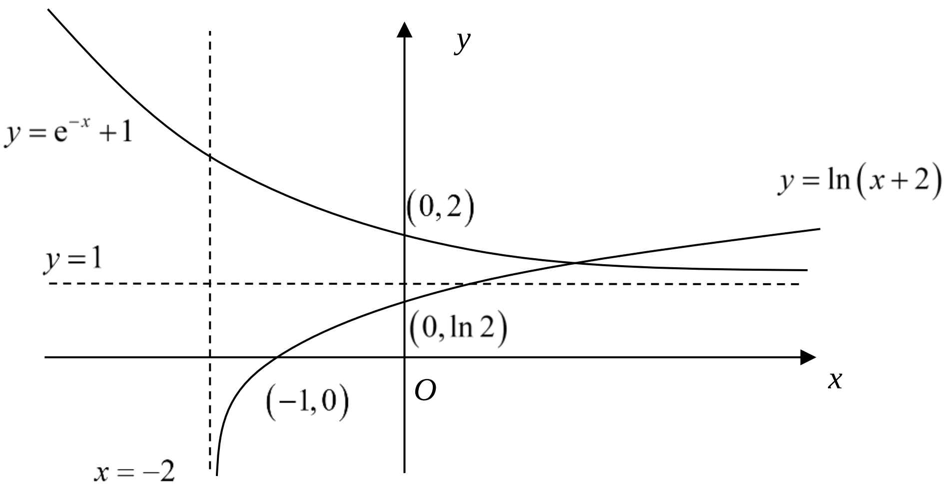
H2 Mathematics (9758)
Chapter 1 Graphing Techniques
Discussion Questions Solutions

<u>Level 1</u>

1 The curve C_1 and C_2 have equations $y=\mathrm{e}^{-x}+1$ and $y=\ln(x+2)$ respectively.

(i) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes.

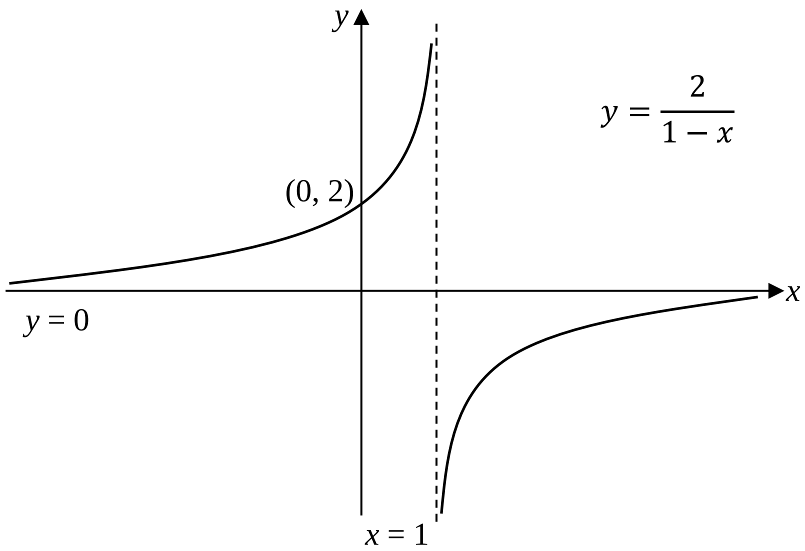
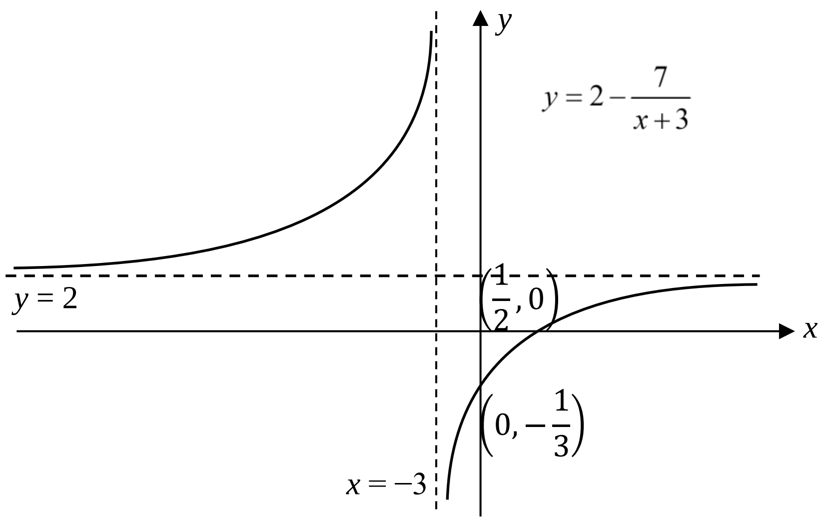
(ii) Use your calculator to determine the coordinates of the intersection point(s).

1	**Solution**
(i)	$y=\mathrm{e}^{-x}+1$ As $x\to\infty$, $y\to 1$ Therefore, $y=1$ is a horizontal asymptote. As $x\to-\infty$, $y\to\infty$ $y=\ln(x+2)$ As $x\to-2$, $y\to-\infty$ Therefore, $x=-2$ is a vertical asymptote. 
(ii)	Using GC, the intersection point is $(1.44, 1.24)$ to 3 s.f

2 Write down the equations of asymptotes of the following graphs:

(a) $y=\dfrac{2}{1-x}$ **(b)** $y=2-\dfrac{7}{x+3}$ **(c)** $y=x-1-\dfrac{1}{x-2}$

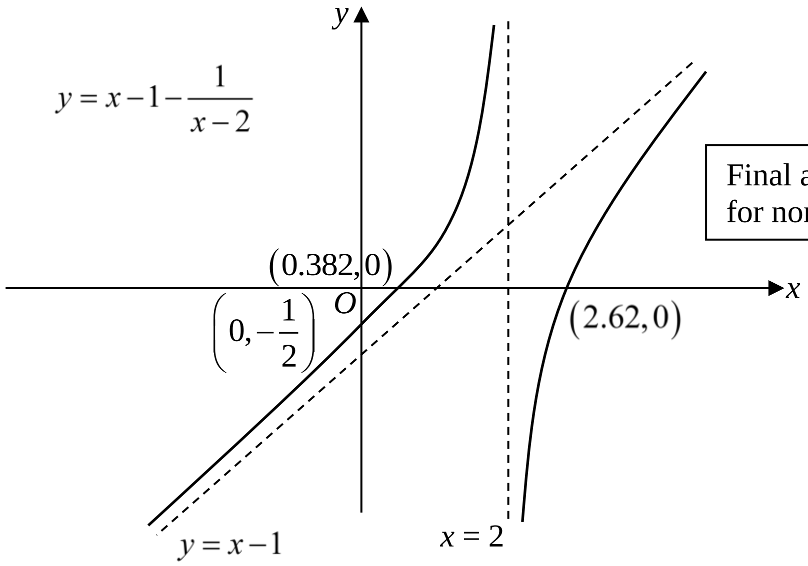
Hence, sketch the graphs on separate diagrams, indicating equations of asymptotes and coordinates of axial intercepts and turning points (if any).

