1 Find the exact equation of the tangent to the curve

$$\ln y = (11-5x)^2$$

at the point where $x = 2$. [5]

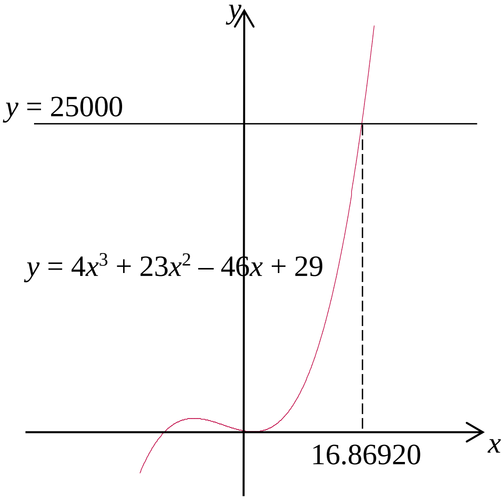
Solutions
$\ln y = (11-5x)^2$ Differentiating with respect to x: $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2(11-5x)(-5) = -10(11-5x)$ At $x = 2,\ y = \mathrm{e},\ \frac{\mathrm{d}y}{\mathrm{d}x} = -10\mathrm{e}$ Equation of the tangent at $x = 2$ $y - \mathrm{e} = -10\mathrm{e}(x-2)$ $y = -10\mathrm{e}x + 21\mathrm{e}$

2 The first four terms of a sequence are given by $u_1 = 10$, $u_2 = 61$, $u_3 = 206$ and $u_4 = 469$. It is given that u_n is a cubic polynomial.

(a) Find u_n in terms of n. [3]

Solutions
Since u_n is a cubic polynomial, $u_n = an^3 + bn^2 + cn + d$ for some $a, b, c, d \in \mathbb{R}$. $u_1 = a+b+c+d = 10 \quad ---(1)$ $u_2 = 8a+4b+2c+d = 61 \quad ---(2)$ $u_3 = 27a+9b+3c+d = 206 \quad ---(3)$ $u_4 = 64a+16b+4c+d = 469 \quad ---(4)$ Using GC to solve (1), (2), (3) and (4), $a = 4$, $b = 23$, $c = -46$, $d = 29$ $\therefore u_n = 4n^3 + 23n^2 - 46n + 29$

(b) Find the range of values of n for which u_n is greater than 25 000. [2]

Solutions
$u_n = 4n^3 + 23n^2 - 46n + 29 > 25\ 000$    From GC, $n > 16.9$ i.e. $n \geq 17$

3 Vectors **a** and **b** are such that **a**·**b** = –1. It is also given that (**a**×**b** + **a**) is perpendicular to (**a**×**b** + **b**).

(a) Show that |**a**×**b**| = 1. [3]

Solutions
Given that $(\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a})$ is perpendicular to $(\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b})$ $\Rightarrow (\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a}).(\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b}) = 0$ $(\underset{\sim}{a} \times \underset{\sim}{b}).(\underset{\sim}{a} \times \underset{\sim}{b}) + (\underset{\sim}{a} \times \underset{\sim}{b}).\underset{\sim}{b} + \underset{\sim}{a}.(\underset{\sim}{a} \times \underset{\sim}{b}) + \underset{\sim}{a}.\underset{\sim}{b} = 0$ $\lvert \underset{\sim}{a} \times \underset{\sim}{b} \rvert^2 + 0 + 0 + \underset{\sim}{a}.\underset{\sim}{b} = 0$ [since $\underset{\sim}{a} \times \underset{\sim}{b}$ is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$] $\lvert \underset{\sim}{a} \times \underset{\sim}{b} \rvert^2 + 0 + 0 - 1 = 0$ $\lvert \underset{\sim}{a} \times \underset{\sim}{b} \rvert^2 = 1$ $\lvert \underset{\sim}{a} \times \underset{\sim}{b} \rvert = 1$ since $\lvert \underset{\sim}{a} \times \underset{\sim}{b} \rvert \geq 0$ (shown)

