown) **<u>OR</u>** Since $\mathbf{p}\cdot\mathbf{q}=2$ and $\lvert\mathbf{p}\cdot\mathbf{q}\rvert$ is the length of projection of **q** onto **p**, M lies on OP produced, and $OM=2OP$. $\therefore \overrightarrow{OM}=2\mathbf{p}$ (shown)	

4(iii)	$\overrightarrow{OM} = 2\mathbf{p}$ $\overrightarrow{OR} = \mathbf{r} = k(\mathbf{p}-\mathbf{q})$, $\overrightarrow{OQ} = \mathbf{q}$ $\overrightarrow{QM} = 2\mathbf{p}-\mathbf{q}$ $\overrightarrow{QR} = k(\mathbf{p}-\mathbf{q})-\mathbf{q} = k\mathbf{p}+(-1-k)\mathbf{q}$ Area of triangle *RQM* $=\frac{1}{2}\left\|\overrightarrow{QM}\times\overrightarrow{QR}\right\|$ $=\frac{1}{2}\left\|(2\mathbf{p}-\mathbf{q})\times(k\mathbf{p}+(-1-k)\mathbf{q})\right\|$ $=\frac{1}{2}\left\|(-2-2k)\mathbf{p}\times\mathbf{q}-k\mathbf{q}\times\mathbf{p}\right\|$ (since $\mathbf{p}\times\mathbf{p}=\mathbf{q}\times\mathbf{q}=\mathbf{0}$) $=\frac{1}{2}\left\|(-2-2k)\mathbf{p}\times\mathbf{q}+k\mathbf{p}\times\mathbf{q}\right\|$ $=\frac{1}{2}\left\|(-2-k)\mathbf{p}\times\mathbf{q}\right\|$ $=\frac{\|2+k\|}{2}\left\|\mathbf{p}\times\mathbf{q}\right\|$ $a=\frac{\|2+k\|}{2}$ When $k=-2$, points *R*, *Q* and *M* are collinear.	
5(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x^2} = -2x\mathrm{e}^{-x^2}$ $\int x^3\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \int x^2\left(x\,\mathrm{e}^{-x^2}\right)\mathrm{d}x$ $= x^2\left(-\frac{1}{2}\mathrm{e}^{-x^2}\right)-\int\left(-\frac{1}{2}\mathrm{e}^{-x^2}\right)(2x)\,\mathrm{d}x$ $=-\frac{1}{2}x^2\,\mathrm{e}^{-x^2}+\int x\,\mathrm{e}^{-x^2}\,\mathrm{d}x$ $=-\frac{1}{2}x^2\,\mathrm{e}^{-x^2}-\frac{1}{2}\mathrm{e}^{-x^2}+c$ $=-\frac{1}{2}\mathrm{e}^{-x^2}\left(x^2+1\right)+c$	

