National Junior College
2016 – 2017 H2 Mathematics

Topic 3: Functions

Key Questions to Answer:

1. What is a function?
   - *What kinds of relationships hold between x and* f*(x)?*
   - *What is the difference between a function and a relation?*

2. How do you define a function?
   - *What does function notation look like?*
   - *When are two functions equal?*

3. How do you determine the range of a function?
   - *Is it sufficient to check the endpoints of the domain? Why?*

4. What is a one-one function?
   - *What are the other possible type(s) of relations?*
   - *What makes a one-one function different from the rest?*

5. What is an inverse function?
   - *What is the condition for an inverse function to exist?*
   - *Why is this condition necessary?*
   - *How do you find the rule of the inverse function?*
   - *How is the domain and range of the inverse function related to that of the original function?*
   - *What is the graphical relationship between a function and its inverse?*

6. What is a composite function?
   - *What is the condition for a composite function to exist?*
   - *Why is this condition necessary?*
   - *How do you find the rule of a composite function?*
   - *What is the domain of a composite function?*
   - *How do you find the range of a composite function?*
   - *In finding the range, is it sufficient to check the endpoints of the domain? Why?*

Before we begin the topic on Functions, it is important for students to be familiar with the use of set notation to represent a range of values.

§1 Set Notation

Recall that a set is a collection of distinct elements. The curly brackets "{...}" represent the phrase "the set of ...". When several sets are being discussed simultaneously, they can be denoted using capital letters.

$$A=\{0,\ 2,\ 4\} \qquad B=\{0,\ 2,\ 4,\ ...\} \qquad S_1=\{0\} \qquad S_2=\{\ \}$$

Set operators:

Notation	Meaning	Usage
$\in$	(element) that belongs to (set)	$2 \in \{0, 2, 4\}$
$\cup$	union (of two sets)	$\{0\} \cup \{2\} = \{0, 2\}$
$\setminus$	(set) take away (set)	$\{0, 2\} \setminus \{2\} = \{0\}$
$\cap$	intersection (of two sets)	$\{0, 2, 4\} \cap \{0, 2, \} = \{0, 2\}$

By convention, particular symbols are reserved for the most important sets of numbers:

Symbol	Meaning
$\mathbb{R}$	Set of real numbers (see Remark 1 below).
$\mathbb{R}^+$	Set of positive real numbers (does not include zero).
$\mathbb{R}^-$	Set of negative real numbers (does not include zero).
$\mathbb{R}^+ \cup \{0\}$	Set of non-negative real numbers.

Remarks:

1. A number that can be represented on the number line is a real number. This means that for any real number x, $-\infty < x < \infty$.
2. By definition, zero is neither positive nor negative.

WONDER
What are the symbols that represent the set of integers, rational numbers and natural numbers respectively?

Answer: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N}$.

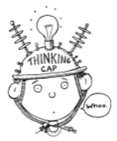
1.1 Representation of Sets of Values

The use of set builder notation and/or interval notation is helpful to represent a range of values as a set.

Range of values	Equivalent representation as a set of values	
	Set Builder Notation	Interval Notation
$-2 < x < 3$	$\{x \in \mathbb{R} \mid -2 < x < 3\}$	$(-2, 3)$
$-2 \le x < 3$	$\{x \in \mathbb{R} \mid -2 \le x < 3\}$	$[-2, 3)$
$-2 \le x \le 3$	$\{x \in \mathbb{R} \mid -2 \le x \le 3\}$	$[-2, 3]$
$x \le -2$	$\{x \in \mathbb{R} \mid x \le -2\}$	$(-\infty, -2]$
$x > 3$	$\{x \in \mathbb{R} \mid x > 3\}$	$(3, \infty)$

In the use of interval notation such as $[a, b)$ or (a, b), note that

- $a < b$,
- we use round brackets if we want to exclude the extreme values of the range, and
- we use square brackets if we want to include the extreme values of the range.



UNDERSTAND

Can we write $\mathbb{R}\backslash\{0\}$ as $(-\infty, 0)\cup(0, \infty)$? Why or why not?

Answer: Yes. Both are sets that correspond to the same interval.

§2 Special Type of Relations: Functions

2.1 Representations of a Relation

A relation is an association between two sets of data (in our case, they are usually numbers).

A relation can be represented in the following ways.

(i) Numerically (as a table of values)

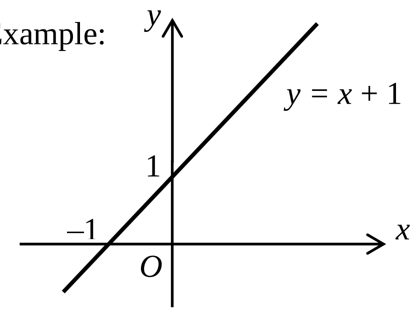
Example:

x	$y = x+1$
0	1
1	2
2	3
3	4

Note: This form of representation is useful especially when the inputs are a set of a finite number of discrete values.

(ii) Graphically (using a curve or line)

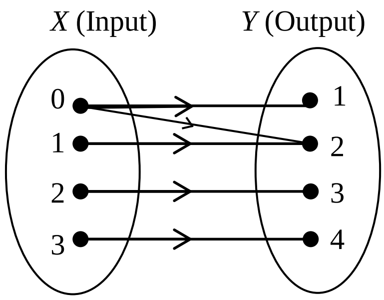
Example:



Note: This form of representation is useful when the input is an interval of real numbers.

(iii) Diagrammatically (using set diagram)

Example:



(iv) Algebraically (as an equation)

Example:

$y = x^2$

$x^2 + y^2 = 4$

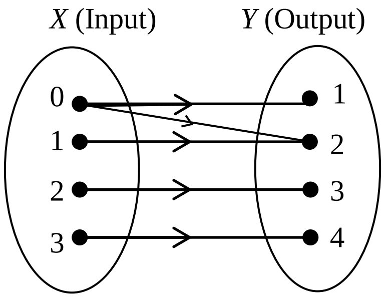
2.2 Relations versus Functions

Example 2.2.1

x	$y = x+1$
0	1
1	2
2	3
3	4

The table above illustrates a relation R (i.e. $y = x+1$) between the elements of two sets of numbers {0,1,2,3} and {1,2,3,4}, namely x and y. It is obvious that R maps 0 to 1, 1 to 2, 2 to 3, and 3 to 4. Since each *input* (i.e. x–value) is mapped to exactly one *output* (i.e. y–value), this relation is a function.

Example 2.2.2



The relation depicted in the set diagram above is not a function as the input "0" is mapped to two outputs, namely "1" and "2".

In this topic, our focus is to study functions, which is defined (not so formally) as follows.

A **function** is a relation that has exactly one output for every possible input.

A formal definition for function is given in **Section 2.3**.

2.3 Rule, Domain and Range of a Function

We shall introduce the formal definition of a function.

Definition 2.3.1 (Function) A relation $\mathrm{f} : X \to Y$ is a function if and only if for each element $x \in X$, there exists <u>exactly</u> one element $y \in Y$ such that $\mathrm{f}(x) = y$.

In the above definition, the element x is mapped to the element y, and y is said to be the **image** of x.

A function f can be expressed algebraically as

$$\mathrm{f} : x \mapsto \mathrm{f}(x), \quad x \in X,$$

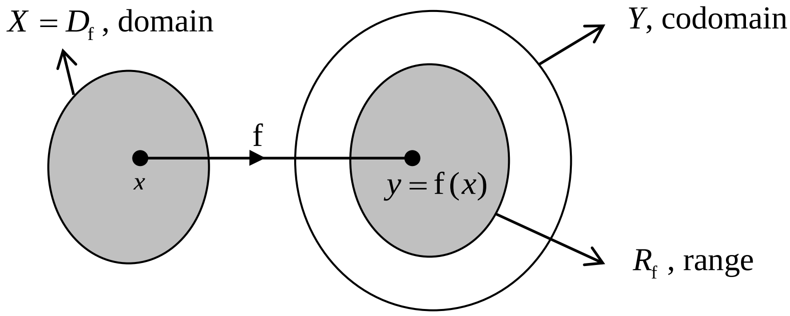
where $\mathrm{f}(x)$ is the **rule** of the function f, and the set X is the **domain** ("inputs") of the function f.

The domain of function f is denoted by D_{f}. Alternatively, a function f can be expressed as

$$y = \mathrm{f}(x), \; x \in X \qquad \text{or} \qquad \mathrm{f} : X \to Y, x \mapsto \mathrm{f}(x),$$

where the set Y is the **codomain** of the function f (usually it is the set of real numbers $\mathbb{R}$).

It is not necessary for all the elements of Y to be the image of some $x \in X$. The **range** ("outputs") of the function f is the subset of Y which contains all the possible images of all the elements of X under f. It is usually denoted by R_{f} .



Example 2.3.1

Explain why the relations shown in Figures A and B are functions. Also, give two reasons why the relation shown in Figure C is not a function.

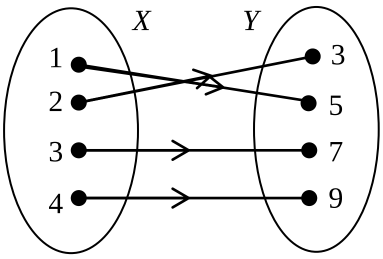


Figure A

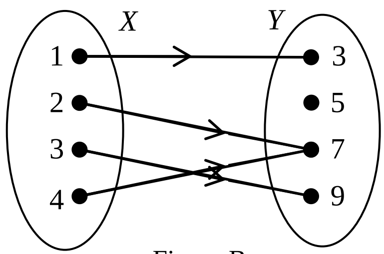


Figure B

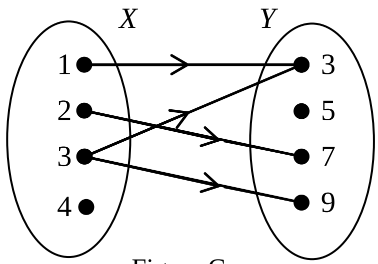


Figure C

Solution:

- The relation as shown in Figure A satisfies the definition of a function.
- The relation as shown in Figure B also satisfies the definition of a function, even though no element of X maps to the element 5 in Y, and two elements of X map to the element 7 in Y.
- The relation as shown in Figure C is not a function, as
  - the element 3 in X maps to two elements in Y, and
  - the element 4 in X does not map to any element in Y.

Example 2.3.2

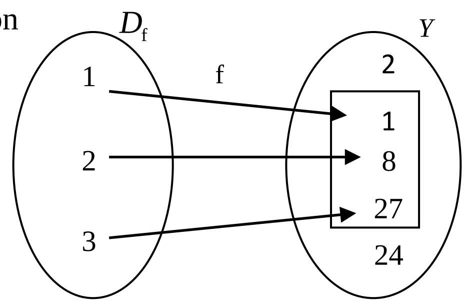
The function with rule $\mathrm{f}(x) = x^3$ defined on domain $\mathbb{R}^+$ can be expressed as

$$\mathrm{f}(x) = x^3,\ x \in \mathbb{R}^+, \qquad \text{or} \qquad \mathrm{f}: x \mapsto x^3,\ x \in \mathbb{R}^+, \qquad \text{or} \qquad \mathrm{f}: \mathbb{R}^+ \to \mathbb{R},\ x \mapsto x^3.$$

Example 2.3.3

Let f be the function with rule $\mathrm{f}(x) = x^3$ defined on $D_{\mathrm{f}} = \{1, 2, 3\}$.