2	**Solution**
(a)	Asymptotes: $x=1, y=0$ 
(b)	$y=2-\dfrac{7}{x+3}$ Asymptotes: $x=-3, y=2$ 

Recall

1. **Asymptotes** (if any)
2. **Shape** of the graph
3. **Maximum**/minimum turning points (if any)
4. **Intercepts** with the axes (if any)
5. **Labelling** of the graph and axes
6. **End** points (if any)

(c) Asymptotes: $x=2, y=x-1$



Final answer must be 3 s.f. for non-exact values.

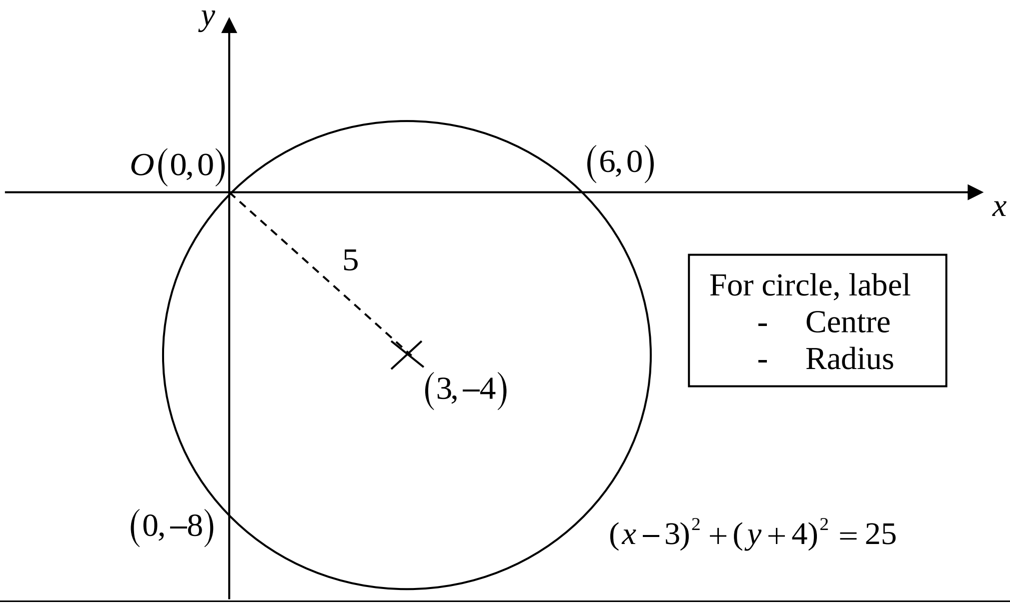
<u>Note</u>
For questions that require **<u>exact</u>** coordinates, students need to find axial intercepts by algebraic method (without GC). i.e. Put $y=0$ to find x-intercept and put $x=0$ to find y-intercept. For this question, the x-intercepts are $\left(\dfrac{3-\sqrt{5}}{2}, 0\right)$ and $\left(\dfrac{3+\sqrt{5}}{2}, 0\right)$.

3 Sketch, on separate diagrams, the following graphs:

(a) $(x-3)^2+(y+4)^2=25$ **(b)** $x^2+\dfrac{y^2}{4}=1$

(c) $\dfrac{y^2}{4}-\dfrac{x^2}{16}=1$ **(d)** $\dfrac{(x-2)^2}{4}-y^2=1$

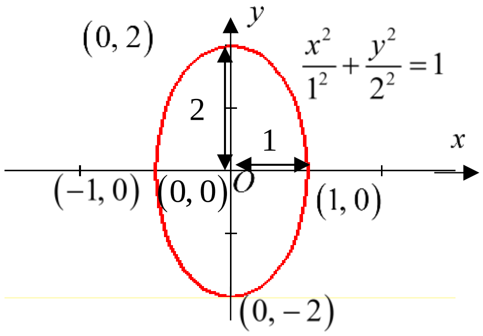
indicating clearly the main relevant features of the graph.

Q3	Solution
(a)	

For circle, label
- Centre
- Radius

(b)

$$x^2+\frac{y^2}{4}=1 \quad \Rightarrow \quad \frac{x^2}{1^2}+\frac{y^2}{2^2}=1$$

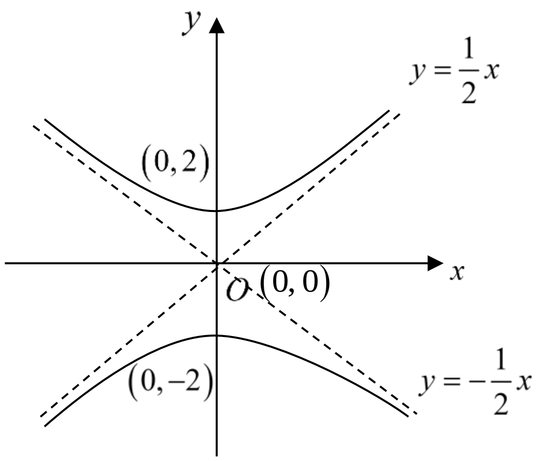


For ellipse, label
- Centre
- Length of both horizontal/vertical semi-major/semi-minor axis