(b) Hence find the angle between the direction of **a** and the direction of **b**. [3]

Solutions
Let the angle between the direction of $\underset{\sim}{a}$ and the direction of **b** be θ. $\lvert \underset{\sim}{a} \times \underset{\sim}{b} \rvert = \lvert \underset{\sim}{a} \rvert \lvert \underset{\sim}{b} \rvert \sin\theta = 1 \quad ---(1)$ $\underset{\sim}{a} \cdot \underset{\sim}{b} = \lvert \underset{\sim}{a} \rvert \lvert \underset{\sim}{b} \rvert \cos\theta = -1 \quad ---(2)$ $\frac{(1)}{(2)}: \quad \tan\theta = -1$ Since $\tan\theta < 0 \quad \Rightarrow \theta$ is obtuse and lies in the second quadrant. $\therefore \theta = 180° - \tan^{-1}\lvert -1 \rvert = 135°$

4 **(a)** Find $\int \cos px \cos qx \, \mathrm{d}x$, where p and q are constants such that $p \neq q$ and $p \neq -q$. [2]

Solutions
$\int \cos px \cos qx \, \mathrm{d}x$ $= \frac{1}{2} \int \cos(p+q)x + \cos(p-q)x \, \mathrm{d}x$ $= \frac{1}{2}\left[\frac{\sin(p+q)x}{p+q} + \frac{\sin(p-q)x}{p-q}\right] + C,$ where C is an arbitrary constant

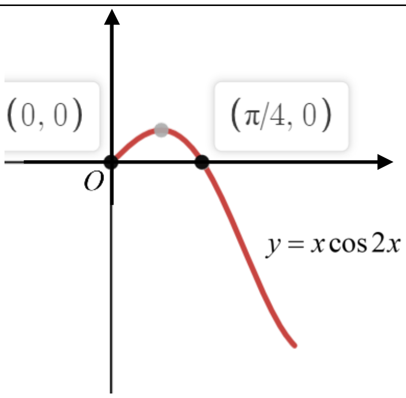
(b) Given that $n \neq 0$, show that $\int x \cos nx \, \mathrm{d}x = \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} + c$, where c is an arbitrary constant.
[3]

Solutions
$\int x \cos nx \, \mathrm{d}x = x\left(\frac{\sin nx}{n}\right) - \int \frac{\sin nx}{n} \, \mathrm{d}x$ $= \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} + c \qquad$ (shown) where c is an arbitrary constant

(c) Using the result in part **(b)** show that, for all positive integers n, the value of $\int_0^{\pi} x\cos nx\,\mathrm{d}x$ can be expressed as $\frac{k}{n^2}$, where the possible value(s) of k are to be determined. [2]

Solutions
$\int_0^{\pi} x\cos nx\,\mathrm{d}x = \left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$ $= \left(\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)$ $= 0 + \frac{\cos n\pi}{n^2} - \left(\frac{1}{n^2}\right)$ $[\because \sin n\pi = 0$ for all positive integers $n]$ $= \frac{(-1)^n}{n^2} - \frac{1}{n^2}$ $\left[\because \cos n\pi = (-1)^n \text{ for all positive integers } n\right]$ $= \begin{cases} \frac{-2}{n^2} & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$ $\therefore k = 0 \text{ or } k = -2$

(d) Using the result in part **(b)** find the exact value of $\int_0^{\pi} \lvert x\cos 2x \rvert\,\mathrm{d}x$. [3]

Solutions
   From the graph (above),

$$|x\cos 2x| = \begin{cases} x\cos 2x, & 0 \le x \le \frac{\pi}{4} \\ -x\cos 2x, & \frac{\pi}{4} < x \le \frac{\pi}{2} \end{cases}$$

$$\int_0^{\frac{\pi}{2}} |x\cos 2x|\,dx$$
$$= \int_0^{\frac{\pi}{4}} x\cos 2x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -x\cos 2x\,dx$$
$$= \left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{\frac{\pi}{4}} - \left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= \left[\frac{\pi}{8} + 0 - 0 - \frac{1}{4}\right] - \left[0 - \frac{1}{4} - \frac{\pi}{8} - 0\right]$$
$$= \frac{\pi}{4}$$