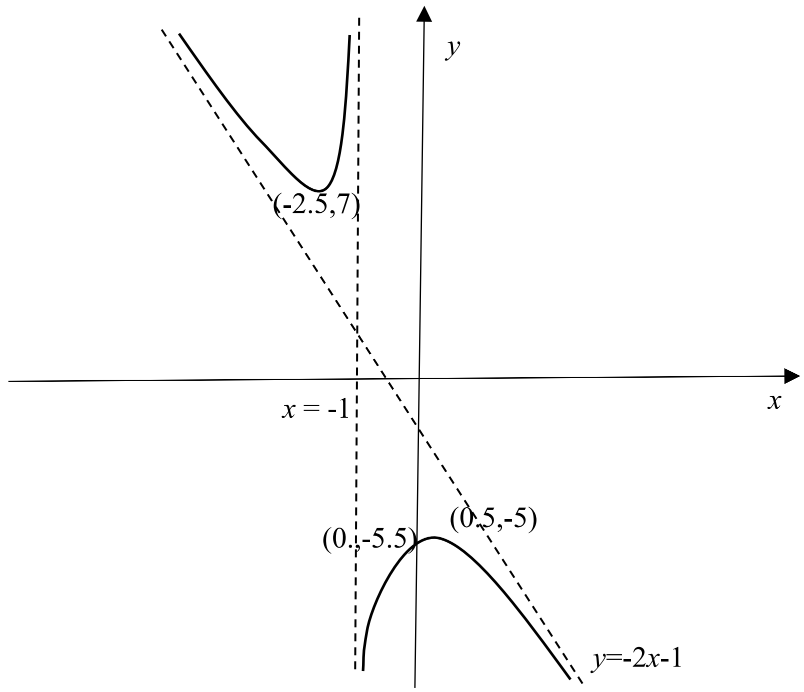
5(ii)	$z = e^{-x^2} y$ $\dfrac{dz}{dx} = \dfrac{d}{dx}\left(e^{-x^2} y\right)$ $\quad = -2xe^{-x^2} y + e^{-x^2} \dfrac{dy}{dx}$ $\dfrac{dy}{dx} = \dfrac{\frac{dz}{dx} + 2xe^{-x^2} y}{e^{-x^2}} = e^{x^2} \dfrac{dz}{dx} + 2xy$ For $\dfrac{dy}{dx} - 2xy = x^3$, $\left(e^{x^2} \dfrac{dz}{dx} + 2xy\right) - 2xy = x^3$ $\dfrac{dz}{dx} = x^3 e^{-x^2}$ $z = \int x^3 e^{-x^2} dx = -\frac{1}{2}x^2 e^{-x^2} - \frac{1}{2}e^{-x^2} + c$ $e^{-x^2} y = -\frac{1}{2}x^2 e^{-x^2} - \frac{1}{2}e^{-x^2} + c$ $y = -\frac{1}{2}x^2 - \frac{1}{2} + ce^{x^2} = -\frac{1}{2}\left(x^2 + 1\right) + ce^{x^2}$	
6(i)	$z^4 - kz^3 + k^3 z - k^4 = 0$ $(z-k)(z+k) = z^2 - k^2$ $z^4 - kz^3 + k^3 z - k^4 = \left(z^2 - k^2\right)\left(z^2 + cz + d\right)$ Comparing coefficients of $z^0 : -k^4 = -k^2 d \Rightarrow d = k^2$ $z^1 : k^3 = -k^2 c \Rightarrow c = -k$ $\therefore z^4 - kz^3 + k^3 z - k^4 = \left(z^2 - k^2\right)\left(z^2 - kz + k^2\right)$ For $z^2 - kz + k^2$, $z = \dfrac{k \pm \sqrt{k^2 - 4k^2}}{2} = \dfrac{k}{2} \pm \dfrac{\sqrt{3}k}{2}i$ The other two roots of the equation are $z_3 = \dfrac{k}{2} + \dfrac{\sqrt{3}k}{2}i$ (shown) and $z_4 = \dfrac{k}{2} - \dfrac{\sqrt{3}k}{2}i$ (as $\text{Im}(z_3) > 0$ and $\text{Im}(z_4) < 0$).	

6(ii)	$\lvert z_3\rvert=\sqrt{\left(\frac{k}{2}\right)^2+\left(\frac{\sqrt{3}k}{2}\right)^2}=k$ $\arg z_3=\tan^{-1}\frac{\left(\sqrt{3}k/2\right)}{\left(k/2\right)}=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$	
6(iii)	$\arg\left(\frac{z_3}{-1+\mathrm{i}}\right)^n=n\left[\arg\left(z_3\right)-\arg\left(-1+\mathrm{i}\right)\right]$ $=n\left[\frac{\pi}{3}-\frac{3\pi}{4}\right]$ $=-\frac{5\pi}{12}n$ For $\left(\frac{z_3}{-1+\mathrm{i}}\right)^n$ to be purely imaginary, $-\frac{5\pi}{12}n=\left(2k+1\right)\frac{\pi}{2}$, where k is an integer $n=-\frac{6}{5}\left(2k+1\right)$ For the two smallest positive values of *n*, consider $k=-3:n=-\frac{6}{5}\left(-6+1\right)=6$ $k=-8:n=-\frac{6}{5}\left(-16+1\right)=18$	
7(i)	 The graphs of $y=\mathrm{f}\left(x\right)$ and $y=\mathrm{f}^{-1}\left(x\right)$ are the reflections of each other about the line $y=x$.	