(c)

$$\frac{y^2}{4}-\frac{x^2}{16}=1$$

$$\Rightarrow \frac{y^2}{2^2}-\frac{x^2}{4^2}=1$$



Oblique asymptotes:

$$\frac{y^2}{2^2}-\frac{x^2}{4^2}=0$$

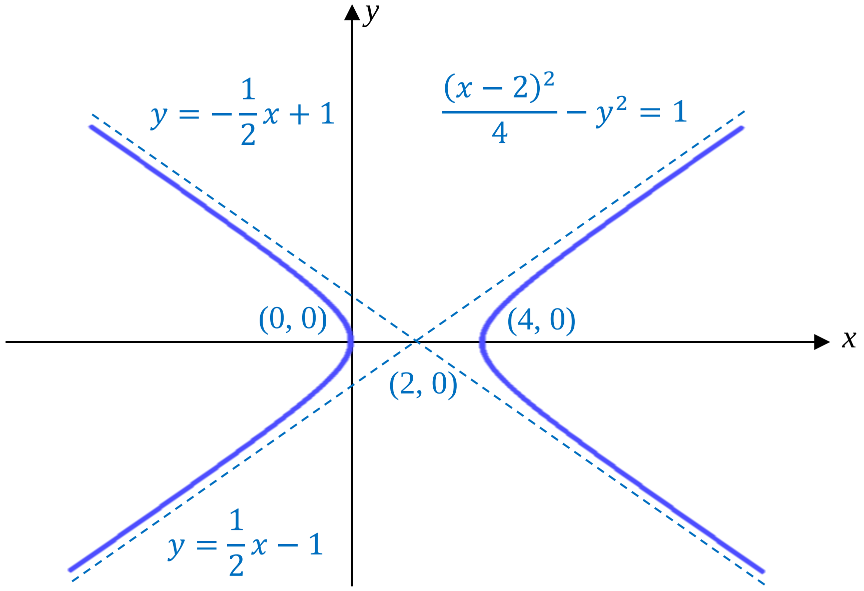
$$\frac{y^2}{2^2}=\frac{x^2}{4^2}$$

$$y=\pm\frac{1}{2}x$$

$y=\frac{1}{2}x$ and $y=-\frac{1}{2}x$

To find equation of oblique asymptotes of hyperbola,
1) Change RHS from "1" to "0" (From standard form)
2) Make y the subject
3) Label correctly on diagram by looking at their gradient

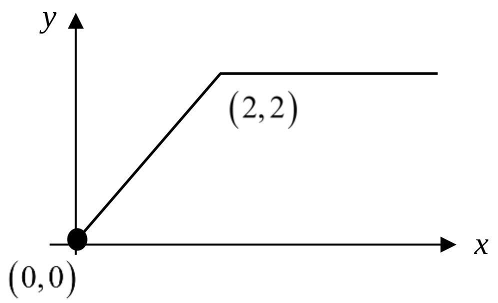
For hyperbola, label
- Centre of hyperbola (Intersection of oblique asymptotes)
- Coordinates of vertices
- Equation of oblique asymptotes

(d)	To obtain the equations of the asymptotes: $\frac{(x-2)^2}{4}-y^2=0$ $y^2=\frac{(x-2)^2}{4}$ $y=\pm\frac{(x-2)}{2}$ $\Rightarrow y=\frac{1}{2}x-1$ or $y=-\frac{1}{2}x+1$ 

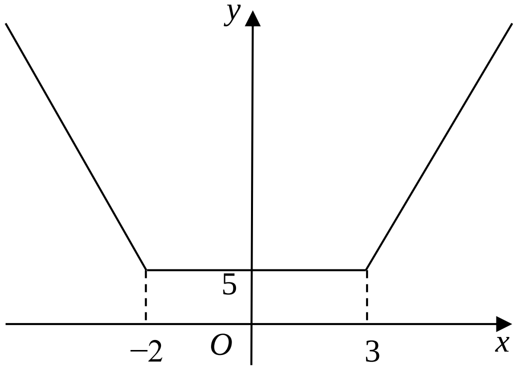
4 Sketch, on separate diagrams, the following graphs:

(a) $y=\begin{cases} x, & \text{if} \quad 0\le x\le 2, \\ 2, & \text{if} \quad x>2. \end{cases}$

(b) $y=\lvert x-3\rvert+\lvert x+2\rvert$

4	**Solution**
(a)	

(b)



Exploration for higher order thinking:

For $|x-3|=\begin{cases} x-3 & ,x\geq 3 \\ -(x-3) & ,x<3 \end{cases}$

For $|x+2|=\begin{cases} x+2 & ,x\geq -2 \\ -(x+2) & ,x<-2 \end{cases}$

$\therefore$ **When** $x<-2$**,**

$$\begin{aligned}|x-3|+|x+2|&=[-(x-3)]+[-(x+2)]\\&=1-2x\end{aligned}$$

$\therefore$ **When** $-2\leq x\leq 3$**,**

$$\begin{aligned}|x-3|+|x+2|&=[-(x-3)]+(x+2)\\&=5\end{aligned}$$

$\therefore$ **When** $x>3$**,**

$$\begin{aligned}|x-3|+|x+2|&=(x-3)+(x+2)\\&=2x-1\end{aligned}$$

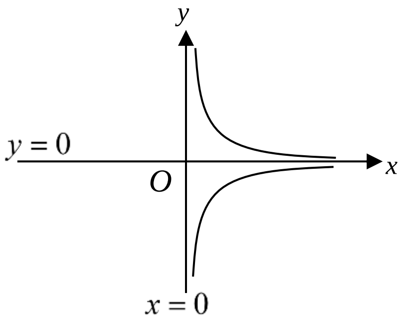
Piecewise graph:

$y=|x-3|+|x+2|$

$$\Rightarrow y=\begin{cases} 1-2x & ,x<-2 \\ 5 & ,-2\leq x\leq 3 \\ 2x-1 & ,x>3 \end{cases}$$

5 **(i)** Sketch the curve C defined by the parametric equations $x=\dfrac{1}{t^2}, y=2t$ where t is a non-zero real parameter.

(ii) Find the Cartesian equation of the curve C.

