NCHS 2023 AM Prelim P1 solutions for students

Q1 (a)

Area of triangle

$$=\frac{1}{2}\left(\frac{1}{2+\sqrt{6}}\right)^2 \sin 60°$$

$$=\frac{1}{2}\left(\frac{1}{4+6+4\sqrt{6}}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$=\left(\frac{1}{10+4\sqrt{6}}\right)\left(\frac{\sqrt{3}}{4}\right)$$

$$=\left(\frac{1}{2}\right)\left(\frac{1}{5+2\sqrt{6}}\right)\left(\frac{\sqrt{3}}{4}\right)$$

$$=\left(\frac{1}{5+2\sqrt{6}}\right)\left(\frac{\sqrt{3}}{8}\right)$$

$$=\left(\frac{\sqrt{3}}{8}\right)\left(\frac{1}{5+2\sqrt{6}}\right)\left(\frac{(5-2\sqrt{6})}{5-2\sqrt{6}}\right)$$

$$=\left(\frac{\sqrt{3}}{8}\right)\left(\frac{5-2\sqrt{6}}{25-24}\right)$$

$$=\left(\frac{\sqrt{3}}{8}\right)\left(\frac{5-2\sqrt{6}}{1}\right)$$

$$=\frac{5\sqrt{3}}{8}-\frac{\sqrt{18}}{4}$$

$$=\frac{5\sqrt{3}}{8}-\frac{3\sqrt{2}}{4}$$

Q1(b)

<u>Mtd 1</u>

$$let\ y = \sqrt{x+3}$$

$$y = \frac{1}{y} - \frac{5}{6}$$

$$y^2 = 1 - \frac{5}{6}y$$

$$y^2 - 1 + \frac{5}{6}y = 0$$

$$6y^2 + 5y - 6 = 0$$

$$(3y-2)(2y+3) = 0$$

$$y = \frac{2}{3}\ or\ y = -\frac{3}{2}$$

$$\sqrt{x+3} = \frac{2}{3}\ or\ \sqrt{x+3} = -\frac{3}{2}\ (rej)$$

$$x + 3 = \frac{4}{9}$$

$$x = -\frac{23}{9}$$

Or

$$6(x+3) = 6 - 5\sqrt{x+3}$$

$$6x + 18 - 6 = -5\sqrt{x+3}$$

$$(6x+12)^2 = \left(-5\sqrt{x+3}\right)^2$$

$$36x^2 + 144x + 144 = 25(x+3)$$

$$36x^2 + 144x + 144 = 25x + 75$$

$$36x^2 + 119x + 69 = 0$$

$$(4x+3)(9x+23) = 0$$

$$x = -\frac{3}{4}\ (rej) or -\frac{23}{9}$$

Q2(a)

$$\frac{d}{dx}(x^5 lnx^2) = x^5\left(\frac{1}{x^2}\right)(2x) + 5x\text{^}4 lnx^2$$

$$= 2x^4 + 5x^4 lnx^2 = 2x^4 + 10x^4 lnx$$

2(b) Method 1

$$\int x^4 lnx\, dx = \frac{1}{10}\int 10x^4 lnx + 2x^4 - 2x^4 dx$$

$$= \frac{1}{10}\left[x^5 lnx^2 - \frac{2x^5}{5}\right] + c$$

$$= \frac{1}{5}x^5 lnx - \frac{1}{25}x^5 + c$$

2(b) Method 2

$$\int 10x^4 lnx + 2x^4 dx = x^5 lnx^2 + c$$

$$10\int x^4 lnxdx + \int 2x^4 dx = x^5 lnx^2 + c$$

$$10\int x^4 lnxdx = x^5 lnx^2 + c - \frac{2x^5}{5} + c1$$

$$= x^5 lnx^2 - \frac{2x^5}{5} + c2$$

$$\int x^4 lnxdx = \frac{(x^5 lnx^2)}{10} - \frac{x^5}{25} + c3$$

$$or\ \frac{(x^5 lnx)}{5} - \frac{x^5}{25} + c3$$

Q3(a)

$$-9 - 2x^2 - 4x$$
$$= -2x^2 - 4x - 9$$
$$= -2(x^2 + 2x) - 9$$
$$= -2[(x+1)^2 - 1] - 9$$
$$= -2(x+1)^2 - 7$$

$$max\ value = -7$$

Q3(b)