7(ii)	$R_g = [-4, \infty)$ From graph, $R_{fg} = [0, 4)$	
7(iii)	$g(a) = f^{-1}(0) = -4$	
7(iv)	Let $y = x^2 + 6x + 5$ $y = x^2 + 6x + 5$ $y = (x+3)^2 - 4$ $x + 3 = \pm\sqrt{y+4}$ $x = -3 - \sqrt{y+4}$ (rej $+\sqrt{y+4}$ as $x < -4$) $h^{-1}(x) = -3 - \sqrt{x+4}$ $D_{h^{-1}} = (-3, \infty)$	
8(i)	At $t = \frac{\pi}{2}$, $x = \frac{\pi}{2} - \sin\frac{\pi}{2} = \frac{\pi}{2} - 1$ $y = \sin^2\frac{\pi}{2} = 1$ $A\left(\frac{\pi}{2} - 1, 1\right)$	
8(ii)	Area = $\int_0^{\frac{\pi}{2}-1} y_C \, dx - \frac{1}{2}\left(\frac{\pi}{2} - 1\right)(1)$ $\int_0^{\frac{\pi}{2}-1} y_C \, dx = \int_0^{\frac{\pi}{2}} (\sin^2 t)\left(\frac{dx}{dt}\right) dt$ $= \int_0^{\frac{\pi}{2}} (\sin^2 t)(1 - \cos t) \, dt$ $= \int_0^{\frac{\pi}{2}} \sin^2 t - \sin^2 t \cos t \, dt$ $= \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2t}{2} \, dt - \left[\frac{1}{3}\sin^3 t\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} - \left[\frac{1}{3}\sin^3 t\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{3} = \frac{\pi}{4} - \frac{1}{3}$ Therefore area = $\left(\frac{\pi}{4} - \frac{1}{3}\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{1}{6}$	

8(iii)	$x = t - \sin t,\ y = \sin^2 t$ $\sin t = \sqrt{y} \quad (\because \sin t \geq 0)$ $t = \sin^{-1}\sqrt{y} \quad \left(\text{for } 0 \leq t \leq \frac{\pi}{2}\right)$ $\Rightarrow x = \sin^{-1}\sqrt{y} - \sqrt{y}$	
8(iv)	Volume = $\frac{1}{3}\pi\left(\frac{\pi}{2}-1\right)^2(1) - \pi\int_0^1 (x_C)^2\,\mathrm{d}y$ (the volume of region bounded by the line and curve gives a cone) $= \frac{1}{3}\pi\left(\frac{\pi}{2}-1\right)^2 - \pi\int_0^1\left(\sin^{-1}\sqrt{y}-\sqrt{y}\right)^2\,\mathrm{d}y$ $= 0.25261 \approx 0.253$ (3 s.f.)	
9(a)	Let $D = \mathrm{f}(x) - \mathrm{g}(x)$. $\frac{\mathrm{d}D}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{f}(x)-\mathrm{g}(x)\right) = \mathrm{f}'(x) - \mathrm{g}'(x)$ At $x = c$, distance is maximum, i.e. $\left.\frac{\mathrm{d}D}{\mathrm{d}x}\right\|_{x=c} = 0 \Rightarrow \mathrm{f}'(c) - \mathrm{g}'(c) = 0 \Rightarrow \mathrm{f}'(c) = \mathrm{g}'(c)$ (shown) From the diagram, at $x = c$, $y = \mathrm{f}(x)$ is concave downwards, i.e. $\mathrm{f}''(c) < 0$ $y = \mathrm{g}(x)$ is concave upwards, i.e. $\mathrm{g}''(c) > 0$ Therefore at $x = c$, $\frac{\mathrm{d}^2D}{\mathrm{d}x^2} = \mathrm{f}''(c) - \mathrm{g}''(c) < 0$. Hence the distance is a maximum.	
9(b)(i)		