5	**Suggested Solutions**
(i)	$x=\dfrac{1}{t^2}\ ,\ \ y=2t$  **Note:** When drawing the parametric equations in the GC, go to $\boxed{\text{window}}$ and adjust the **Tmin** and **Tmax** In this case, since "*t* is a non-zero real parameter", you have to include negative values, hence you can set Tmin = −10 and Tmax = 10 **<u>Note:</u>** Students are advised to sketch the graph in parametric mode rather than convert parametric to Cartesian before drawing the graph which can be tedious or not possible in some cases.
(ii)	$x=\dfrac{1}{t^2}\ ,\ \ y=2t$ $t=\dfrac{y}{2}$ $x=\dfrac{1}{t^2}=\dfrac{1}{\left(\dfrac{y}{2}\right)^2}=\dfrac{4}{y^2}$ Cartesian equation of C is $xy^2=4$

Level 2

6 H2 Specimen Paper 2006/1/9 Modified

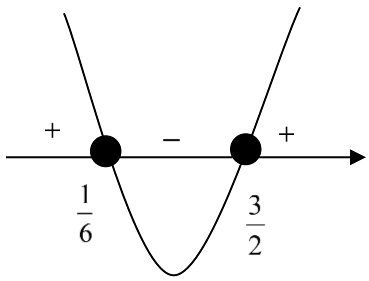
Consider the curve $y = \dfrac{3x-6}{x(x+6)}$.

(i) State the coordinates of any points of intersection with the axes.

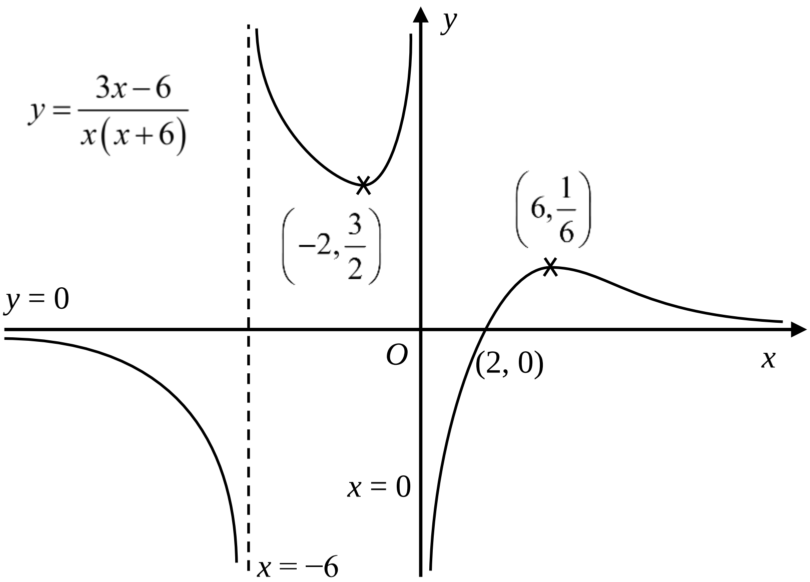
(ii) State the equations of the asymptotes.

(iii) Prove, using an algebraic method, that $\dfrac{3x-6}{x(x+6)}$ cannot lie between two certain values (to be determined).

(iv) Draw a sketch of the curve, $y = \dfrac{3x-6}{x(x+6)}$, indicating clearly the main relevant features of the curve.

6	Suggested Solutions
(i)	$(2,0)$
(ii)	Asymptotes: $x=0, x=-6$, $y=0$
(iii)	We first find the region in which the graph can lie i.e. it would have **intersection(s)** with a horizontal line $y=k, k\in\mathbb{R}$ drawn in that region. Hence, the equation $k=\dfrac{3x-6}{x(x+6)}$ will have **real roots**. Then we take the complement. $y=\dfrac{3x-6}{x(x+6)}$ Let $y=k$, $k=\dfrac{3x-6}{x(x+6)}$ $kx^2+(6k-3)x+6=0$ For a quadratic equation to have real roots, discriminant $\geq$ 0  $(6k-3)^2-4k(6)\geq 0$ $36k^2-60k+9\geq 0$ $(6k-1)(2k-3)\geq 0$ $k\leq\dfrac{1}{6}$ or $k\geq\dfrac{3}{2}$ Therefore, $\dfrac{3x-6}{x(x+6)}$ cannot lie between $\dfrac{1}{6}$ and $\dfrac{3}{2}$.

(iv)



<u>Note:</u>

- Students need to make connections between parts **(i) – (iii)** and **(iv)** to get the correct shape on the right-hand side.
- In part **(i)**, there is an intercept at (2,0) so the graph must cut the x-axis at (2,0).
- In part **(ii)**, there is a horizontal asymptote $y = 0$ so the graph must approach the x-axis at the extreme ends.
- From **(iii)**, there is no graph between $y = \dfrac{1}{6}$ and $y = \dfrac{3}{2}$. So, there should be some graph in the interval $0 \le y \le \dfrac{1}{6}$. In fact, there is a max turning point with y-coordinate $\dfrac{1}{6}$.

Students may change the window settings to see the shape on the right side:

Xmin = 0
Xmax = 20
Ymin = 0
Ymax = 1

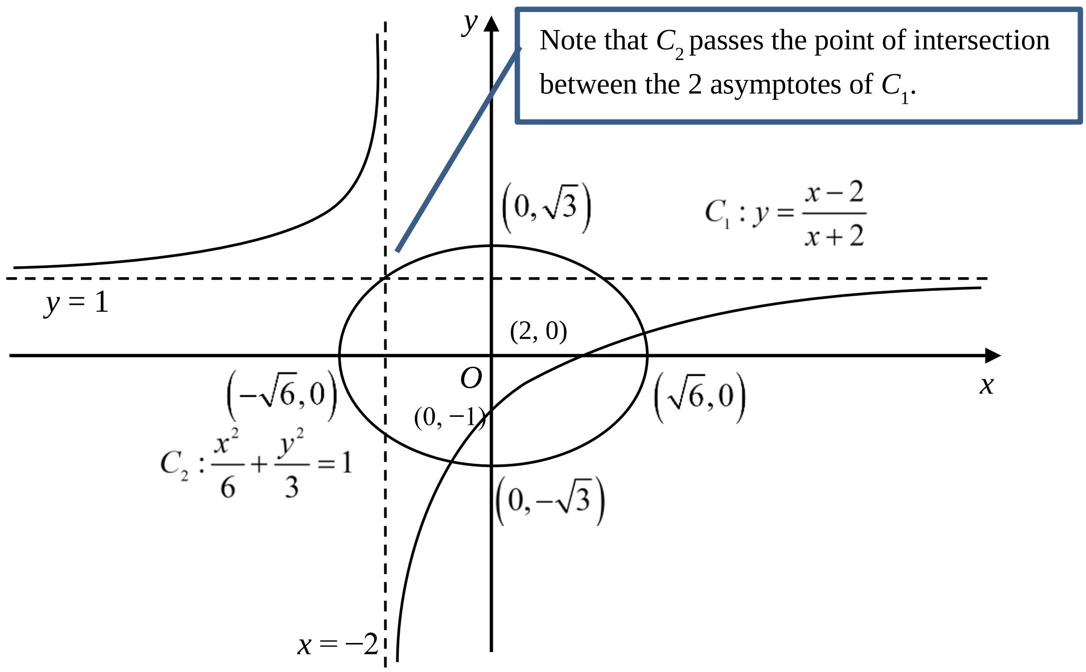
7 **N2009/1/6**

The curve C_1 has equation $y=\dfrac{x-2}{x+2}$. The curve C_2 has equation $\dfrac{x^2}{6}+\dfrac{y^2}{3}=1$.

(i) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

(ii) Show algebraically that the x-coordinates of the points of intersection of C_1 and C_2 satisfy the equation $2(x-2)^2=(x+2)^2(6-x^2)$. [2]

(iii) Use your calculator to find these x-coordinates. [2]

7	Suggested Solutions
(i)	$C_1: y=\dfrac{x-2}{x+2}=1-\dfrac{4}{x+2}$ Asymptotes: $y=1,\ x=-2$ $x=0, y=-1$ $y=0, x=2$ Axial intercepts: (0, –1), (2, 0). C_2 is an ellipse with centre (0,0) and x-intercepts $\left(\pm\sqrt{6},0\right)$ and y-intercepts $\left(0,\pm\sqrt{3}\right)$ Note: **Exact coordinates** are required. 

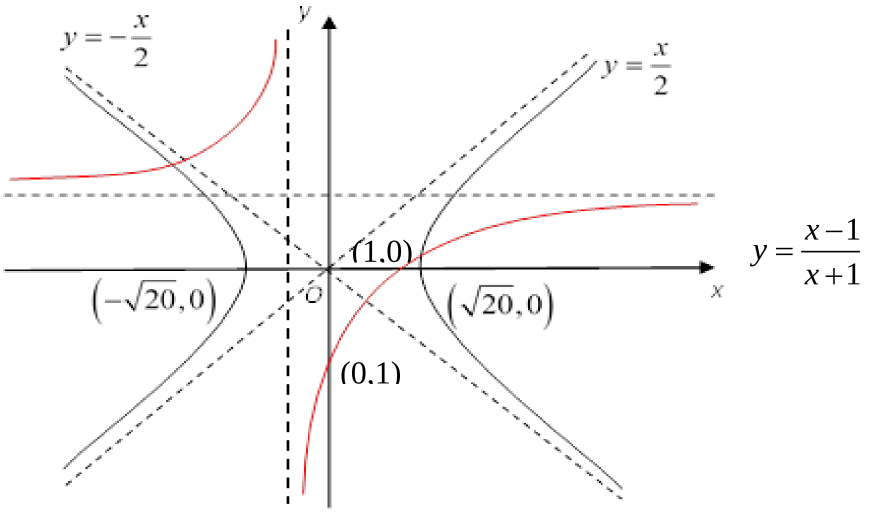
(ii)	$C_1 : y = \frac{x-2}{x+2} \Rightarrow y^2 = \left(\frac{x-2}{x+2}\right)^2$ $C_2 : \frac{x^2}{6} + \frac{y^2}{3} = 1 \Rightarrow y^2 = \frac{1}{2}\left(6 - x^2\right)$ The point of intersection between C_1 and C_2 is $\left(\frac{x-2}{x+2}\right)^2 = \frac{1}{2}\left(6-x^2\right)$ $2(x-2)^2 = (x+2)^2(6-x^2)$ Note: "Show algebraically" so students should show ALL steps clearly.
(iii)	<u>Method 1</u>: By sketching the two graphs using GC and finding the intersection points, the required x- coordinates are $x = –0.515$ and $x = 2.45$. <u>Method 2</u>: By sketching the graph of $y = 2\left(x-2\right)^2 - \left(x+2\right)^2\left(6-x^2\right)$ using GC and finding the roots of the equation, the x coordinates are $x = –0.515$ and $x = 2.45$. **<u>Note:</u>** Errors were usually seen when the equation given was wrongly multiplied out to give a quartic equation in x resulting in 4 answers.

8 2013/NJC Promo/12(b)(modified)

The curve C_3 has equation $y=\frac{x-1}{x+1}$. The curve C_4 has equation $\frac{x^2}{20}-\frac{y^2}{5}=1$.

Sketch C_3 and C_4 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

Hence find the number of solutions to the equation $x^2-\frac{4(x-1)^2}{(x+1)^2}=20$. [2]

8	**Suggested Solutions**
	 Substitute $y=\frac{x-1}{x+1}$ into $\frac{x^2}{20}-\frac{y^2}{5}=1$. $\frac{x^2}{20}-\frac{\left(\frac{x-1}{x+1}\right)^2}{5}=1\Rightarrow x^2-4\left(\frac{x-1}{x+1}\right)^2=20$ Number of solutions = 2 **<u>Note:</u>** "**Hence**, find the number of solutions to the equation $x^2-\frac{4(x-1)^2}{(x+1)^2}=20$." So students should think of **how to use the previous part** i.e. the sketch. The technique is to think of how to manipulate the expression to involve the two graphs that have been sketched and conclude that we are finding intersection points between the two graphs.

9 2018/MI/Promo/8

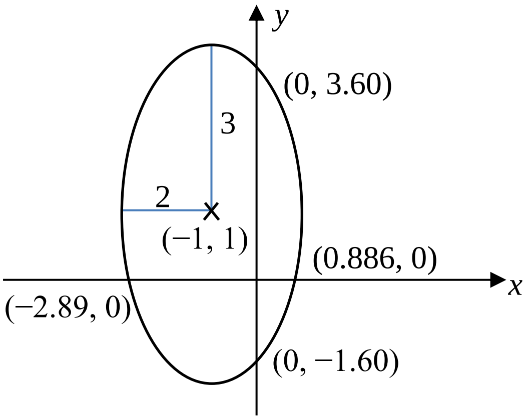
The curve C has equation $9x^2+18x+4y^2-8y=23$.