5 **(a)** Use the method of difference to show that $\sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \ln\left(\frac{n+1}{n}\right) - \ln 2$. [3]

Solutions

$$\sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \sum_{r=2}^{n}\left[\ln(r-1) - 2\ln r + \ln(r+1)\right]$$

$$= \left[\begin{array}{l} \ln 1 - 2\ln 2 + \ln 3 \\ +\ln 2 - 2\ln 3 + \ln 4 \\ +\ln 3 - 2\ln 4 + \ln 5 \\ +\ldots \\ +\ln(n-3) - 2\ln(n-2) + \ln(n-1) \\ +\ln(n-2) - 2\ln(n-1) + \ln(n) \\ +\ln(n-1) - 2\ln n + \ln(n+1) \end{array}\right]$$

$= 0 - \ln 2 - \ln n + \ln(n + 1)$

$= \ln\left(\frac{n+1}{n}\right) - \ln 2$ (Shown)

(b) Show that the corresponding infinite series is convergent and state the sum to infinity. [2]

Solutions

$$\sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \ln\left(\frac{n+1}{n}\right) - \ln 2$$
$$= \ln\left(1 + \frac{1}{n}\right) - \ln 2$$

As $n \to \infty$, $1+\frac{1}{n} \to 1$,

$\sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] \to \ln 1 - \ln 2 = -\ln 2$, a finite unique value.

Therefore, the series is convergent and the sum to infinity = –ln 2

(c) Show that $\sum_{r=10}^{20} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \ln\left(\frac{a}{b}\right)$, where *a* and *b* are integers to be found. [2]

Solutions

$$\sum_{r=10}^{20} \ln\left[\frac{(r-1)(r+1)}{r^2}\right]$$

$$= \sum_{r=2}^{20} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] - \sum_{r=2}^{9} \ln\left[\frac{(r-1)(r+1)}{r^2}\right]$$

$$= \left[\ln\left(\frac{21}{20}\right) - \ln 2\right] - \left[\ln\left(\frac{10}{9}\right) - \ln 2\right]$$

$$= \ln\left(\frac{21}{20} \times \frac{9}{10}\right)$$

$$= \ln\frac{189}{200}$$

Therefore, *a* = 189 and *b* = 200

6 **(a)** Using double angle formulae, show $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [2]

Solutions

$$\cos^4\theta$$

$$= \left[\cos^2\theta\right]^2$$

$$= \left[\frac{1}{2}(\cos 2\theta + 1)\right]^2, \text{ since } \cos 2\theta = 2\cos^2\theta - 1$$

$$= \frac{1}{4}\left[\cos^2 2\theta + 2\cos 2\theta + 1\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}(\cos 4\theta + 1) + 2\cos 2\theta + 1\right]$$

$$= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) \text{ (shown)}$$

<u>Alternative</u>

$$\frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) = \frac{1}{8}(2\cos^2 2\theta - 1 + 4\cos 2\theta + 3)$$
$$= \frac{2}{8}(\cos^2 2\theta + 2\cos 2\theta + 1)$$
$$= \frac{1}{4}(\cos 2\theta + 1)^2$$
$$= \frac{1}{4}(2\cos^2\theta - 1 + 1)^2$$
$$= \frac{1}{4}(4\cos^4\theta)$$
$$= \cos^4\theta \qquad \text{(shown)}$$