$\mathrm{f}(1) = 1,\ \mathrm{f}(2) = 8$ and $\mathrm{f}(3) = 27$.
Codomain of f, $Y = \{1, 2, 8, 24, 27\}$.
Range of f, $R_{\mathrm{f}} = \{1, 8, 27\}$.



IMPORTANT: A function is defined by both its **rule** **AND** **domain**.

Note that if the domain of a function is not given explicitly in the question, the convention is that the domain is the set of all numbers for which the rule makes sense and defines a real number.

For example, let $\mathrm{f}(x)=\dfrac{1}{x}$. If domain is not specified, then take D_{f} to be $\mathbb{R}\backslash\{0\}$.

This choice of domain is also known as the **maximal domain** of a function.

Example 2.3.4

It is given that $\mathrm{f}(x)=\ln(x+1)$. State the maximal domain and its corresponding range of f.

Solution:

$D_{\mathrm{f}}=(-1,\ \infty)$, $R_{\mathrm{f}}=\mathbb{R}$.

[To help you see this, **sketch the graph** of $y=\ln(x+1)$ and observe the range of values of *y*.]

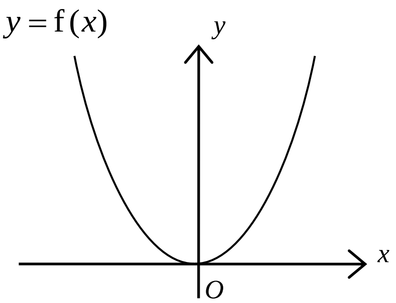
IMPORTANT:

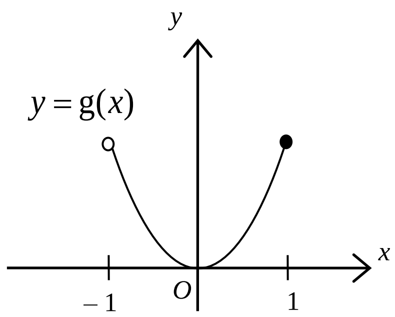
Two functions with the *same rule* but *different domains* are **different** functions.

For instance, consider

$$\mathrm{f}: x \mapsto x^2, \quad x\in\mathbb{R},$$
$$\mathrm{g}: x \mapsto x^2, \quad -1<x\le 1.$$

Even though the rules are the same, that is, $\mathrm{f}(x)=\mathrm{g}(x)=x^2$, the domains D_{f} and D_{g} are different. This has implications on how the corresponding graphs look.





Example 2.3.5

State the domain and find the range of each of the following functions.

(a) $\mathrm{f}: x\mapsto x^2, x\in\mathbb{R}$,

(b) $\mathrm{g}: x\mapsto x^2, x\in[-1,\ 1]$,

(c) $\mathrm{h}: x\mapsto x^2, x\in[-1, 2)$.

Solution:

(a)	(b)	(c)
$D_{\mathrm{f}}=(-\infty,\infty)$, $R_{\mathrm{f}}=[0,\ \infty)$.	$D_{\mathrm{g}}=[-1,\ 1]$, $R_{\mathrm{g}}=[0,\ 1]$.	$D_{\mathrm{h}}=[-1,\ 2)$, $R_{\mathrm{h}}=[0,4)$.

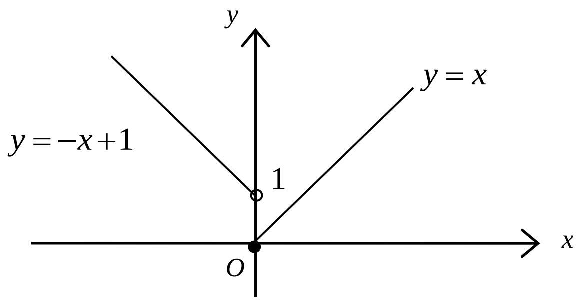
EXPLORE
It is **not** correct (or incomplete) to use only the two endpoints of the domain to find the range of a function. Why?

Answer: Within the interval, the graph of $y=\mathrm{f}\left(x\right)$ may take on other *y*-values.

Example 2.3.6
A piecewise function g is defined as

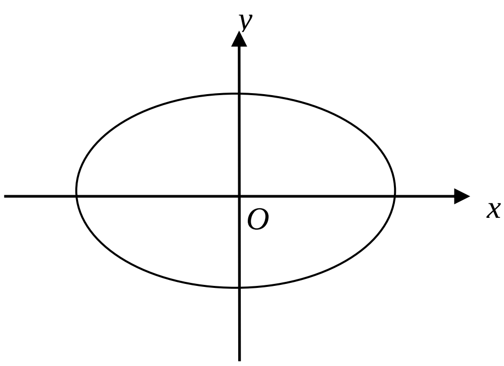
$$\mathrm{g}(x)=\begin{cases}-x+1, & x<0,\\ x, & x\geq 0.\end{cases}$$

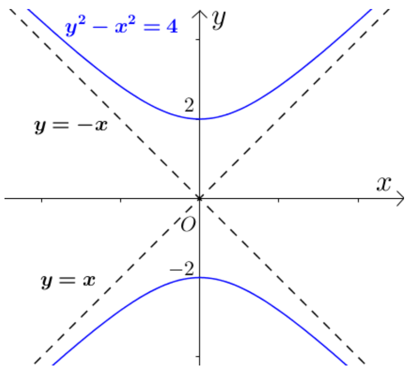
A piecewise function is one that has a different rule for each domain component. The graphical representation of g is as follows.



Read Appendix A1 and A2 on how to use GC to sketch graphs of functions and find their ranges.

WONDER
Explain why the following graphical representations of relations are not functions.





Answer: Choose an input *x* and find output *y*. Then check the definition of function.

2.4 Vertical Line Test (Testing for Functions)

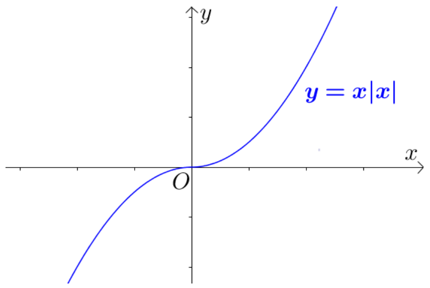
The vertical line test is a **graphical** method for testing whether a relation is a function.

To prove that a relation f is a function:	To prove that a relation f is **NOT** a function:
EITHER Show that **<u>every</u>** vertical line $x = a,\ a \in D_f$ cuts the graph of $y = \mathrm{f}(x),\ x \in D_f$ **<u>exactly once</u>**, OR Show that **<u>every</u>** vertical line $x = a,\ a \in \mathbb{R}$ cuts the graph of $y = \mathrm{f}(x),\ x \in D_f$ **<u>at most once</u>**.	Provide a counter-example, that is, find **<u>one</u>** vertical line such that it cuts the graph of $y = \mathrm{f}(x),\ x \in D_f$ **<u>at least two times</u>**.

Example 2.4.1

Prove that $y = x|x|$ defined on $\mathbb{R}$ is a function.

Solution:



$y = x|x|, x \in \mathbb{R}$ is a function, since from the graph, any vertical line cuts the curve exactly once.

Example 2.4.2

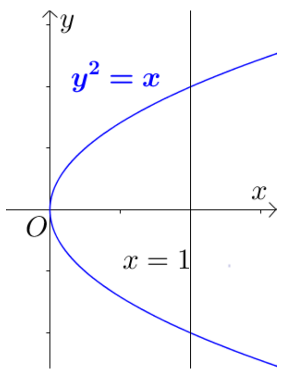
State whether $y^2 = x$ defined on $\mathbb{R}$ is a function, and explain your answer.

Solution:

From the graph, the line $x = 1$ cuts the graph of $y^2 = x$ two times. Hence $y^2 = x$ is not a function.

Alternatively,

When $x = 1$, $y = -1$ and $y = 1$ are two images of x. This contradicts definition of a function.



Remarks:

- When asked to prove a result, you must show ALL steps clearly.
- To prove that the relation is indeed a function, it is necessary to provide a relevant graph.
- To prove that the relation is not a function,
    - you must state explicitly the equation of the vertical line that fails the vertical line test, and
    - you do not have to find all the vertical lines that cut the graph of $y = \mathrm{f}(x),\ x \in D_{\mathrm{f}}$ more than once. (Why?)

UNDERSTAND
How is the vertical line test consistent with the definition of a function?

Answer: When you place a vertical line on the graph, you are choosing a particular x.

CHECK
Does the line $x = 0$ cut the graph shown in **Example 2.3.6** more than once?

Answer: No.

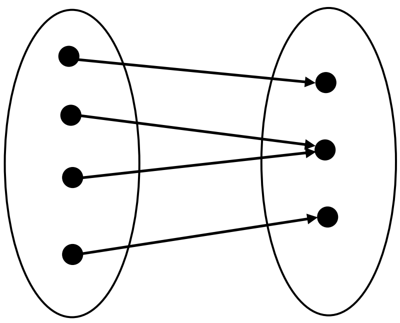
§3 One-One (1-1) Functions

3.1 Definition of One-One Functions

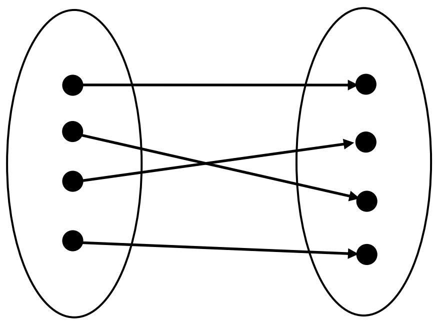
A function f is said to be **one-one** (1-1) or **injective** if no two distinct elements in its domain have the same image under f. In other words,

if k is an image under f, then there is <u>only one</u> value of x that gives $\mathrm{f}(x) = k$.

Example 3.1.1



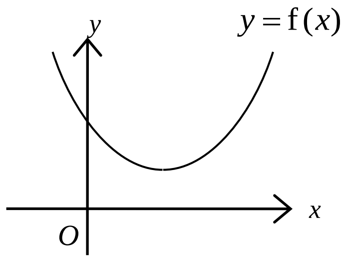
Not a one-one function



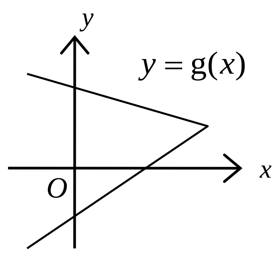
A one-one function

Example 3.1.2

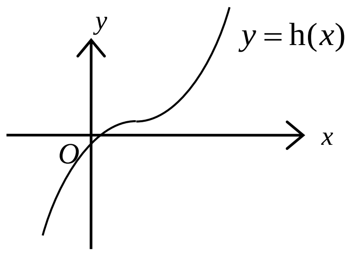
Which of the following are graphs of 1-1 functions?