Hence,

$$-9 - 2x^2 - 4x = -2 - k$$
$$since \text{ max } = -7$$
$$-2 - k < -7$$
$$k > 5$$

Otherwise,

$$-2x^2 - 4x + k - 7 = 0$$
$$discriminant > 0$$
$$(-4)^2 - 4(-2)(k-7) > 0$$
$$16 + 8k - 56 > 0$$
$$8k - 40 > 0$$
$$k > 5$$

Q4

$$(x+1)(x^2-1) = x^3$$

$$3$$
$$x^3 + x^2 - x - 1\sqrt{3x^3 + 0x^2 + 0x + 2}$$
$$-(3x^3 + 3x^2 - 3x - 3)$$
$$-3x^2 + 3x + 5$$

$$\frac{3x^3 + 2}{(x+1)(x^2-1)} = 3 + \frac{3x + 5 - 3x^2}{(x+1)^2(x-1)}$$

$$\frac{3x + 5 - 3x^2}{(x+1)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$3x + 5 - 3x^2 = A(x+1)^2 + B(x-1)(x+1) + c(x-1)$$

$$sub\ x = 1,$$
$$3 + 5 - 3 = A(2)^2$$
$$4A = 5$$
$$A = \frac{5}{4}$$
$$sub\ x = -1,$$
$$-3 + 5 - 3 = C(-2)$$
$$-1 = -2C$$
$$C = \frac{1}{2}$$
$$comparing\ coefficient\ of\ x^2: -3 = A + B$$
$$-3 = \frac{5}{4} + B$$
$$B = -\frac{17}{4}$$
$$\therefore 3 + \frac{5}{4(x-1)} - \frac{17}{4(x+1)} + \frac{1}{2(x+1)^2}$$

Q5(a)

$$\log_7 x + \frac{\log_7 x^2}{\log_7 7^2} = 6$$

$$\log_7 x + \frac{2\log_7 x}{2} = 6$$

$$\log_7 x + \log_7 x = 6$$

$$2\log_7 x = 6$$

$$\log_7 x = 3$$

$$x = 7^3$$

$$= 343$$

Q5(b)

$$x^2 + 8x + 15 > 0 \ and \ x + 4 > 0$$

$$(x+5)(x+3) > 0 \ and \ x > -4$$

$$x > -3, x < -5 \ and \ x > -4$$

$$\therefore x > -3$$

Q6(a)

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2c(-\sqrt{1-c^2})$

$= -2c\sqrt{1-c^2}$

Q6(b)

$$\cos(\theta + 30°)$$

$$= \cos\theta\cos 30° - \sin\theta\sin 30°$$

$$= \left(-\sqrt{1-c^2}\right)\left(\frac{\sqrt{3}}{2}\right) - (c)\left(\frac{1}{2}\right)$$

$$= \frac{-1}{2}c - \frac{\sqrt{3-3c^2}}{2}$$

Q6(c) $-(180° - \theta)$ or $\theta - 180°$

Q7(a)

$$\frac{4\sin\theta + 4\sin^2\theta}{\sec\theta + \tan\theta}$$

$$= 4\sin\theta(1+\sin\theta) \div \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)$$

$$= 4\sin\theta(1+\sin\theta) \div \left(\frac{1+\sin\theta}{\cos\theta}\right)$$

$$= 4\sin\theta(1+\sin\theta) \times \frac{\cos\theta}{1+\sin\theta}$$

$$= 4\sin\theta\cos\theta$$

$$= 2(2\sin\theta\cos\theta)$$

$$= 2\sin 2\theta$$

Q7(b)