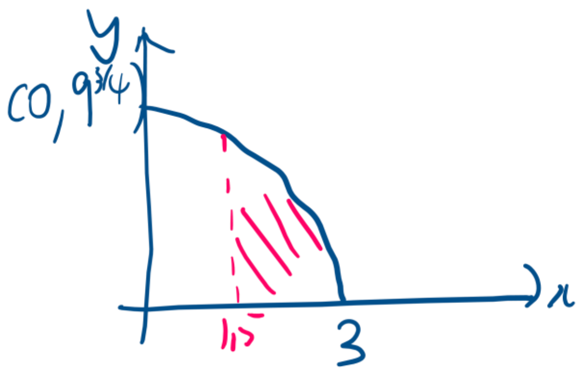
(b) The region R lies in the first quadrant and is bounded by the curve $y^4 = (9 - x^2)^3$, the x-axis and the lines $x = 1.5$ and $x = 3$. R is rotated about the x-axis through 2π radians. Using the substitution $x = 3\sin\theta$, find the exact volume generated. [6]

Solutions

$x = 3\sin\theta \Rightarrow \dfrac{dx}{d\theta} = 3\cos\theta$

$x = 1.5 \Rightarrow \sin\theta = 0.5 \Rightarrow \theta = \dfrac{\pi}{6}$

$x = 3 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \dfrac{\pi}{2}$



Required volume $= \pi\int_{1.5}^{3} y^2 \, dx$

$$= \pi\int_{1.5}^{3} (9 - x^2)^{\frac{3}{2}} \, dx$$
$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (9 - 9\sin^2\theta)^{\frac{3}{2}} \left(\frac{dx}{d\theta}\right) d\theta$$
$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (9 - 9\sin^2\theta)^{\frac{3}{2}} (3\cos\theta) \, d\theta$$

$$= 81\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(1-\sin^2\theta\right)^{\frac{3}{2}}\left(\cos\theta\right)\mathrm{d}\theta$$

$$= 81\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^3\theta\left(\cos\theta\right)\mathrm{d}\theta$$

$$=81\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^4\theta\,\mathrm{d}\theta$$

$$= 81\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{8}\left(\cos 4\theta+4\cos 2\theta+3\right)\mathrm{d}\theta$$

$$= \frac{81}{8}\pi\left[\frac{\sin 4\theta}{4}+2\sin 2\theta+3\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

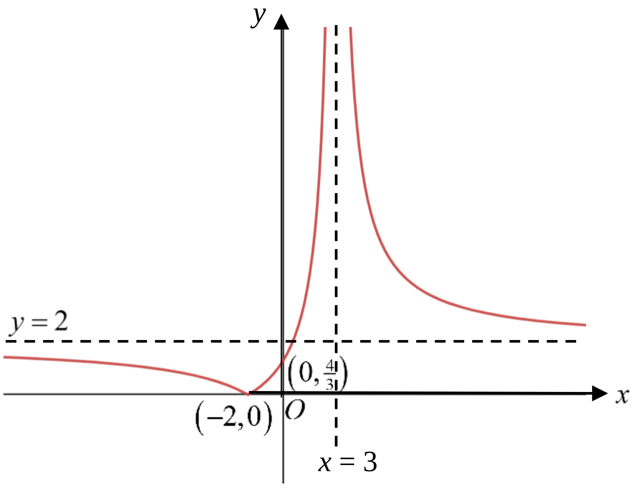
$$= \frac{81}{8}\pi\left[0+0+\frac{3\pi}{2}-\frac{\sqrt{3}}{8}-\sqrt{3}-\frac{\pi}{2}\right]$$

$$= \frac{81}{8}\pi\left[\pi-\frac{9\sqrt{3}}{8}\right]\text{ units}^3$$

7 The function f is defined by

$$\mathrm{f}: x \to \left|\frac{2x+4}{3-x}\right|, \ x \in \mathbb{R}, \qquad x \neq 3.$$

(a) Sketch the graph of $y = \mathrm{f}(x)$, giving the equations of any asymptotes and the coordinates of the points where the curve meets the axes. [3]