(i) By completing the square, show that the equation of C can be expressed as $\frac{(x+1)^2}{2^2}+\frac{(y-1)^2}{3^2}=1$. [2]

(ii) Sketch C, stating the coordinates of any points of intersection with the axes. [3]

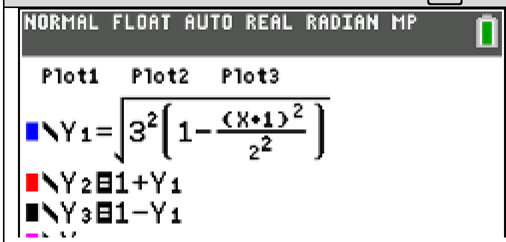
9	Solution
(i)	$9x^2+18x+4y^2-8y=23$ $\Rightarrow 9\left(x^2+2x\right)+4\left(y^2-2y\right)=23$ $\Rightarrow 9\left[(x+1)^2-1\right]+4\left[(y-1)^2-1\right]=23$ $\Rightarrow 9(x+1)^2-9+4(y-1)^2-4=23$ $\Rightarrow \frac{(x+1)^2}{2^2}+\frac{(y-1)^2}{3^2}=1$ Tip: When completing square, you can avoid mistakes by factoring out the coefficient of the squared term first.

(ii)



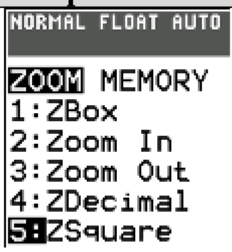
For this question, need to label the intercepts as coordinates since the question asked for them to be labelled.

GC Skills: To draw Conics in $\boxed{\text{Y=}}$ to find intercepts and intersections

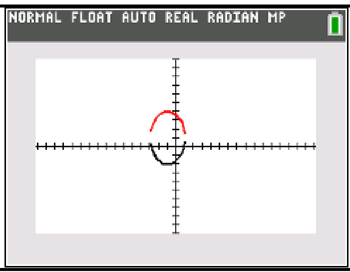


1) Make y the subject and key equation into $\boxed{\text{Y=}}$ on your GC

Note: There is ± when you square root both sides, you can key in 2 separate lines



2) In $\boxed{\text{ZOOM}}$, select **5: ZSquare** to make the x- and y-axis the same scale (so it looks like an ellipse)

10 Sketch, on the **same** diagram, $x^2+y^2-4y=0$ and $x^2-4(y-2)^2=4$, giving the equations of asymptotes and other relevant features. [5]

10	**Suggested Solutions**

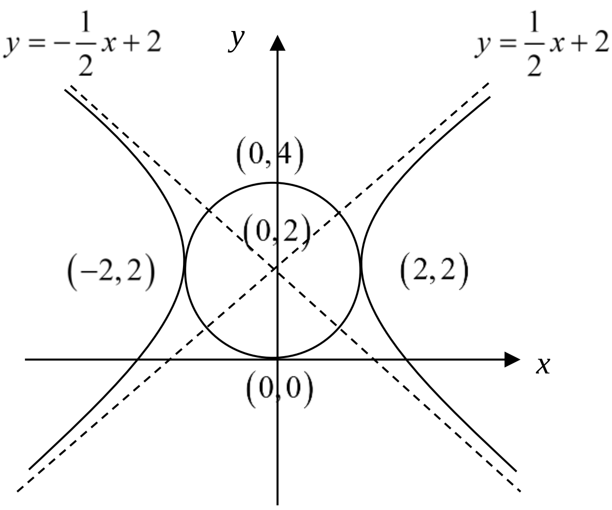
$$x^2+y^2-4y=0$$
$$x^2+(y-2)^2-4=0$$
$$x^2+(y-2)^2=2^2$$

Circle Centre: $(0,2)$ and radius 2

$$x^2-4(y-2)^2=4$$
$$\frac{x^2}{2^2}-\frac{(y-2)^2}{1^2}=1$$

Hyperbola:
Asymptotes:

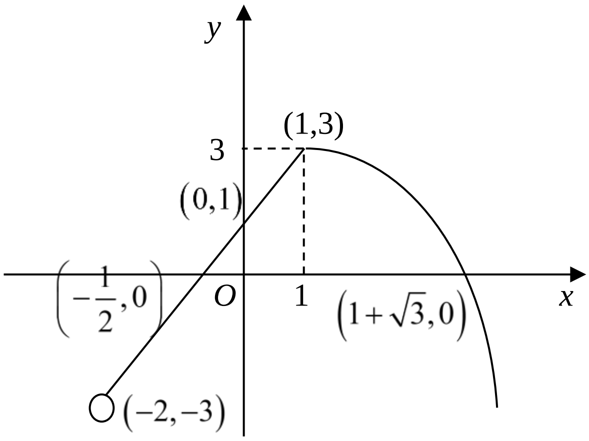
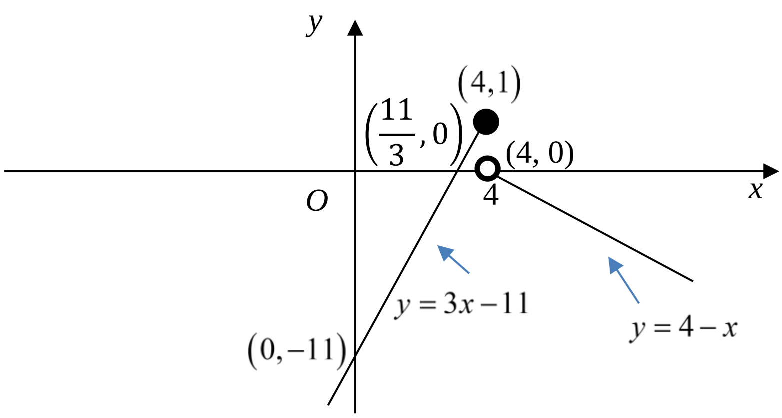
$$\frac{x^2}{2^2}-\frac{(y-2)^2}{1^2}=0$$
$$(y-2)^2=\frac{x^2}{2^2}$$
$$y=2\pm\frac{x}{2}$$