(Answer : No)



(Answer : No)



(Answer: Yes)

<u>For your information only:</u> Formal definition of one-one functions

Given a function $\mathrm{f}: X \to Y$,
f is 1-1 if for all $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $\mathrm{f}(x_1) \neq \mathrm{f}(x_2)$.

3.2 Horizontal Line Test (Testing for 1-1 Functions)

The horizontal line test is a **graphical** method for testing whether a function is 1-1.

To prove that a function f is 1-1:	To prove that a function is **NOT** 1-1:
EITHER Show that **<u>every</u>** horizontal line $y = b,\ b \in R_\mathrm{f}$ cuts the graph of $y = \mathrm{f}(x),\ x \in D_\mathrm{f}$ **<u>exactly once</u>**, OR Show that **<u>every</u>** horizontal line $y = b,\ b \in \mathbb{R}$ cuts the graph of $y = \mathrm{f}(x),\ x \in D_\mathrm{f}$ **<u>at most once</u>**.	Provide a counter-example, that is, find **<u>one</u>** horizontal line such that it cuts the graph of $y = \mathrm{f}(x),\ x \in D_\mathrm{f}$ **<u>at least two times</u>**.

Remark: It is necessary to provide a relevant graph and the statement above to prove that the function is indeed one-one. However we only need to provide a counterexample to show that the function is not one-one, by stating explicitly an equation of a horizontal line that intersects the graph at least twice.

Example 3.2.1

Determine if the following are one-one functions, giving your reasons. If it is not one-one, find a restriction of the function with a maximal domain such that it is one-one and has the same range as the original function.

(i) $\mathrm{f} : x \mapsto x^2, x \in \mathbb{R},$

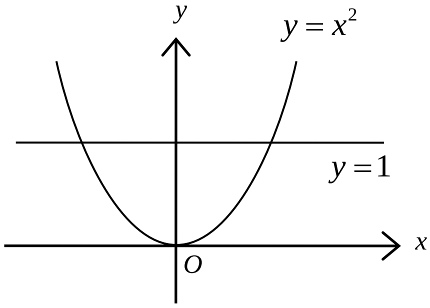
(ii) $\mathrm{g} : x \mapsto x^2 - 4x + 1, \quad x \in \mathbb{R}, x \geq 1,$

(iii) $\mathrm{h} : x \mapsto \dfrac{1-x}{x}, \; x > 0.$

Remark: A restriction of a function f, say g, is a function that has the same rule $\mathrm{g}(x) = \mathrm{f}(x)$, but a smaller domain $D_{\mathrm{g}} \subset D_{\mathrm{f}}$.

Solution:

(i) The horizontal line $y = 1$ cuts the graph of $y = x^2$ twice. Hence, f is not 1-1.

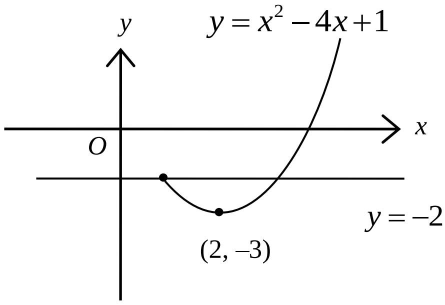


Alternatively, since $\mathrm{f}(-1) = 1 = \mathrm{f}(1)$, f is not 1-1.

Now $\mathrm{R_f} = [0, \infty)$. From the graph, if we restrict the domain to $[0, \infty)$, then every horizontal line will cut the graph exactly once. Furthermore, the range is the same as $\mathrm{R_f}$. Therefore, a required restriction of f is

$$\mathrm{f_1} : x \mapsto x^2, \; x \geq 0.$$

(ii) $\mathrm{g}(x) = x^2 - 4x + 1 = (x-2)^2 - 3, \; x \geq 1.$

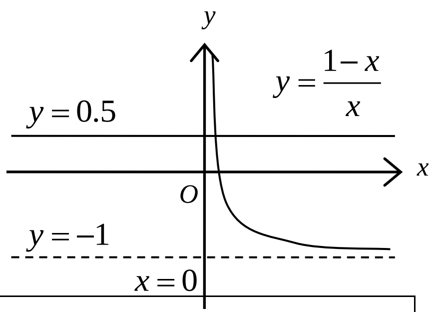


From the graph, the horizontal line $y = -2$ cuts the curve at 2 points. Hence the function is not 1-1.

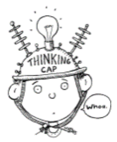
Alternatively, since $\mathrm{f}(1) = -2 = \mathrm{f}(3)$, f is not 1-1.

Now $\mathrm{R_g} = [-3, \infty)$. From the graph, a restriction of g is $\mathrm{g_1} : x \mapsto x^2 - 4x + 1, \; x \geq 2$.

(iii) $\mathrm{h} : x \mapsto \dfrac{1-x}{x}, \; x > 0.$



From the graph, any horizontal line $y = k, \; k > -1$ cuts the graph exactly once. Therefore h is a 1-1 function.



WONDER
Are the answers to **Examples 3.2.1(i)** and **(ii)** unique? Explain.

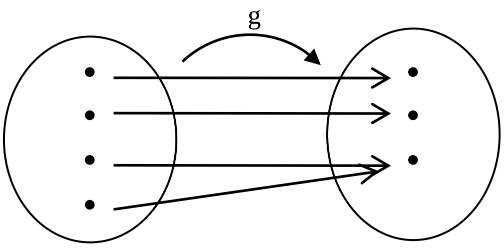
Answer: No for **(i)**, and yes for **(ii)**. For **(i)**, you can also choose the domain as $(-\infty, 0]$.

§4 Inverse Functions

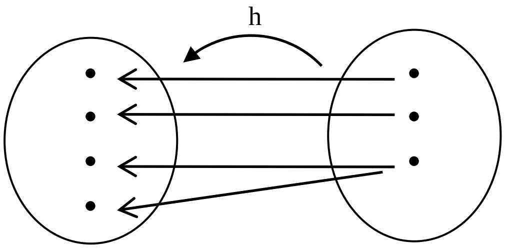
4.1 Inverse OF Function versus Inverse Function

It is very natural to ask if a process is reversible. Similarly for functions, it is very tempting for us to ask if there is a function that reverses what another function does. This leads to the notion of the inverse OF function. (**Caution:** This is not the same as inverse functions.)

Consider the function g as shown in the diagram below (observe that g is not 1-1).

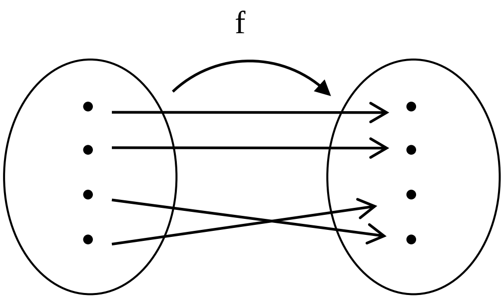


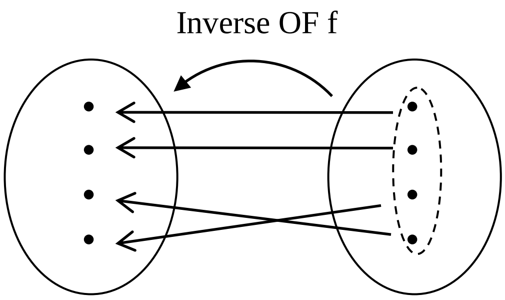
Suppose that h reverses the mapping by function g.



Notice that h, the inverse OF function g, is only a relation. It is **NOT** a function.

We shall consider another example, where the function f and the inverse OF f are shown below (observe that f is 1-1).





In this case, notice that the inverse OF f is a function. Hence we have the following definition.

Definition 4.1.1 (Inverse Function)

Let $\mathrm{f} : X \to Y$ be a function such that $\mathrm{f} : x \mapsto y, x \in X$. Also, let $\mathrm{g} : Y \to X$ be a relation such that x can be obtained from g(y).

Then the relation g is the **inverse of function** f. Additionally, if g is a function, then g is called the **inverse function**, and we write g as f^{-1}.

UNDERSTAND

Explain, in your own words, the similarity and difference between "inverse of function" and "inverse function".

Answer: Similarity – both are "un-doing" a function.
Difference – one of them may not be a function. Which one is it?

CHECK

True or False: The inverse of function f can be written as f^{-1}.

Answer: False.

WONDER

With reference to the domain and range of f, what can you say about the domain and range of f^{-1}?

Answer: Read below.

Important Results:

1. The inverse function f^{-1} exists only if and only if the function f is one-one.
2. The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f, i.e., $D_{\mathrm{f}^{-1}} = R_{\mathrm{f}}$ and $R_{\mathrm{f}^{-1}} = D_{\mathrm{f}}$. (**Exercise:** Use the set diagram to deduce this.)

Example 4.1.2

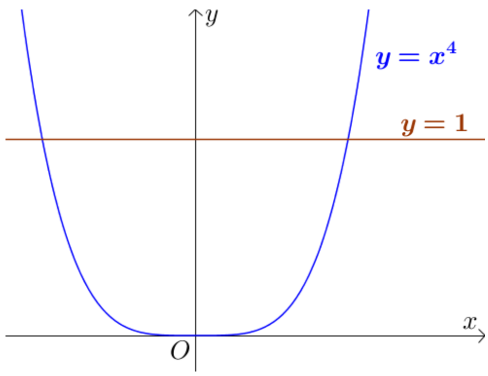
For each of the following functions, determine if the inverse function exist. Justify your answer.

(i) $\mathrm{f}: x \mapsto x^4, \quad x \in \mathbb{R},$

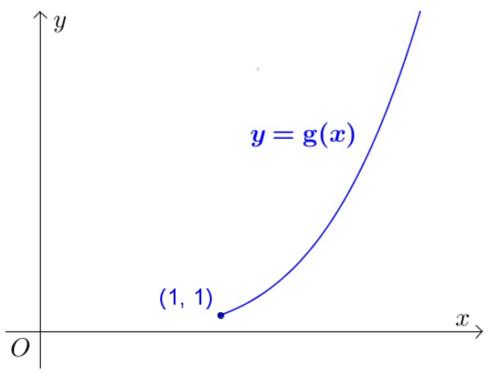
(ii) $\mathrm{g}: x \mapsto x^4, \quad x \in \mathbb{R}, x \geq 1.$

Solution:

(i) The horizontal line $y = 1$ cuts the graph of $y = x^4$ twice. Hence, f is not 1-1. Therefore f^{-1} does not exist.