Solutions
$y = \left\|\frac{2x+4}{3-x}\right\| = \left\|-2+\frac{10}{3-x}\right\|$ Equations of asymptotes: $y = \|-2\| = 2$, $x = 3$ When $x = 0$, $y = \left\|\frac{4}{3}\right\| = \frac{4}{3}$ When $y = 0$, $x = -2$ 

(b) Hence state the range of f. [1]

Solutions
$\mathrm{R_f} = [0, \infty)$

(c) Explain why the function f^2 does not exist. [1]

Solutions
$\mathrm{R_f} = [0, \infty)$, $\mathrm{D_f} = (-\infty, 3) \cup (3, \infty)$ Since $3 \in \mathrm{R_f}$ but $3 \notin \mathrm{D_f}$ Hence $\mathrm{R_f} \not\subset \mathrm{D_f}$ $\therefore \mathrm{f}^2$ does not exist.

The domain of f is further restricted to $x \leq a$, where a is a constant.

(d) State the greatest value of a such that the function f^{-1} exists. [1]

Solutions
Greatest value of $a = -2$

(e) Hence find $\mathrm{f}^{-1}(x)$ and state its domain. [4]

Solutions

When $x \le -2$,

$$\mathrm{f}(x) = \left|\frac{2x+4}{3-x}\right| = -\frac{2x+4}{3-x}$$

Let $y = \mathrm{f}(x)$

$$-3y + xy = 2x + 4$$

$$x(y-2) = 4 + 3y$$

$$x = \frac{4+3y}{y-2}$$

Since $x = \mathrm{f}^{-1}(y)$,

$$\mathrm{f}^{-1}(x) = \frac{4+3x}{x-2},$$

$$\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = [0, 2)$$

8 Do not use a calculator in answering this question.

(a) (i) Express z, where $z = -1 + \sqrt{3}\mathrm{i}$, in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

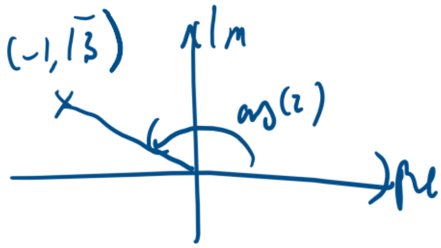
Solutions

$$z = -1 + \sqrt{3}\mathrm{i}$$

$$|z| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\arg(z) = \pi - \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{2}{3}\pi$$

$$z = 2\mathrm{e}^{\mathrm{i}\frac{2}{3}\pi}$$



(ii) Find the smallest positive integer value of n such that $\dfrac{z^n}{\mathrm{i}z^*}$ is purely imaginary. [3]

Solutions

$$\frac{z^n}{\mathrm{i}z^*} = \frac{2^n \mathrm{e}^{\mathrm{i}\frac{2n}{3}\pi}}{\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} 2\mathrm{e}^{\mathrm{i}\left(-\frac{2}{3}\pi\right)}} = 2^{n-1}\mathrm{e}^{\mathrm{i}\left(\frac{2n}{3}+\frac{1}{6}\right)\pi}$$

$$= 2^{n-1}\left(\cos\left[\left(\frac{2n}{3}+\frac{1}{6}\right)\pi\right] + \mathrm{i}\sin\left[\left(\frac{2n}{3}+\frac{1}{6}\right)\pi\right]\right)$$

For purely imaginary,

$$\cos\left[\left(\frac{2n}{3}+\frac{1}{6}\right)\pi\right] = 0$$

$$\left(\frac{2n}{3}+\frac{1}{6}\right)\pi = \frac{(2k+1)\pi}{2}, \text{ where } k \in \mathbb{Z}$$

$$\frac{4n+1}{6} = \frac{2k+1}{2}$$

$$n = \frac{6k+2}{4} = \frac{3k+1}{2}$$

$\therefore$ smallest positive integer value for n is $n = 2$

(b) Find the complex numbers v and w which satisfy the following simultaneous equations.

$$2v+|w|=1$$

$$3v-\mathrm{i}w=-3+4\mathrm{i}$$

Give your answers in the form $a+\mathrm{i}b$, where a and b are real numbers. [5]