11 Sketch the graph of $y = \mathrm{f}(x)$ where

(a) $\mathrm{f}(x) = \begin{cases} 2x+1, & -2 < x \le 1, \\ -x^2+2x+2, & x > 1. \end{cases}$

(b) $\mathrm{f}(x) = \begin{cases} 3x-11, & \text{for } x \in \mathbb{R}, x \le 4, \\ 4-x, & \text{for } x \in \mathbb{R}, x > 4. \end{cases}$

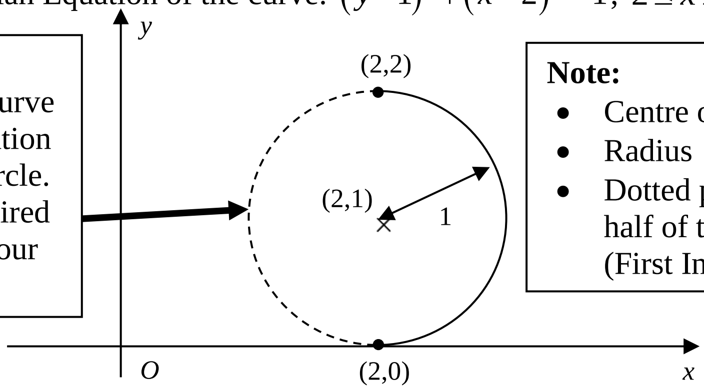
11	**Suggested Solutions**
(a)	 Note: The quadratic curve $y = -x^2 + 2x + 2$ $= 3 - (x-1)^2$ has a turning point at (1, 3), therefore the curved part is half a quadratic curve with max turning point (1, 3)
(b)	**Note:** - Draw the line with gradient +3 for $x \le 4$ and another line with gradient -1 for $x > 4$ - When $x = 4$, $y = 1$. Hence, we use a closed circle for (4,1) and an open circle for (4,0). This is also called a disjoint piecewise function. 

12 **(a)** A curve C has parametric equations $x = 2 + \cos\theta,\ \ y = 1 + \sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

(i) State the range of values of x and y.

(ii) Find a Cartesian equation of curve C. Hence sketch C.

(b) Find the Cartesian equation of the curve whose parametric equations are $x = 2\tan\theta,\ \ y = 3\cos\theta$.

12	**Suggested Solutions**
(a) (i)	Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\quad 0 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ $2 \le x \le 3$ and $0 \le y \le 2$
(ii)	Since $\sin^2\theta + \cos^2\theta = 1$, $(y-1)^2 + (x-2)^2 = 1$ Cartesian Equation of the curve: $(y-1)^2 + (x-2)^2 = 1,\ 2 \le x \le 3$ **Note:** The dotted curve is for illustration that it is a circle. It is not required to show in your solution.  **Note:** • Centre of circle • Radius • Dotted part for excluded half of the circle (First Intro to dotted curve) **<u>Note:</u>** While it is recommended, in general, to sketch the curve in parametric mode if it is given in parametric form, the question has instructed you to convert to Cartesian form and "hence" sketch the curve so if you recognise that it is a circle equation, you may sketch a semi-circle following the restriction in part (i).
(b)	$x = 2\tan\theta \quad \Rightarrow \quad \tan\theta = \frac{x}{2}$ $y = 3\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{y}{3} \quad \Rightarrow \quad \sec\theta = \frac{3}{y}$ $\tan^2\theta + 1 = \sec^2\theta$ $\frac{x^2}{4} + 1 = \frac{9}{y^2}$ Tip: Use trigonometric identities

Level 3

13 **2018/DHS Promo/Q4 (Modified)**

The curve C has equation

$$y=\frac{x^2+2x+1}{x-p},\ x\neq p,$$

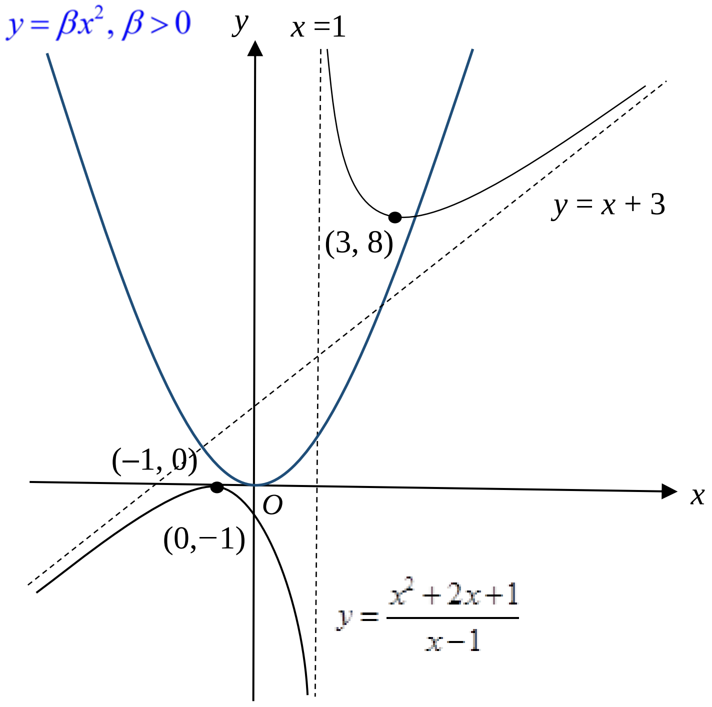
where p is a real constant. It is given that the line $y=x+3$ is an asymptote of C. Show that $p=1$. [2]

(i) Sketch C. [3]

(ii) By adding another graph, deduce that for all positive β, the equation

$$(\beta+1)x^2+2x+1=\beta x^3$$

has exactly one real root. [2]

13	Suggested Solutions
	$y=\dfrac{x^2+2x+1}{x-p},\ x\neq p,$ Given that $y=x+3$ is an asymptote of C, $\dfrac{x^2+2x+1}{x-p}=x+3+\dfrac{k}{x-p}$ $=\dfrac{(x+3)(x-p)+k}{x-p}$ $=\dfrac{x^2+(3-p)x-3p+k}{x-p}$ Comparing coefficient of x: $2=3-p\Rightarrow p=1$ (shown) Since denominator is linear "$x-p$", numerator must be constant for the fraction to be rational **Note:** Long division is difficult to use here due to unknowns
(i)	