(ii) The function g is 1-1 since any horizontal line $y = b,\ b \in \mathbb{R}$ cuts the graph of $y = \mathrm{g}(x)$ at most once. Therefore g^{-1} exists.



Remarks:

- Recall that the horizontal line test is a graphical method that determines whether a function is 1-1.
- Even though we have the result that all one-one functions have an inverse function, it is **NOT** correct to write the following statement:

  "... by the horizontal line test, f^{-1} exists."
- This is because the horizontal line test does not graphically determine whether the inverse function exists.

Reminders:

- It is **INCORRECT** to view "-1" in the notation "f^{-1}" as a power and treat f^{-1} as $\frac{1}{\mathrm{f}}$. You may wish to compare the terms "inverse" and "reciprocal".
- You have seen inverse functions in trigonometry previously. They are the inverse trigonometric functions, e.g. $\cos^{-1} x$ is the arc cosine of x and is **not** equal to $\sec x$, i.e. $\cos^{-1} x \neq \frac{1}{\cos x}$.

4.2 Relationship between Graph of a One-One Function and its Inverse Function

Consider the graph of a one-one function $\mathrm{f}: x \mapsto y,\ x \in D_\mathrm{f}$. When $x = a$, the coordinates of the point on the graph of $y = \mathrm{f}(x)$ is $\left(a, \mathrm{f}(a)\right)$.

We recall from earlier that the inverse function f^{-1} exists, and that

$$D_{\mathrm{f}^{-1}} = R_\mathrm{f} \text{ and } R_{\mathrm{f}^{-1}} = D_\mathrm{f} .$$

This means that the corresponding point on the graph of $y = \mathrm{f}^{-1}(x)$ is $\left(\mathrm{f}(a),\ a\right)$.

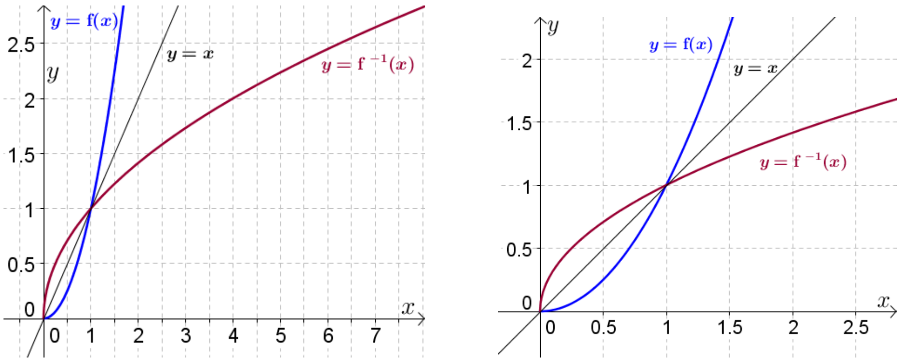
We will visualise 6 points on the graph of $y = \mathrm{f}(x) = x^2,\ x > 0$.

Value of a	$\left(a, \mathrm{f}(a)\right)$, coordinates of points on graph of $y = \mathrm{f}(x)$	$\left(\mathrm{f}(a),\ a\right)$, coordinates of points on graph of $y = \mathrm{f}^{-1}(x)$
1	(1, 1)	(1, 1)
2	(2, 4)	(4, 2)
3	(3, 9)	(9, 3)
4	(4, 16)	(16, 4)
5	(5, 25)	(25, 5)
6	(6, 36)	(36, 6)

Plotting these 6 points on the Cartesian plane will help us see that the coordinates in the right column are reflection of those in the left column about the line $y = x$. That means the **graph of $y = \mathrm{f}^{-1}(x)$ is the reflection of the graph of y = f(x) about the line $y = x$.**

THINK

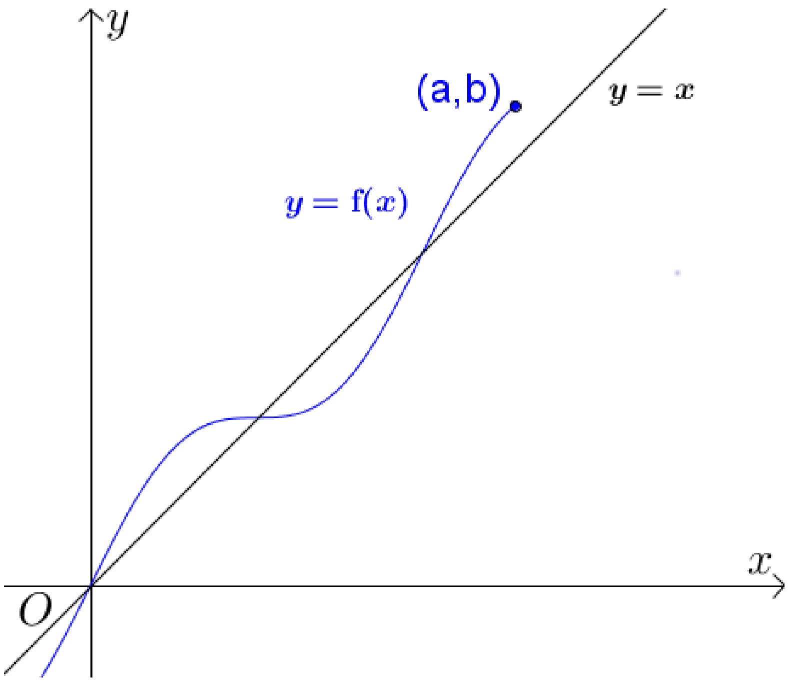
The following two diagrams illustrate both the same graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$. Which diagram is a correct (i.e. convincing) illustration of $y = x$ being the line of symmetry? Why?



Answer: The keyword is "reflection". What does it mean to reflect a graph?

Example 4.2.1 (Do it yourself)

The graphs of y = f(x) and y = x are shown below. Sketch, on the same diagram, the graph of $y = \mathrm{f}^{-1}(x)$, labelling the coordinates of the endpoints.



4.3 Finding the Rule of Inverse Function

Suppose f(x) is the rule for a one-one function f. To find algebraically $\mathrm{f}^{-1}(x)$, the rule of the inverse function, we apply the following steps:

Step 1: Let y = f(x). [**Remark:** $y=\mathrm{f}(x) \Leftrightarrow x=\mathrm{f}^{-1}(y)$.]
Step 2: Make x the subject. [**Remark:** RHS gives $\mathrm{f}^{-1}(y)$.]
Step 3: Replace y on RHS by x. [**Remark:** RHS becomes $\mathrm{f}^{-1}(x)$.]

Example 4.3.1

It is given that $\mathrm{f}: x \mapsto 2x-2,\ x \in \mathbb{R}$. Explain why f^{-1} exists, and find f^{-1} in a similar form.

Solution:

The graph of y = f(x) is a straight line. As such any horizontal line cuts the graph exactly once. Hence f is 1-1 and so f^{-1} exists.

To find $\mathrm{f}^{-1}(x)$: Let $y=\mathrm{f}(x)=2x-2 \quad \Rightarrow \quad x=\dfrac{y+2}{2}=\mathrm{f}^{-1}(y)$.

$$\Rightarrow \quad \mathrm{f}^{-1}(x)=\frac{x+2}{2}.$$

Finally since $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=\mathbb{R}$, we have $\mathrm{f}^{-1}: x \mapsto \dfrac{x+2}{2},\ x \in \mathbb{R}$.

Example 4.3.2

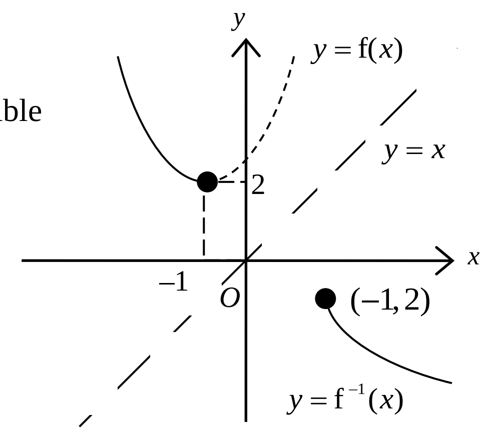
A function f is defined as $\mathrm{f}: x \mapsto x^2+2x+3,\ x \in S$, where $S \subseteq \mathbb{R}$. Find the largest possible domain S in the form $(-\infty, k]$ such that f is one-one. Hence define f^{-1} and sketch the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ on the same diagram.

Solution:

$\mathrm{f}(x)=x^2+2x+3=\left(x+1\right)^2+2$.

From the graph of y = f(x), f is one-one if the largest possible domain is $(-\infty,-1]$ or $\left[-1,\infty\right)$.

$\Rightarrow \quad S=(-\infty,-1]$ and $k=-1$.



To find $\mathrm{f}^{-1}(x)$: Let $y=\mathrm{f}(x)=\left(x+1\right)^2+2,\quad x \le -1$,

$$\Rightarrow \quad y-2=\left(x+1\right)^2$$
$$\Rightarrow \quad x=-1 \pm \sqrt{y-2}.$$

Since $x \le -1$, we have $x=-1-\sqrt{y-2}$.

Finally as $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=[2,\infty)$, $\mathrm{f}^{-1}(x)=-1-\sqrt{x-2},\ x \ge 2$.

Important:

1. It only makes sense to find the rule of the inverse function f^{-1} **after** we establish its existence (i.e. we need to ensure that function f is one-one).

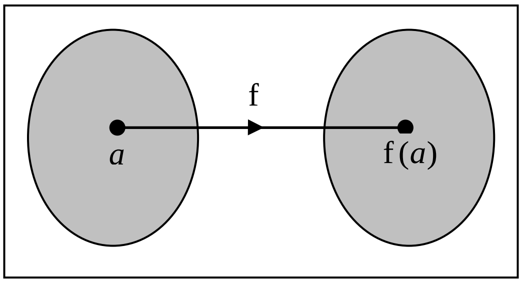
2. You are expected to know how to sketch inverse graphs **without** actually finding the rule of the inverse function (make use of the relationship between a function and its inverse), and to label the respective axial intercepts, range, and domains of the inverse graphs accurately.

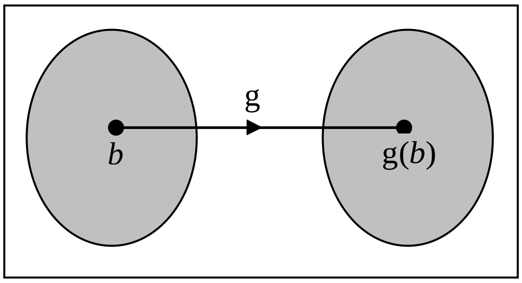
A

Read Appendix A3 on use of GC to sketch graphs of inverse functions.