$$2\sin 2\theta = 0.7$$

$$\sin 2\theta = \frac{7}{20}$$

$$-\pi \leq \theta \leq \pi$$

$$-2\pi \leq 2\theta \leq 2\pi$$

$$\alpha = \sin^{-1}\frac{7}{20}$$

$$= 0.35757$$

$$2\theta \ is \ in \ the \ 1st \ and \ 2nd \ quadrant.$$

$$2\theta = 0.35757, \pi - 0.35757, -(\pi + 0.35757), -(2\pi - 0.35757)$$

$$\theta = 0.179, 1.39, -1.75, -2.96$$

Q8(a)

$$gradient\ of\ DC = \tan 45° = 1$$

$$gradient\ of\ BC = -1$$

$$\frac{y-9}{x-1} = -1$$

$$y - 9 = -x + 1$$

Equation of BC: $y = -x + 10$

$$\frac{y+18}{x-2} = 1$$

$$y + 18 = x - 2$$

Equation of AB: $y = x - 20$

$$-x + 10 = x - 20$$

$$2x = 30$$

$$x = 15$$

$$y = -15 + 10 = -5 \quad \therefore B(15, -5)(shown)$$

Q8(b)

$$midpt\ AC = midpt\ DB$$

$$\left(\frac{2+1}{2}, \frac{9-18}{2}\right) = \left(\frac{x+15}{2}, \frac{-5+y}{2}\right)$$

$$3 = x + 15$$
$$x = -12$$
$$9 - 18 = -5 + y$$
$$y = -4$$
$$D(-12, -4)$$

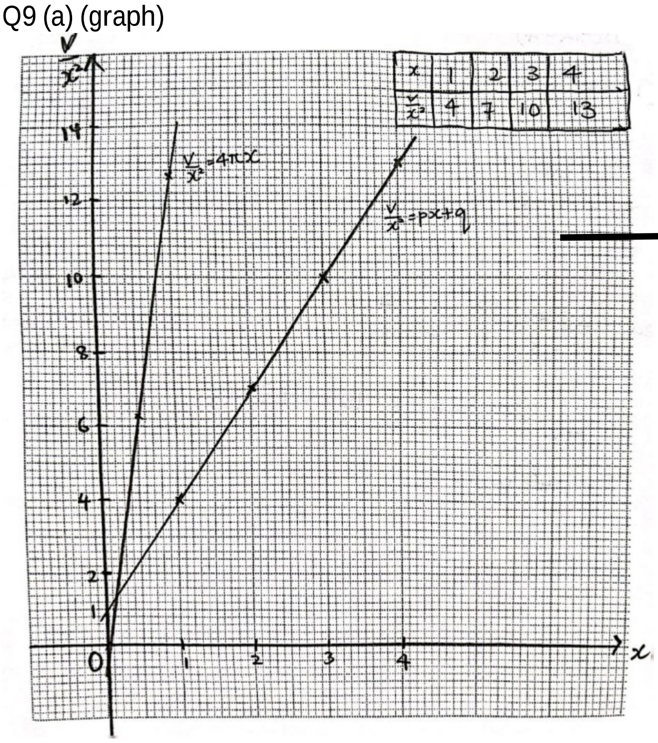
$$B\ is\ the\ midpt\ of\ DE$$

$$(15, -5) = \left(\frac{-12+x}{2}, \frac{-4+y}{2}\right)$$

$$30 = -12 + x$$
$$x = 42$$
$$-10 = -4 + y$$
$$y = -6$$
$$E(42, -6)$$

$$Area = \frac{1}{2}\begin{vmatrix} 2 & 42 & 1 & 2 \\ -18 & -6 & 9 & -18 \end{vmatrix}$$
$$= \frac{1}{2}(-12 + 378 - 18 - 18 + 6 + 758)$$
$$= 546\ units^2$$