Solutions
$2v+\|w\|=1---(1)$ $3v-\mathrm{i}w=-3+4\mathrm{i}---(2)$ From (1): $\|w\|=1-2v---(3)$ From (2): $w=\dfrac{4\mathrm{i}-3-3v}{-\mathrm{i}}\times\dfrac{\mathrm{i}}{\mathrm{i}}=-4+(-3-3v)\mathrm{i}---(*)$ From (1), we observe that v is a real number hence $\|w\|=\sqrt{(-4)^2+(-3-3v)^2}---(4)$ $(3)=(4)$: $1-2v=\sqrt{(-4)^2+(-3-3v)^2}$ $(1-2v)^2=16+(-3-3v)^2$ $1-4v+4v^2=16+9+18v+9v^2$ $5v^2+22v+24=0$ $(5v+12)(v+2)=0$ $v=-\dfrac{12}{5}$, or $v=-2$ Substitute $v=-\dfrac{12}{5}$ into (*) $w=-4+\left[-3-3\left(-\dfrac{12}{5}\right)\right]\mathrm{i}=-4+\dfrac{21}{5}\mathrm{i}$ Subst $v=-2$ into (*) $w=-4+[-3-3(-2)]\mathrm{i}=-4+3\mathrm{i}$

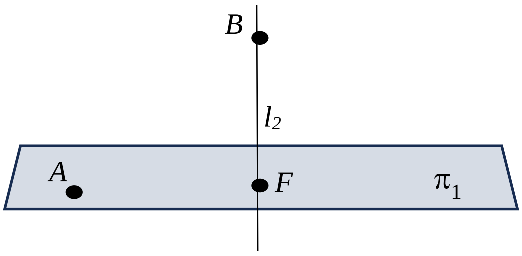
9 The line l_1 contains the point A with coordinates (–3, 1, 2) and is parallel to the vector $\begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$, where a is a constant. The line l_2 has equation $\dfrac{x+2}{3} = \dfrac{y-1}{2} = z-5$. It is given that l_1 and l_2 cross at the point B.

(a) Find the value of a and the coordinates of B. [5]

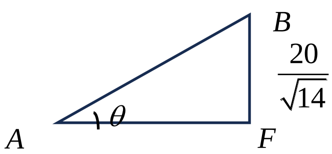
Solutions
l_1: $\mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}, \lambda \in \mathbb{R}$ l_2: $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$ Let $\begin{pmatrix} -3+2\lambda \\ 1+\lambda \\ 2+a\lambda \end{pmatrix} = \begin{pmatrix} -2+3\mu \\ 1+2\mu \\ 5+\mu \end{pmatrix}$, for some $\lambda, \mu \in \mathbb{R}$ $\Rightarrow$ $2\lambda - 3\mu = 1$ …(1) $\lambda - 2\mu = 0$ …(2) $a\lambda - \mu = 3$ …(3) Using GC for (1) and (2), $\lambda = 2$, $\mu = 1$ From (3): $2a - 1 = 3$ $a = 2$ When $\mu = 1$, $\begin{pmatrix} -2+3\mu \\ 1+2\mu \\ 5+\mu \end{pmatrix} = \begin{pmatrix} -2+3 \\ 1+2 \\ 5+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ Hence the coordinates of B are (1, 3, 6)

(b) The plane π_1 contains the point A and is perpendicular to l_2.