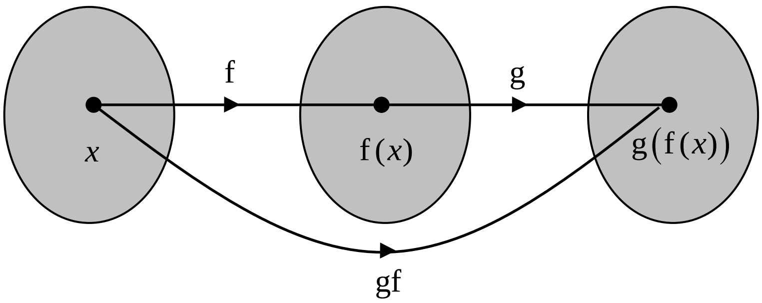
§5 Composite Functions

Suppose that there are two functions, say f and g.





It is natural to ask if we can first apply f to an input x to obtain the "output" of f, i.e. f(x), and then apply g to this output.



If it is possible, we can denote

$$\mathrm{g}\left(\mathrm{f}(x)\right) = \mathrm{gf}(x),\ x \in D_{\mathrm{f}}\ ,$$

where gf (or $\mathrm{g} \circ \mathrm{f}$) is the composite function.

In this particular case, function f is first applied to x to obtain f(x), the "output" of f, and then g is applied to f(x).

WONDER
How would you denote the composite function where g is applied on x to obtain an "output", and then f is applied on this "output"?

Answer: fg (note that this is not the same as gf, and we shall see this later).

Example 5.0.1

The functions f and g are defined as

$$\text{f}: x \mapsto x+1, \quad x \in \mathbb{R}, \qquad \text{g}: x \mapsto x^2, \quad x \in \mathbb{R}.$$

Find fg(2) and gf(2).

Solution:

$$\text{fg}(2) = \text{f}\left(\text{g}(2)\right) = \text{f}(4) = 5; \qquad \text{gf}(2) = \text{g}\left(\text{f}(2)\right) = \text{g}(3) = 9.$$

5.1 Existence of Composite Functions

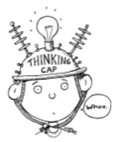
Consider the following functions:

$$\text{f}: x \mapsto -x, \quad x \in \mathbb{R}, \qquad \text{g}: x \mapsto \ln x, \quad x \in (0, \infty).$$

We would like to ask if the composite function gf exist. While gf(x) is defined for certain values of x, say $x = -3$, it is easy to see that when $x = 2$,

$$\text{gf}(2) = \text{g}(-2) = \text{undefined}.$$

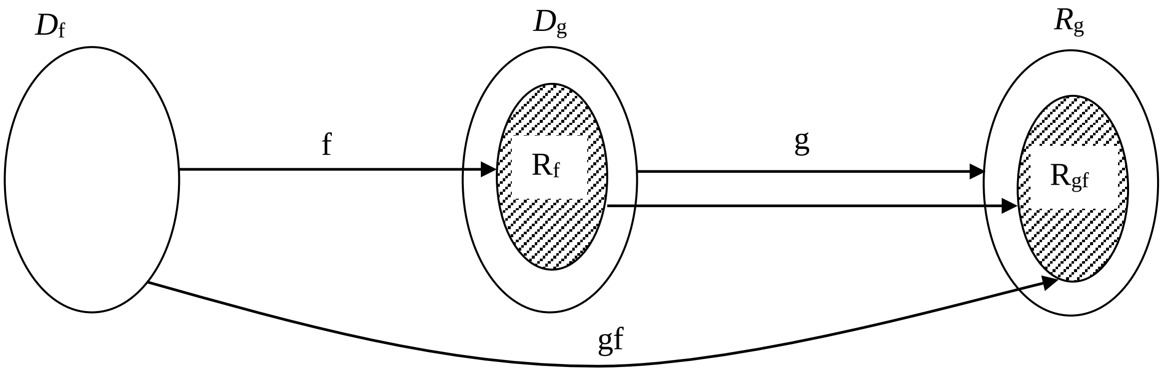
Hence we would like to know the condition for which a composite function gf exists. To answer this question, we consider two cases: $R_\text{f} \subseteq D_\text{g}$ and $R_\text{f} \not\subseteq D_\text{g}$.



WONDER
Use the above illustration to explain why we would want to consider these two cases.

Answer: The two cases arise because g(value) can either be defined or undefined.

<u>Case I:</u> $R_\text{f} \subseteq D_\text{g}$



Observations:

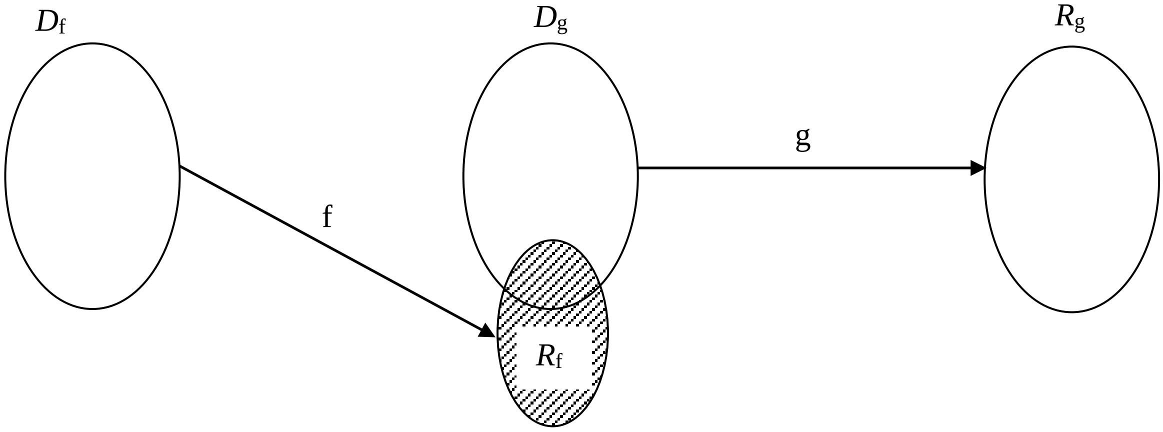
(i) The composite function gf exists. (ii) $D_\text{gf} = D_\text{f}$. (iii) $R_\text{gf} \subseteq R_\text{g}$.

WONDER

In Case I, What can you say about R_{gf} when $R_{\mathrm{f}} = D_{\mathrm{g}}$?

Answer: $R_{\mathrm{gf}} = R_{\mathrm{g}}$. Why?

<u>Case II:</u> $R_{\mathrm{f}} \not\subseteq D_{\mathrm{g}}$



Observations:

(i) We can find an element of R_{f} that is not an element of D_{g} and hence, no image is defined under g. Therefore the composite function gf does not exist.

(ii) If we want gf to exist, we must 'reduce' R_{f} so that $R_{\mathrm{f}} \subseteq D_{\mathrm{g}}$. We can do so by restricting the domain of function f so that $R_{\mathrm{f}} \subseteq D_{\mathrm{g}}$. The restricted range of f will be a subset of $D_{\mathrm{g}} \cap R_{\mathrm{f}}$. Then it will be as in Case I, except that we are working with the restriction of function f, instead of f itself.

5.2 Determining the Rule of a Composite Function

Example 5.2.1

The functions f and g are defined as

$$\mathrm{f} : x \mapsto x+1, \qquad x \in \mathbb{R},$$
$$\mathrm{g} : x \mapsto x^2, \qquad x \in \mathbb{R}.$$

Assuming that fg and gf exists, find fg(x) and gf(x).

Solution:

$$\mathrm{fg}(x) = \mathrm{f}\left(\mathrm{g}(x)\right) = \mathrm{f}\left(x^2\right) = x^2 + 1; \qquad \mathrm{gf}(x) = \mathrm{g}\left(\mathrm{f}(x)\right) = \mathrm{g}(x+1) = (x+1)^2.$$

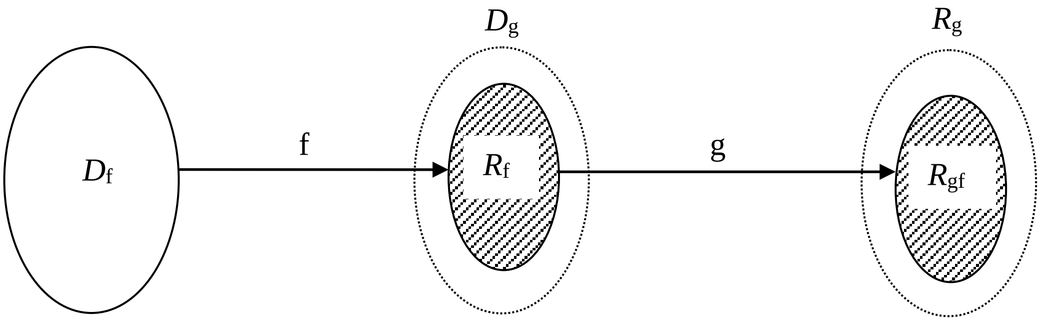
Remark: **Examples 5.0.1** and **5.2.1** illustrate the fact that generally gf and fg are **NOT** equivalent, that is, composition of functions are not commutative.

5.3 Determining the Range of a Composite Function

There are typically 2 methods of determining the range of a composite function, say gf.

<u>Method 1:</u> Using the graphs of both $y = \text{f}(x)$ and $y = \text{g}(x)$

- The key idea of this method is to obtain R_f first graphically from $y = \text{f}(x)$, and then use R_f to find R_gf graphically from $y = \text{g}(x)$.



- You may wish to recall how to find the range of a function using its rule and domain (refer to Page 6) and using GC (refer to Appendixes A1 and A2).
.

<u>Method 2:</u> Sketching the graph of the composite function $y = \text{gf}(x),\ x \in D_\text{gf}$.

- The GC may be used to sketch the composite function (Beware of the limitations of GC!)
- In certain situations, for example, there is an unknown constant in the rule of gf, algebraic methods are required.
- The range of the composite function is determined from the graph.

A

Read Appendix A4 on use of GC to sketch graphs of composite functions.

UNDERSTAND

The 2 methods above are **NOT** necessary to find R_gf when $R_\text{f} = D_\text{g}$, because $R_\text{gf} = R_\text{g}$.
Similarly when $R_\text{g} = D_\text{f}$, $R_\text{fg} = R_\text{f}$.

Example 5.3.1

The functions f and g are defined as

$$\mathrm{f}: x \mapsto \mathrm{e}^{1-x}, \qquad x \in \mathbb{R}, x > 0,$$
$$\mathrm{g}: x \mapsto \ln(x-1), \qquad x \in \mathbb{R}, x > 1.$$

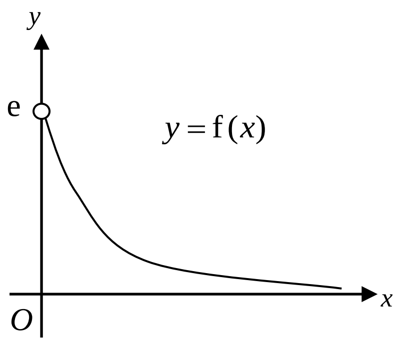
(i) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$, and state the range of f and g.