Q9 (a) (graph)



$$v = x(px^2 + qx)$$

$$v = px^3 + qx^2$$

$$\frac{v}{x^2} = px + q$$

$$plotted\ points$$

$$join\ with\ a\ straight\ line$$

$$gradient = p$$

$$= \frac{10-1}{3-0} = 3$$

$$q = 1$$

Q9(b)

$$\frac{V}{x^2} = 3x + 1$$

$$V = x(3x^2 + x)$$

$$radius = 2x$$

$$base\ area = \pi(2x)^2 = 4\pi x^2$$

$$4\pi x^2 = 3x^2 + x$$

$$(4\pi - 3)x^2 - x = 0$$

$$x[(4\pi - 3)x - 1] = 0$$

$$x = \frac{1}{4\pi - 3} = 0.1045$$

$$= 0.105$$

Q9(c)

$$V = x(4\pi x^2)$$

$$= 4\pi x^3$$

$$draw\ \frac{V}{x^2} = 4\pi x$$

The volume of the cylinder is expressed in terms of x and then rearranged into the same linear form as the equation of the straight line drawn.

The x − coordinate of the intersection therefore gives the value of x for which the solid is a cylinder with height half its base radius.

x	0	0.5	1
$\frac{V}{x^2}$	0	6.3	12.6

$$From\ graph, x = 0.1$$

This is close to the x value obtained in part (b).

*verify means to confirm if something is accurate

Q10a

$$a = 2.5\cos\left(\frac{1}{2}t\right) \times \frac{1}{2}$$

$$= 1.25\cos\left(\frac{1}{2}t\right)\ or\ \frac{5}{4}\cos\left(\frac{1}{2}t\right)$$

$$when\ t = \frac{\pi}{2},$$

$$a = 1.25\cos\left(\frac{\pi}{2} \times \frac{1}{2}\right)$$

$$= 0.88388$$

$$= \frac{0.884m}{s^2 or}\ or\ \frac{5\sqrt{2}}{8} m/s^2$$

Q10b

$$when\ v = o,$$

$$2.5\sin\left(\frac{1}{2}t\right) = 0$$

$$\sin\left(\frac{1}{2}t\right) = 0$$

$$\frac{1}{2}t = \pi$$

$$t = 2\pi$$

$$s = -2.5\cos\left(\frac{1}{2}t\right) \times 2 + c$$

$$= -5\cos\left(\frac{1}{2}t\right) + c$$

$$when\ t = 0, s = 0,$$

$$0 = -5\cos(0) + c$$

$$c = 5$$

$$s = -5\cos\left(\frac{1}{2}t\right) + 5$$

$$when\ t = 2\pi,$$

$$s = -5\cos(\pi) + 5$$

$$= -5(-1) + 5 = 10m$$

Q10c

$$When\ Jean\ returns\ for\ the\ 4th\ time, s = 0$$

$$-5\cos\left(\frac{1}{2}t\right) + 5 = 0$$

$$\cos\left(\frac{1}{2}t\right) = 1$$

$$\frac{1}{2}t = 0, 2\pi, 4\pi, 6\pi, 8\pi$$

$When\ Jean\ returns\ for\ the\ 4th\ time, t = 2 \times 8\pi = 16\pi.$

$$\therefore k = 16\pi$$

$$When\ t > 16\pi,$$
$$0.25t > 4\pi$$
$$0.25 - 4\pi > 0$$
$$v > 0$$

$Jean\ will\ always\ be\ moving\ away\ from\ O\ when\ t > 16\pi$

$$or\ \frac{dv}{dt} = 0.25 > 0$$

$This\ implies\ velocity\ is\ an\ increasing\ function\ starting\ from\ v = 0\ at\ t = 16\pi\ so\ it\ will\ never\ become\ zero. Hence\ Jean\ will\ not\ be\ turning\ back\ towards\ O.$