(ii)	$(\beta+1)x^2+2x+1=\beta x^3$ $x^2+2x+1=\beta x^3-\beta x^2$ $x^2+2x+1=\beta x^2(x-1)$ $\dfrac{x^2+2x+1}{x-1}=\beta x^2$ **Learning point 1:** Manipulate the equation till 1 side of it has the expression $\dfrac{x^2+2x+1}{x-1}$ which is the graph drawn earlier When the graph of $y=\beta x^2$, $\beta>0$ is inserted, there is only **one intersection point**, thus $(\beta+1)x^2+2x+1=\beta x^3$ **<u>has exactly one real root.</u>** **Learning point 2:** To explain why the equation has exactly one real root, show that the graph inserted intersects the previous graph at exactly one point

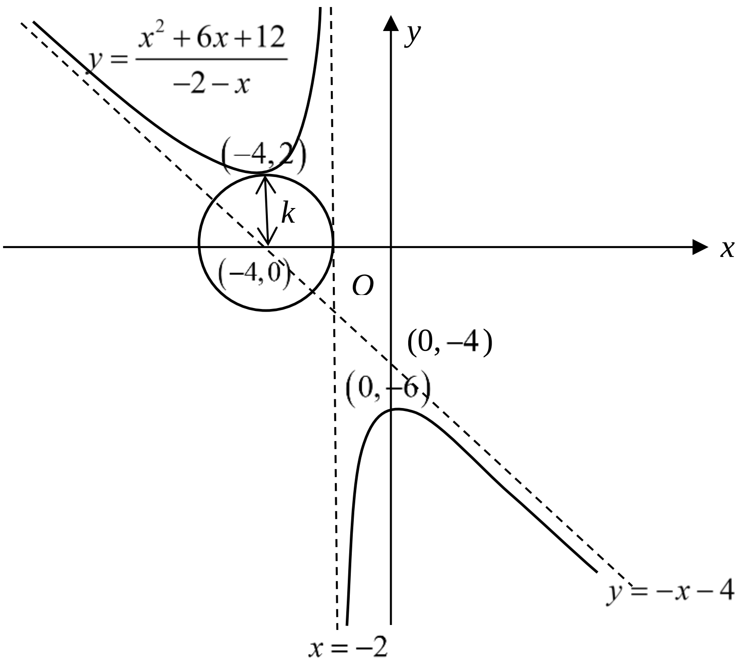
14 2013/PJC Prelim/2/5

The curve C has equation $y=\dfrac{x^2+ax+b}{c-x}$. The vertical asymptote of C is $x=-2$, and the coordinates of the turning points are $(-4,2)$ and $(0,-6)$.

(i) Find the values of a, b and c. [3]

(ii) Sketch C, stating the equations of the asymptotes. [2]

(iii) By drawing an appropriate graph on the sketch of C, find the range of values of k $(k>0)$ such that the equation $(x+4)^2+\left(\dfrac{x^2+ax+b}{c-x}\right)^2=k^2$ has no real roots. [2]

14	Suggested Solutions
(i)	Given $x=-2$ is a vertical asymptote, $c=-2$ C passes through $(-4,2)$ and $(0,-6)$ $2=\dfrac{(-4)^2+a(-4)+b}{-2-(-4)}$ $-4a+b=-12 \quad -----(1)$ $-6=\dfrac{(0)^2+a(0)+b}{-2-(0)}$ $b=12$ Substitute $b=12$ into (1), $a=6$ $\therefore\ a=6,\ b=12,\ c=-2$ Note: Avoid long division with unknowns
(ii)	

(iii)	$(x+4)^2+\left(\frac{x^2+ax+b}{c-x}\right)^2=k^2$ $[x-(-4)]^2+(y)^2=k^2$ Graph to be inserted is a circle with radius k units, centred at $(-4,0)$. $\therefore$ For the equation to have no real roots, $0<k<2$ **<u>Note:</u>** Students should think of **how to use the previous part** i.e. the sketch. The technique is to think of how to manipulate the expression to involve two graphs one of which has been sketched and the other to be inserted and conclude that we are looking at intersection points between the two graphs.

15 2013/NJC Promo/12(a)(modified)

The curve C_1 has parametric equations

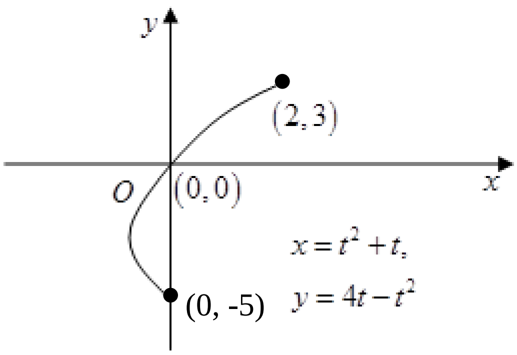
$$x=t^2+t, \quad y=4t-t^2, \quad -1\le t\le 1.$$

(i) Sketch C_1, labelling the coordinates of the end-points and the axial intercepts (if any) of this curve. [2]

(ii) The curve C_2 is defined parametrically by the equations

$$x=t^2+t, \quad y=4t-t^2, \quad t\in\mathbb{R}.$$

Find a Cartesian equation of C_2. [2]

Qn 15	Suggested Solutions
(i)	 Remember to find coordinates of endpoints and use open/closed circles to indicate inclusion/exclusion of points. Do this by looking at range of t and GC "$-1\le t\le 1$".
(ii)	$x+y=5t \Rightarrow t=\frac{x+y}{5}$ Substitute $t=\frac{x+y}{5}$ into $x=t^2+t$ $\Rightarrow x=\left(\frac{x+y}{5}\right)^2+\left(\frac{x+y}{5}\right)$ $\Rightarrow 25x=(x+y)^2+5(x+y)$ $\Rightarrow (x+y)^2=20x-5y$ $\Rightarrow (x+y)^2=5(4x-y)$ Alternate form: $y=\pm5\sqrt{x+\frac{1}{4}}-\frac{5}{2}-x$; $x=10\pm5\sqrt{4-y}-y$ Think of possible operations to eliminate t: i.e. $+-\times\div$ No need to make y the subject unless stated in the question