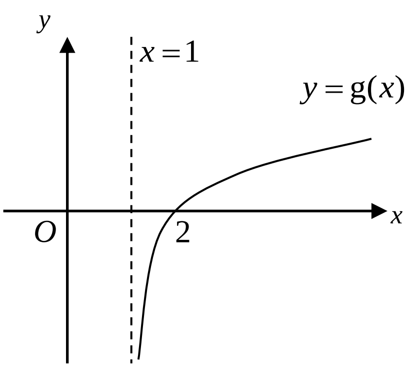
(ii) State why the composite function fg does not exist.

(iii) By restricting the domain of g to (a, ∞), where a is a real number, find the least value of a such that the composite function fg exists. Define fg in similar form. State the range of fg.

Solution:

(i)





From the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$, $R_\mathrm{f} = (0, \mathrm{e})$ and $R_\mathrm{g} = \mathbb{R}$.

(ii) Since $\mathbb{R} = R_\mathrm{g} \not\subseteq D_\mathrm{f} = (0, \infty)$, fg does not exist.

(iii) For the composite function to exist, we must restrict the domain of g such that the range of the restriction function is a subset of D_f. Since we want the least value of a, we will use the 'largest' possible subset of D_f and R_g, i.e. $R_\mathrm{g} \cap D_\mathrm{f} = (0, \infty)$.

From the graph of $y = \mathrm{g}(x)$, we see that if we restrict the domain to $(2, \infty)$, then the range of the restriction of g is $(0, \infty)$, which is D_f.

Therefore least value of $a = 2$.

$$\mathrm{fg}(x) = \mathrm{f}\left(\mathrm{g}(x)\right) = \mathrm{f}\left(\ln(x-1)\right) = \mathrm{e}^{1-\ln(x-1)} = \frac{\mathrm{e}}{x-1} \qquad \Rightarrow \qquad \mathrm{fg}: x \mapsto \frac{\mathrm{e}}{x-1}, x > 2.$$

$R_\mathrm{fg} = (0, \mathrm{e})$.

Reminder: When defining a function, the rule and domain of the function must be stated.

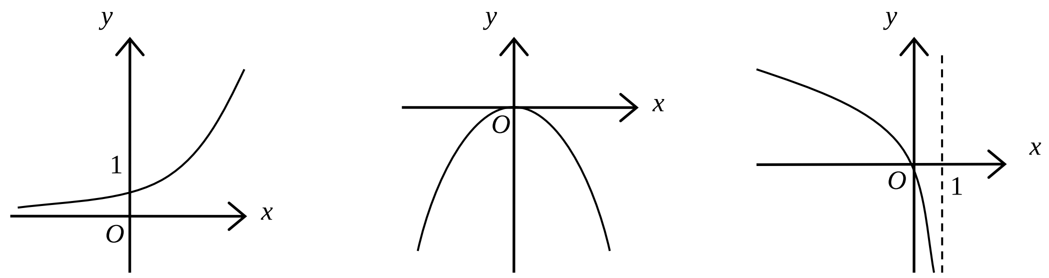
Example 5.3.2

It is given that

$$\begin{aligned} &\mathrm{f}: x \mapsto \mathrm{e}^x, && x \in \mathbb{R}, \\ &\mathrm{g}: x \mapsto -x^2, && x \in \mathbb{R}, \\ &\mathrm{h}: x \mapsto \ln(1-x), && x < 1. \end{aligned}$$

Determine whether the composite functions fg, gh, hg and hf exist. If the composite function exists, define the function and state its range. If the composite function does not exist, find the maximal domain such that it exists.

Solution:



$\mathrm{f}(x) = \mathrm{e}^x$ $\qquad$ $\mathrm{g}(x) = -x^2$ $\qquad$ $\mathrm{h}(x) = \ln(1-x)$

$D_\mathrm{f} = \mathbb{R},\ R_\mathrm{f} = (0, \infty)$ $\qquad$ $D_\mathrm{g} = \mathbb{R},\ R_\mathrm{g} = (-\infty, 0]$ $\qquad$ $D_h = (-\infty, 1),\ R_h = \mathbb{R}$

Check fg:

Since $R_\mathrm{g} \subseteq D_\mathrm{f}$, fg exists.

$$\Rightarrow \quad \mathrm{fg}(x) = \mathrm{f}\left(-x^2\right) = \mathrm{e}^{-x^2}, x \in \mathbb{R}.$$

To find range of fg (using Method 1):

From graph of $y = \mathrm{g}(x)$: $\mathbb{R} \xrightarrow{\mathrm{g}} (-\infty, 0]$

From graph of $y = \mathrm{f}(x)$: $(-\infty, 0] \xrightarrow{\mathrm{f}} (0, 1]$

Therefore,

$$\mathbb{R} \xrightarrow{\mathrm{g}} (-\infty, 0] \xrightarrow{\mathrm{f}} (0, 1].$$

$$R_\mathrm{fg} = (0, 1]$$

Check gh:

Since $R_\mathrm{h} \subseteq D_\mathrm{g}$, gh exists.

$$\Rightarrow \quad \mathrm{g\,h}(x) = \mathrm{g}\left(\ln(1-x)\right)$$
$$= -\left(\ln(1-x)\right)^2, x \in (-\infty, 1).$$

To find range of gh (using Method 1):

$$(-\infty, 1) \xrightarrow{\mathrm{h}} \mathbb{R} \xrightarrow{\mathrm{g}} (-\infty, 0].$$

$$R_\mathrm{gh} = (-\infty, 0]$$

Note: $R_\mathrm{h} = D_g$, so $R_\mathrm{gh} = R_g$.

Check hg:

Since $R_\mathrm{g} \subseteq D_\mathrm{h}$, hg exists.

$$\Rightarrow \quad \mathrm{hg}(x) = \mathrm{h}\left(-x^2\right) = \ln\left(1 + x^2\right), x \in \mathbb{R}.$$

To find range of hg (using Method 1):

$$\mathbb{R} \xrightarrow{\mathrm{g}} (-\infty, 0] \xrightarrow{\mathrm{h}} [0, \infty).$$

$$R_\mathrm{hg} = [0, \infty)$$

Check hf:

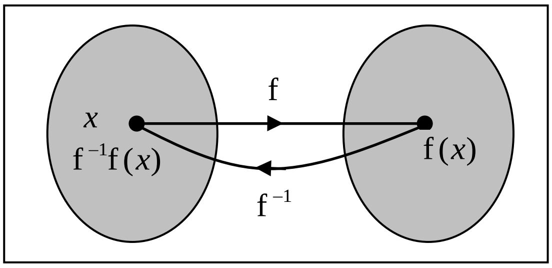
Since $R_\mathrm{f} \not\subseteq D_\mathrm{h}$, hf does not exist.

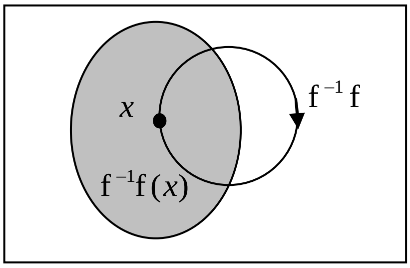
$R_\mathrm{f} \cap D_\mathrm{h} = (0, 1)$. Therefore, we need to find the largest possible subset of D_f which gives $(0, 1)$ under f.

From the graph of $y = \mathrm{f}(x)$, we see that we need $x < 0$ order to obtain $0 < y < 1$.

Therefore maximal domain is $(-\infty, 0)$.

5.4 Composition of a One-One Function and its Inverse Function





Recall that the inverse function of f reverses what the function f does. As such, if f^{-1} is applied to $\mathrm{f}(x)$, then we will get x. In other words, we obtain $\mathrm{f}^{-1}\mathrm{f}(x)=x$ without performing any algebraic manipulation. Similarly, we can also deduce that $\mathrm{ff}^{-1}(x)=x$.

THINK

Since $\mathrm{f}^{-1}\mathrm{f}(x)=\mathrm{ff}^{-1}(x)=x$, can we conclude that the two functions $\mathrm{f}^{-1}\mathrm{f}$ and ff^{-1} are indeed the same? Explain.

Answer: No. Do ask yourself what defines a function.

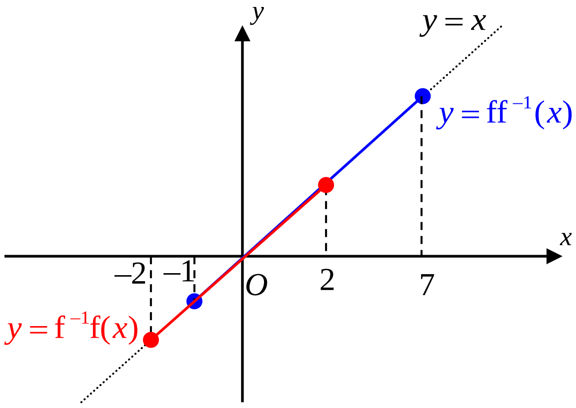
Example 5.4.1

It is given that f is a one-one function defined by $\mathrm{f}: x \mapsto 2x+3, -2 \le x \le 2$. Find

(i) f^{-1}, (ii) $\mathrm{f}^{-1}\mathrm{f}$, (iii) ff^{-1}.

Sketch the graphs of $y=\mathrm{f}^{-1}\mathrm{f}(x)$ and $y=\mathrm{ff}^{-1}(x)$ on a single diagram. Hence solve $\mathrm{f}^{-1}\mathrm{f}(x)=\mathrm{ff}^{-1}(x)$.

Solution:



$D_{\mathrm{f}}=[-2,2]$, $R_{\mathrm{f}}=[-1,7]$.

(i) Let $y=2x+3$

$$\Rightarrow x=\frac{y-3}{2}$$

$$\Rightarrow \mathrm{f}^{-1}(x)=\frac{x-3}{2},\ x\in[-1,7].$$

(ii) $\mathrm{f}^{-1}\mathrm{f}(x)=\mathrm{f}^{-1}\left(\mathrm{f}(x)\right)=x,\ x\in D_{\mathrm{f}}=[-2,2]$.

(iii) $\mathrm{ff}^{-1}(x)=\mathrm{f}\left(\mathrm{f}^{-1}(x)\right)=x,\ x\in D_{\mathrm{f}^{-1}}=[-1,7]$.

(Last part) $\mathrm{f}^{-1}\mathrm{f}(x)=\mathrm{ff}^{-1}(x)\Rightarrow -1\le x\le 2$.

Remarks:

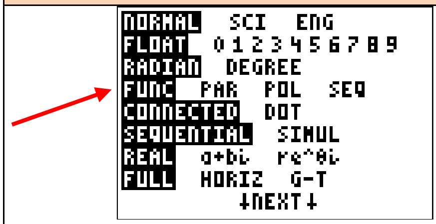
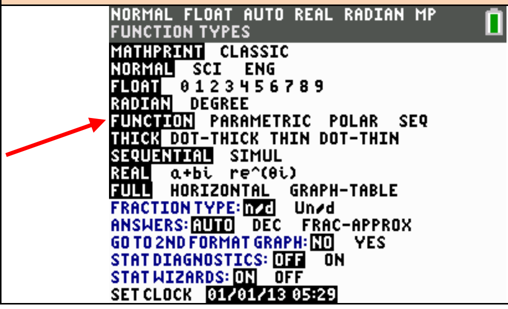
(i) $\mathrm{f}^{-1}\mathrm{f}$ and ff^{-1} are generally two different functions because the domains might be different. Therefore, care must be taken when you sketch these graphs.