Q10d

$$total\ distance$$

$$= 20 \times 4 + \int_{16\pi}^{60} 0.25t - 4\pi\, dt$$

$$= 80 + \left[\frac{0.25t^2}{2} - 4\pi t\right]_{16\pi}^{60}$$

$$= 80 + \frac{1}{8}(60)^2 - 4\pi(60) - \frac{1}{8}(16\pi)^2 + 4\pi(16\pi)$$

$$= 91.845m$$
$$= 91.8m$$

Q11a

$$for\ y\ increasing, \frac{dy}{dx} > 0$$

$$\frac{dy}{dx} = 3x^2e^{-\frac{1}{2}x}\left(-\frac{1}{2}\right) + 6xe^{-\frac{1}{2}x}$$

$$= e^{-\frac{1}{2}x}\left(-\frac{3x^2}{2} + 6x\right)$$

$$Since\ e^{-\frac{1}{2}x} > 0\ for\ x \in \mathbb{R},$$

$$-\frac{3x^2}{2} + 6x > 0$$

$$x\left(-\frac{3}{2}x + 6\right) > 0$$

$$0 < x < 4$$

Q11b

1st derivative test

$$when\ x = 0, y = 3(0)^2e^{-\frac{1}{2}(0)} = 0$$

x	0^{-1}	0	0^+
$\frac{dy}{dx}$	<0	0	>0
Sketch			

Second derivative test

$$When\ x = 0, \frac{dy}{dx} = 0. \therefore x = 0\ is\ a\ stationary\ pt.$$

$$\frac{d^2y}{dx^2} = e^{-\frac{1}{2}x}(3x + 6) - \frac{1}{2}e^{-\frac{1}{2}x}\left(-\frac{3}{2}x^2 + 6x\right)$$

$$when\ x = 0,$$

$$\frac{d^2y}{dx^2} = 6 > 0\ \therefore \min pt$$

$$when\ x = 0, y = 0$$

$$\therefore (0,0)\ \text{is}\ \min pt$$

Q11c

$$0 < x < 4$$

Q11d

$$when\ x = 1, \frac{dy}{dx} = e^{-\frac{1}{2}}\left(-\frac{3}{2} + 6\right)$$

$$= \frac{9}{2}e^{-\frac{1}{2}}$$

$$gradient = \frac{-2e^{\frac{1}{2}}}{9}$$

$$when\ x = 1, y = 3e^{-\frac{1}{2}},$$

$$y - 3e^{-\frac{1}{2}} = \frac{-2e^{\frac{1}{2}}}{9}(x - 1)$$

$$y = -\frac{2e^{\frac{1}{2}}}{9}x + \frac{2e^{\frac{1}{2}}}{9} + 3e^{-\frac{1}{2}}$$

$$or\ y = -0.366x + 2.19$$

Q12a

$Since\ D\ \&\ T\ are\ the\ midpts\ of\ BA\ and\ BC,$

$by\ midpt\ theorem, DF // AC.$

$Since\ there\ is\ one\ pair\ of\ parallel\ sides, ADFE\ is\ a\ trapezium$

Q12b

$\angle EFC = \angle FAE (alt\ segment\ theorem)$

$\angle FAE = \angle AFD (alt\ \angle s, DF \parallel AE)$

$\therefore \angle EFC = \angle AFD$

$180° - \angle ADF = \angle FEA\ (\angle s\ in\ opp\ segment)$

$\angle FEC = 180° - \angle FEA (adj\ \angle s\ on\ a\ str. line)$

$= 180° - (180° - \angle ADF)$

$= \angle ADF$

or

$\angle DFB = \angle DAF (alt\ segment\ theorem)$

$\angle DFB = \angle FCE (corrs\ \angle s, AC \parallel DF)$

$\therefore \angle DAF = \angle FCE$

$\therefore \Delta DFA\ is\ similar\ to\ \Delta EFC\ (AA\ Similarity\ test)$

Q12c

$\frac{DA}{EC} = \frac{DF}{EF}$ (corresponding sides of similar triangles are proportional)

$$DF \times EC = DA \times EF$$

$$By\ midpt\ theorem, DF = \frac{1}{2}AC$$

$$\frac{1}{2}AC \times EC = DA \times EF$$