9(b)(ii)	From (a), we know that the maximum will occur when $A'(c) = B'(c)$, and c will satisfy the equation $A'(x) = B'(x)$. $\frac{d}{dx}\left(-0.16x^3 + 12x^2\right) = \frac{d}{dx}\left(5000\log_3(x+9) - 10\,000\right)$ $-0.48x^2 + 24x = \frac{5000}{(x+9)\ln 3}$ Therefore $P = -0.48, Q = 24, R = \frac{5000}{\ln 3}$. Using GC to solve the equation gives $x = 11.906\,(5 \text{ s.f.}) = 11.9\,(3 \text{ s.f.})$ $x = 46.296\,(5 \text{ s.f.}) = 46.3\,(3 \text{ s.f.})$	
9(b)(iii)	$A(11.906) - B(11.906) = -2404.8$ $A(46.296) - B(46.296) = 1580.9$ Therefore the furthest distance between them is 2404.8m (or 2400m) and Ben is in front of Andy.	
10(i)	$y = \frac{-4x^2 - 6x - 11}{2x + 2}$ $y(2x+2) = -4x^2 - 6x - 11$ $4x^2 + (6+2y)x + (11+2y) = 0$ $D < 0$ $(6+2y)^2 - 4(4)(11+2y) < 0$ $36 + 24y + 4y^2 - 176 - 32y < 0$ $4y^2 - 8y - 140 < 0$ $y^2 - 2y - 35 < 0$ $(y-7)(y+5) < 0$ $-5 < y < 7$	

10(ii)		
10(iii)	$(x+1)^2(2x+2)^2-a^2(-4x^2-8x-13)^2=a^2(2x+2)^2$ $\frac{(x+1)^2(2x+2)^2}{a^2(2x+2)^2}-\frac{a^2(-4x^2-8x-13)^2}{a^2(2x+2)^2}=\frac{a^2(2x+2)^2}{a^2(2x+2)^2}$ $\frac{(x+1)^2}{a^2}-\frac{(-4x^2-6x-11-2x-2)^2}{(2x+2)^2}=1$ $\frac{(x+1)^2}{a^2}-\left(\frac{-4x^2-6x-11}{2x+2}-1\right)^2=1$ Hence consider the intersections of the curve *C* with the hyperbola $\frac{(x+1)^2}{a^2}-(y-1)^2=1$. Centre of hyperbola: $(-1,1)$ Equations of asymptotes of hyperbola: $y-1=\pm\frac{1}{a}(x+1)$, which intersect at $(-1,1)$. Comparing $y-1=-\frac{1}{a}(x+1)$ with the oblique asymptote of *C* ($y=-2x-1$): $\therefore -\frac{1}{a}\geq -2 \Rightarrow a\geq\frac{1}{2}$ (for *a* positive)	