(ii) $\mathrm{ff}^{-1}(x)=x,\ D_{\mathrm{ff}^{-1}}=D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}$; $\quad \mathrm{f}^{-1}\mathrm{f}(x)=x,\ D_{\mathrm{f}^{-1}\mathrm{f}}=D_{\mathrm{f}}$.

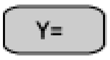
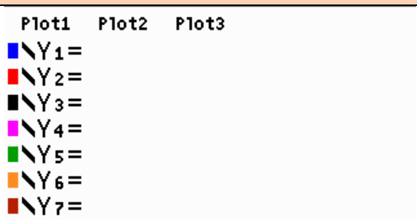
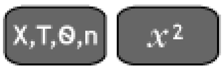
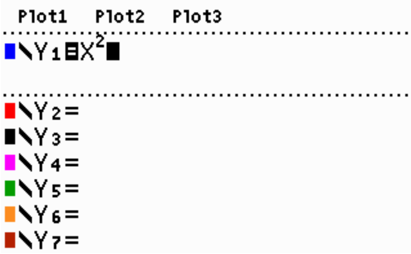
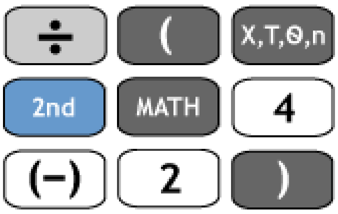
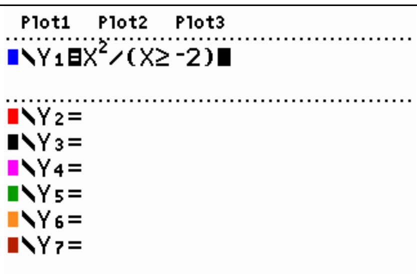
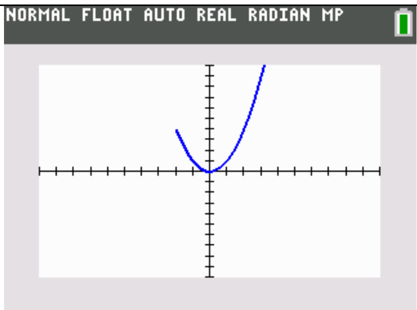
Appendix: Use of Graphing Calculators in Functions

A1 Sketching Functions

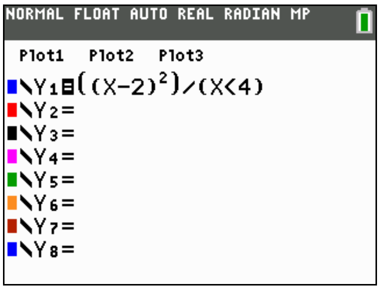
Before we begin, always check the mode by pressing MODE key on your calculator. Since we are sketching graphs of the form $y = \text{f}(x)$, we use the FUNCTION mode.

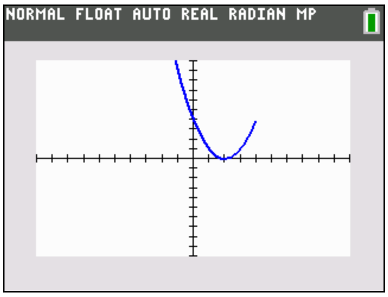
TI-84 Plus	TI-84 C Plus Edition
	

EXAMPLE: Sketch the graph of $\text{f} : x \mapsto x^2, x \geq -2$ using GC.

Key Press	Screen Shot	Steps/Notes/Descriptions
		**Input window:** You can plot a maximum number of 10 functions at any one time.
		**Entering the rule of f:** The four symbols shown in first key press correspond to the four different variables. In FUNCTION mode, the variable X will appear as you press the button.
		**Entering the domain of f:** The inequality signs can be found by pressing TEST, i.e. 2nd MATH. Performing a division of this syntax sets the domain (see explanation on next page).
		**Plotting f:** Sometimes you need to adjust WINDOW in order to see some features that are not inside the current window. Alternatively you can use ZOOM.

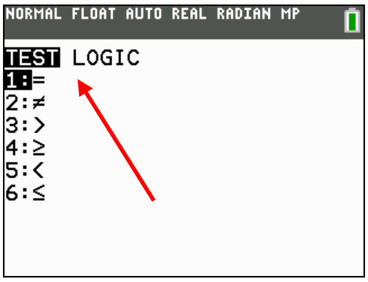
EXERCISE: Sketch the graph of $\mathrm{f}: x \mapsto (x-2)^2, x<4$. You may find the following screenshots helpful.

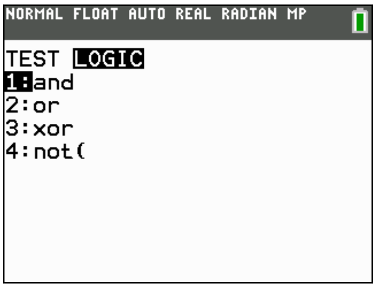




Remarks:

- If the rule of the function contains more than one term, then the rule must be placed inside the parentheses, i.e. $\left((x-2)^2\right)(x<4)$.
- If the domain is given as $-2 \le x < 2$, then enter "(X<2 and X ≥ -2)". The "and" can be found in TEST > LOGIC tab.



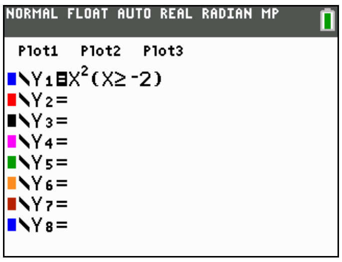


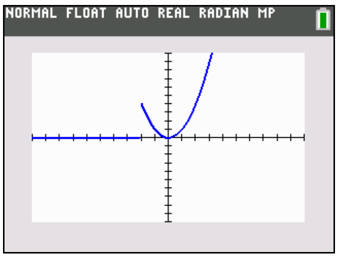
Explanation of Syntax:

The syntax "$(\mathrm{X} \ge -2)$" or the like can be seen to take the value "1" when the condition inside the parentheses is true and the value "0" if it is false. Since a function is unchanged when it is divided by 1, and a function is undefined when it is divided by 0, we have

$$\frac{\mathrm{X}^2}{(\mathrm{X} \ge -2)} = \begin{cases} \mathrm{X}^2 & \text{if } \mathrm{X} \ge -2 \\ \text{undefined} & \text{if } \mathrm{X} \not\ge -2. \end{cases}$$

EXERCISE: Justify why the two GC screenshots correspond to each other
(Please view this in colour – there is a blue horizontal line.)





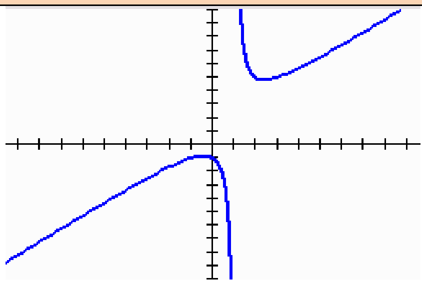
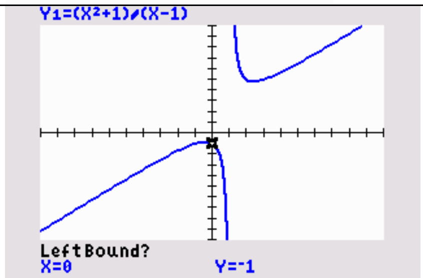
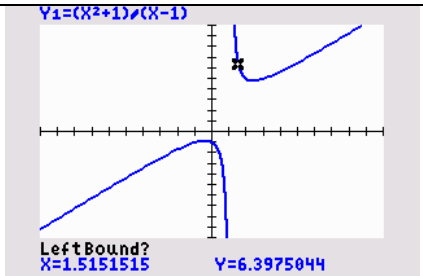
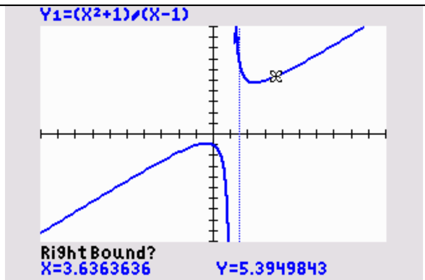
Challenge: By using $\mathrm{Y}_1 = -\mathrm{X}(\mathrm{X} \le 0)$ and $\mathrm{Y}_2 = \mathrm{X}^2(\mathrm{X} \ge 0)$, you can plot

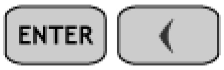
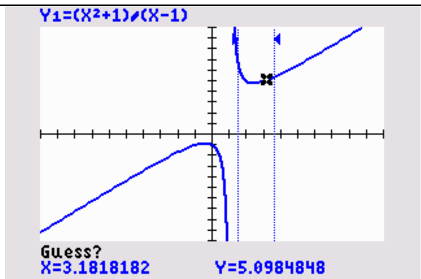
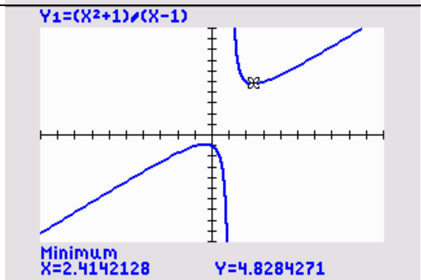
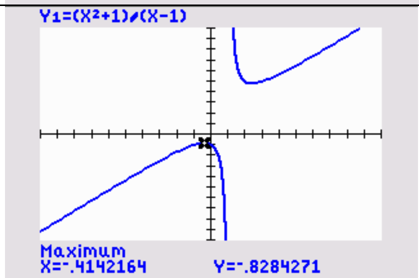
$$\mathrm{f}(x) = \begin{cases} -x, & x \le 0, \\ x^2, & x \ge 0. \end{cases}$$

Are you able to use **only** Y_1 to plot the same function on your GC?

A2 Finding Range of Functions

EXAMPLE. Find the range of $\mathrm{f}: x \mapsto \dfrac{x^2+1}{x-1},\ x \in \mathbb{R},\ x \neq 1$ using GC.

Key Press	Screen Shot	Steps/Notes/Descriptions
Y= CLEAR (x^2 (X,T,Θ,n x^2 + 1) ÷ (X,T,Θ,n − 1) GRAPH		Key in the function and plot the graph. Since the domain consists of all real values except for the singularity at $x=1$, we can ignore the singularity. Observe that there is a minimum and maximum turning point. We need to find the *y*-values of the two turning points.
2nd TRACE	CALCULATE 1:value 2:zero 3:minimum 4:maximum 5:intersect 6:dy/dx 7:∫f(x)dx	Press 2[nd] TRACE to access the CALCULATE menu. We need the MINIMUM and MAXIMUM functions in this menu.
3		The GC will prompt you for three inputs after selecting MINIMUM: • Left Bound • Right Bound, • Guess.
ENTER 〉		Use the direction key to move the cursor to a point on the left of the minimum point. Then press ENTER. The chosen point and the minimum point should be connected via the graph. Alternatively you can enter an appropriate value for the *x*-coordinate of the Left Bound.
ENTER 〉		Upon enter it will ask for a Right Bound. Move the cursor to a point on the right of the minimum point. Press ENTER.

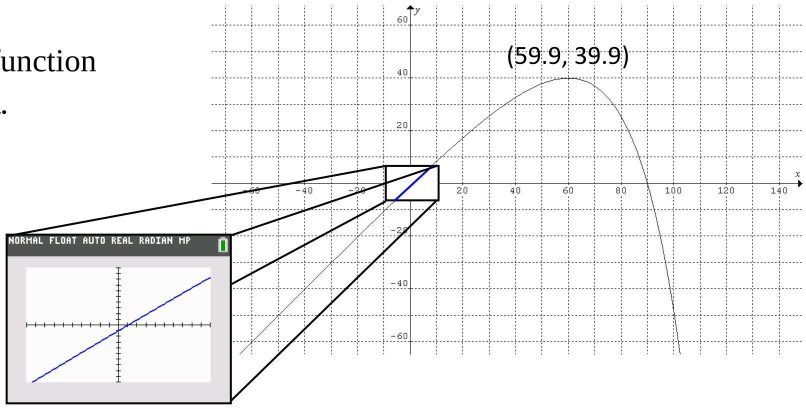
<table>
<tr><td>
</td><td>
</td><td>Now choose move the cursor to a point near the minimum point. Then press ENTER.</td></tr>
<tr><td>
</td><td>
</td><td>The coordinates of the minimum point is now shown on the screen:
(2.41, 4.83).</td></tr>
<tr><td></td><td>
</td><td>Repeat the same steps to find the coordinates of the MAXIMUM point:
$(-0.414,\ -0.828)$.
Since the range consists of all possible y-values that the graph takes, the range of f is:
$(-\infty,-0.828]\cup[4.83,\infty)$.</td></tr>
</table>

The most important thing when finding range is to adjust the window or zoom to ensure that you did not leave out any features of the graph. For example, there might be a turning point for large values of x. You might need to analyse the function to get a rough picture of the graph. A good start is to check the signs of the function when $|x|$ is large, i.e. when x tends to positive or negative infinity.

EXERCISE. Find the range for the function

$$f : x \mapsto x - e^{0.05x},\ x \in \mathbb{R}.$$

Answer: $(-\infty, 39.9]$.



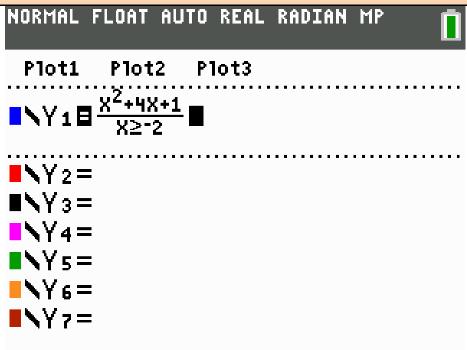
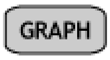
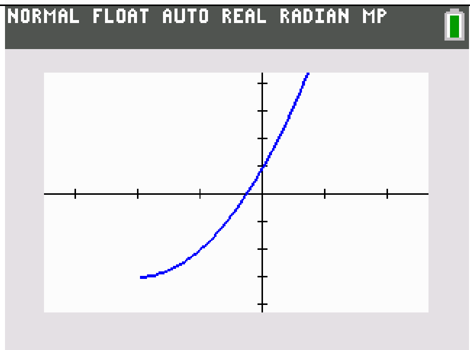
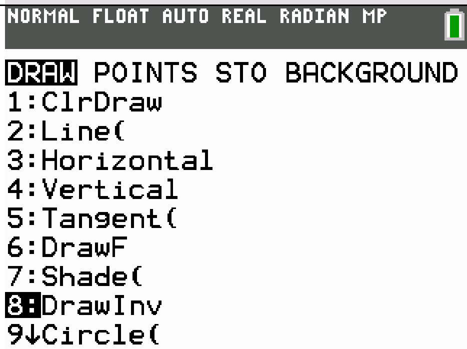
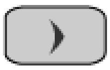
Learning Point:

"Partial truth can be a complete lie".

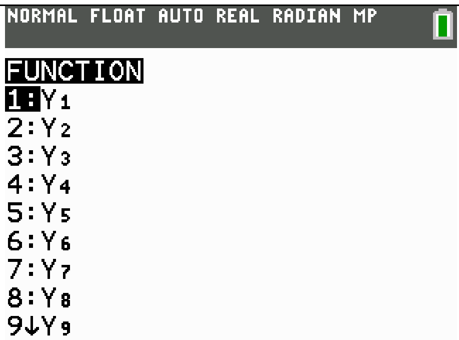
A3 Sketching Inverse Functions

There are a few ways to sketch the inverse function. The method introduced here is an 'artificial' method because it sort of draws on top of the graph. As you are not really plotting it, you will not be able to perform calculations on the inverse graph (such as finding the axial intercepts).

EXAMPLE. Sketch the function $f : x \mapsto x^2 + 4x + 1, x \geq -2$ and its inverse on the same diagram.

Key Press	Screen Shot	Steps/Notes/Descriptions
		Key in the function
		Adjust WINDOW to get a better picture of the graph.
 		Access the DRAW menu. Browse through the objects that you can draw. You will find 8: DrawInv Press ENTER and you will be prompted to enter a function.
		
 		Access the VARS > Y-VARS. Since we are in FUNC mode, select Function.

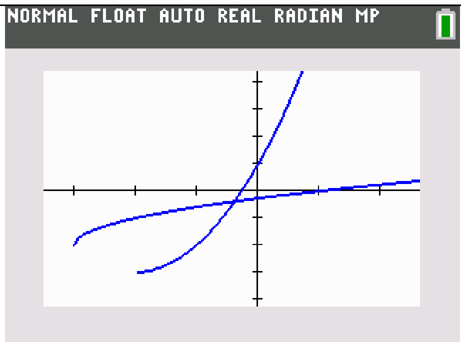
ENTER



Choose Y_1 since the function is entered into Y_1. Press ENTER.

ENTER

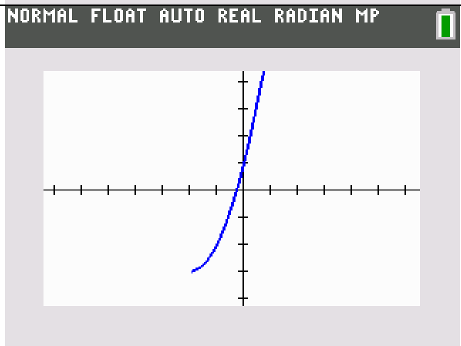
ENTER



Note that the graph that was newly drawn does not look like a reflection of the original graph about the line $y = x$.

The reason for this is because the scale is different for both axes. To rectify this problem, we use **ZSquare** under the ZOOM menu.

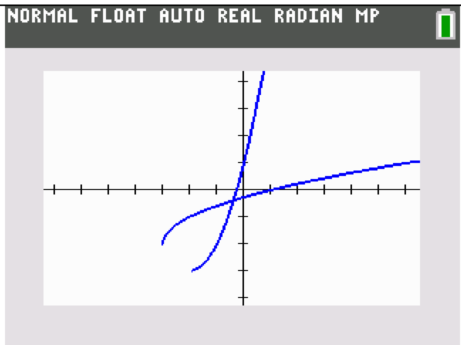
ZOOM 5



You should see that the axes markings are equally spaced for both axes. However, the graph for the inverse function is gone. This is exactly what is meant by “artificial”. The inverse is not plotted as a function but rather is drawn as a picture. Hence you should repeat DrawInv Y_1.

Since you have already entered the command previously, you can easily recall it by pressing 2^{nd} ENTER which gives ENTRY. You can repeated press ENTRY to recall the pass commands you have entered.

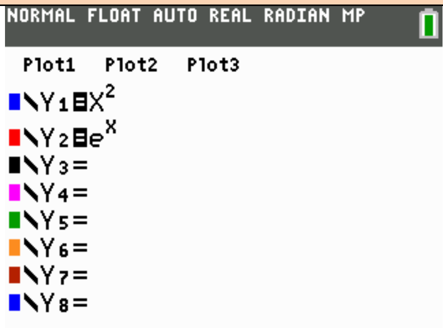
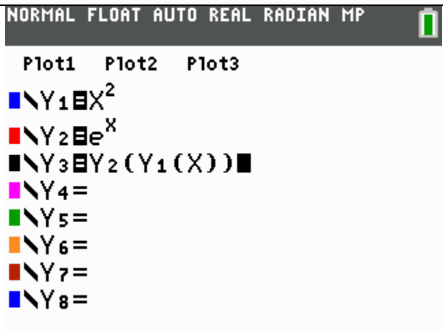
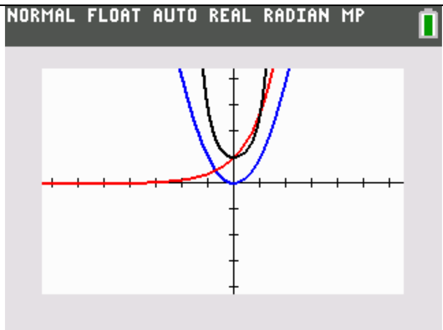
2nd ENTER
ENTER



Now you should get the correct graphs. Therefore if you want to draw the inverse function, do remember to use ZSquare so that the scale for both axes are the same.

A4 Sketching Composite Functions

EXAMPLE. It is given that $\mathrm{f}: x \mapsto x^2,\ x \in \mathbb{R}$ and $\mathrm{g}: x \mapsto \mathrm{e}^x,\ x \in \mathbb{R}$. Sketch $y = \mathrm{gf}(x)$ on GC.

Key Press	Screen Shot	Steps/Notes/Descriptions
		Key in the two functions into Y_1 and Y_2. It does not matter which one you key as f and g.
VARS,), ENTER, ⌄, ENTER, (, VARS,), ENTER, ENTER, (, X,T,θ,n,),)		Enter gf, the composite function into Y_3. Since the functions f and g are assigned to Y_2 and Y_1 respectively, we enter the composite function as $Y_2(Y_1(X))$.
GRAPH		

Remark: If you wish to see only the graph of the composite function gf, you can hide the other graphs from the input window. Move the cursor to the "=" signs and press ENTER. The "equal" signs will no longer be highlighted. The GC will not show any graph that does not have the "equal" sign highlighted.