10(iv)	$y=\frac{-4x^2-6x-11}{2x+2}=-2x-1-\frac{9}{2x+2}$ 1. Reflection in the y-axis (or x-axis) $y=2x+\frac{9}{2x}\rightarrow y=-2x-\frac{9}{2x}$ 2. Translation 1 unit in the negative x-direction $y=-2x-\frac{9}{2x}\rightarrow y=-2\left(x+1\right)-\frac{9}{2\left(x+1\right)}$ 3. Translation 1 unit in the positive y-direction $y=-2\left(x+1\right)-\frac{9}{2\left(x+1\right)}\rightarrow y=-2x-1-\frac{9}{2x+2}$ **OR** 1. Translation 1 unit in the positive x-direction $y=2x+\frac{9}{2x}\rightarrow y=2\left(x-1\right)+\frac{9}{2\left(x-1\right)}$ 2. Reflection in the ***y*-axis** $y=2\left(x-1\right)+\frac{9}{2\left(x-1\right)}\rightarrow y=-2-2x-\frac{9}{2x+2}$ 3. Translation 1 unit in the positive y-direction $y=-2-2x-\frac{9}{2x+2}\rightarrow y=-1-2x-\frac{9}{2x+2}$	
11(i)	Day 3 – 8 new squares Day 4 – 64 $=8^2$ new squares Day N – 8^{N-2} new sqaures	
11(ii)	Day 3 $-\left(\frac{1}{3}\right)^2 x$ Day 4 $-\left(\frac{1}{3}\right)^3 x$ Day $N-\left(\frac{1}{3}\right)^{N-1} x$	
11(iii)	Total perimeter of new squares drawn on Day N is $\left(8\right)^{N-2}\left(\frac{1}{3}\right)^{N-1}\left(4x\right)=\frac{4}{8}\left(\frac{8}{3}\right)^{N-1}x=\frac{1}{2}\left(\frac{8}{3}\right)^{N-1}x=\frac{x}{2}\left(\frac{8}{3}\right)^{N-1}$	
11(iv)	Total perimeter of all the squares the artist draws by the end of the Nth day is	

	$4x+\frac{x}{2}\left(\frac{8}{3}\right)^{2-1}+\frac{x}{2}\left(\frac{8}{3}\right)^{3-1}+\frac{x}{2}\left(\frac{8}{3}\right)^{4-1}+\ldots+\frac{x}{2}\left(\frac{8}{3}\right)^{N-1}$ $=4x+\frac{x}{2}\left(\frac{8}{3}\right)^{1}+\frac{x}{2}\left(\frac{8}{3}\right)^{2}+\frac{x}{2}\left(\frac{8}{3}\right)^{3}+\ldots+\frac{x}{2}\left(\frac{8}{3}\right)^{N-1}$ $=4x+\frac{\frac{x}{2}\left(\frac{8}{3}\right)^{1}\left(\left(\frac{8}{3}\right)^{N-1}-1\right)}{\left(\left(\frac{8}{3}\right)-1\right)}$ $=4x+\frac{\frac{4x}{3}\left(\left(\frac{8}{3}\right)^{N-1}-1\right)}{\frac{5}{3}}$ $=4x+\frac{4x}{5}\left(\left(\frac{8}{3}\right)^{N-1}-1\right)$ $=\frac{4x}{5}\left(\frac{8}{3}\right)^{N-1}+\frac{16}{3}x$	
11(v)	Total number of squares drawn by the end of Day 6 is $1+8^0+8^1+8^2+8^3+8^4=4682$ The director will pay for the 1st, 4th, 7th, 10th, …, 4681st square, a total of 1561 squares. He will be paying \$10 for the 1st square, \$13 for the 4th square, \$16 for the 7th square, and so on. $\frac{1561}{2}\left[2(10)+(1561-1)3\right]$ $=\frac{1561}{2}\left[20+1560\times 3\right]$ $=3668350$ The amount of money the director will personally pay is \$3,668,350.	
12(i)	$\sin\theta=\frac{1}{\sqrt{1}\sqrt{4^2+3^2+1^2}}\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}4\\3\\1\end{pmatrix}=\frac{1}{\sqrt{26}}$ $\theta=\sin^{-1}\frac{1}{\sqrt{26}}=11.310\approx 11.3°$	

12(ii)	Let the point on the flight path closest to the peak be *N*. Let the peak be represented by point *M*. $\overrightarrow{OM}=\begin{pmatrix}18000\\4500\\2000\end{pmatrix}$ Since *N* lies on the flight path, $\overrightarrow{ON}=\begin{pmatrix}2000\\500\\0\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\1\end{pmatrix}$ for some λ. $\overrightarrow{NM}=\begin{pmatrix}16000-4\lambda\\4000-3\lambda\\2000-\lambda\end{pmatrix}$ $\overrightarrow{NM}\cdot\begin{pmatrix}4\\3\\1\end{pmatrix}=0$ $\begin{pmatrix}16000-4\lambda\\4000-3\lambda\\2000-\lambda\end{pmatrix}\cdot\begin{pmatrix}4\\3\\1\end{pmatrix}=0$ $78000=26\lambda\Rightarrow\lambda=3000$ $\therefore\overrightarrow{ON}=\begin{pmatrix}2000\\500\\0\end{pmatrix}+3000\begin{pmatrix}4\\3\\1\end{pmatrix}=\begin{pmatrix}14000\\9500\\3000\end{pmatrix}$ $N(14000,9500,3000)$. $\left\lvert\overrightarrow{NM}\right\rvert=\left\lVert\begin{pmatrix}4000\\-5000\\-1000\end{pmatrix}\right\rVert=\sqrt{4000^2+5000^2+1000^2}=6481\text{ m}$	
12(iii)	$l_A:\mathbf{r}=\begin{pmatrix}2000\\500\\0\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\1\end{pmatrix}$ $l_B:\mathbf{r}=\begin{pmatrix}1400\\2900\\0\end{pmatrix}+\mu\begin{pmatrix}4\\2\\h\end{pmatrix}$ For the flight paths to cross each other, there should be solutions to the set of linear equations:	

	$\begin{cases} 2000+4\lambda=1400+4\mu \\ 500+3\lambda=2900+2\mu \\ \lambda=h\mu \end{cases} \Rightarrow \begin{cases} 4h\mu-4\mu=-600 \\ 3h\mu-2\mu=2400 \end{cases}$ $\Rightarrow \begin{cases} h\mu-\mu=-150 \\ 3h\mu-2\mu=2400 \end{cases}$ Solving simultaneously, $h\mu=2700, \mu=2850$ $h=\frac{2700}{2850}=\frac{18}{19}=0.947$	
12(iv)	$l_A: \mathbf{r}=\begin{pmatrix}2000\\500\\0\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\1\end{pmatrix}, \overrightarrow{OB}=\begin{pmatrix}1400\\2900\\0\end{pmatrix}$ $\overrightarrow{AB}=\begin{pmatrix}-600\\2400\\0\end{pmatrix}=600\begin{pmatrix}-1\\4\\0\end{pmatrix}$ $\overrightarrow{AB}\times\begin{pmatrix}4\\3\\1\end{pmatrix}=\begin{pmatrix}-1\\4\\0\end{pmatrix}\times\begin{pmatrix}4\\3\\1\end{pmatrix}=\begin{pmatrix}4\\1\\-19\end{pmatrix}$ $\mathrm{r}\cdot\begin{pmatrix}4\\1\\-19\end{pmatrix}=\begin{pmatrix}2000\\500\\0\end{pmatrix}\cdot\begin{pmatrix}4\\1\\-19\end{pmatrix}=8500$ $\therefore 4x+y-19z=8500$ $l_B: \mathbf{r}=\begin{pmatrix}1400\\2900\\0\end{pmatrix}+\mu\begin{pmatrix}4\\2\\1\end{pmatrix}$ $\begin{pmatrix}4\\2\\1\end{pmatrix}\cdot\begin{pmatrix}4\\1\\-19\end{pmatrix}=-1\neq 0$ This implies that the direction vector of the flight path of Aircraft Bravo is not perpendicular to the normal of the found plane, and hence does not lie in the plane containing the flight path of Aircraft Alpha. (As such, the flight path of Aircraft Bravo will not cross the flight path of Aircraft Alpha.)	