(i) Find the shortest distance from B to π_1. [3]

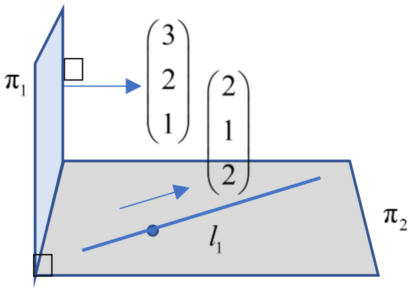
Solutions
 Shortest distance, *AF* $= \left\lvert \overrightarrow{AB} \bullet \hat{\underset{\sim}{n}} \right\rvert$ $= \dfrac{\left\lvert \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\rvert}{\sqrt{9+4+1}}$ $= \dfrac{20}{\sqrt{14}}$

(ii) Hence find the acute angle between l_1 and π_1. [2]

Solutions
 $\left\lvert \overrightarrow{AB} \right\rvert = \sqrt{4^2+2^2+4^4} = 6$ Let the required acute angle be θ. $\sin\theta = \dfrac{\frac{20}{\sqrt{14}}}{6}$ $\theta = 62.983° \approx 63.0°$ (1dp)

(c) The plane π_2 is perpendicular to π_1 and contains l_1. Find a cartesian equation of π_2. [3]

Solutions



$$\begin{pmatrix}2\\1\\2\end{pmatrix}\times\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}1-4\\-(2-6)\\4-3\end{pmatrix}=\begin{pmatrix}-3\\4\\1\end{pmatrix}$$

$$\pi_2\text{: } \mathbf{r}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=\begin{pmatrix}-3\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=9+4+2=15$$

$\Rightarrow$ Cartesian equation of π_2 is $-3x+4y+z=15$

10 The mass of a person depends both on daily rate of energy intake and on daily rate of energy expenditure.

In this question, mass is in kg, time is in days, and energy intake and energy expenditure are measured in Calories per day.

The rate of change of a person's mass with respect to time is proportional to the difference between energy intake and energy expenditure.

Andrew has a mass of M kg and his energy intake is fixed at C Calories per day. For every kg of his mass, he expends 30 Calories per day.

(a) Show that $\dfrac{\mathrm{d}M}{\mathrm{d}t}=k(C-30M)$ where t is time and k is a constant. [1]

Solutions

Given that Andrew has a mass of M kg at t days,

$$\frac{\mathrm{d}M}{\mathrm{d}t}\propto(\text{energy intake } - \text{ energy expenditure})$$

$$\therefore \frac{\mathrm{d}M}{\mathrm{d}t}=k(C-30M)$$

where k is the constant of proportionality

Andrew's initial mass is 110 kg.

(b) Find the energy intake such that the maintains his mass at 110 kg. [1]

Solutions
If no change in mass, $\frac{\mathrm{d}M}{\mathrm{d}t}=0$, $k\left(C-30M\right)=0$ $k\neq 0$, $C=30M$ When M = 110, $C=30\ (110)=3300$ Energy intake is 3300 Calories per day

As part of a new health plan, Andrew fixes his energy intake at 80% of the value found in part **(b)**.

(c) By solving the differential equation in part **(a)**, show that Andrew's mass while he is on the plan satisfies the equation $M = 88 + 22\mathrm{e}^{-30kt}$. [4]

Solutions
$C=0.8(3300)=2640$ $\frac{\mathrm{d}M}{\mathrm{d}t}=k\left(2640-30M\right)$ Integrating both sides with respect to t: $\int \frac{1}{2640-30M}\,\mathrm{d}M=k\int 1\,\mathrm{d}t$ $-\frac{1}{30}\ln\left\|2640-30M\right\|=kt+D$, where D is an arbitrary constant $\ln\left\|2640-30M\right\|=-30kt-30D$ $\left\|2640-30M\right\|=\mathrm{e}^{-30kt-30D}=A\mathrm{e}^{-30kt}$, $A=\mathrm{e}^{-30D}$ $2640-30M=B\mathrm{e}^{-30kt}$, $B=A$ or $-A$ $M=\frac{1}{30}\left(2640-B\mathrm{e}^{-30kt}\right)$ $=88-\frac{B}{30}\mathrm{e}^{-30kt}$ When $t=0$, $M=110$, $110=88-\frac{B}{30}\mathrm{e}^{-30k(0)}$ $\frac{B}{30}=88-100=-22$ $M=88+22\mathrm{e}^{-30kt}$ (shown)

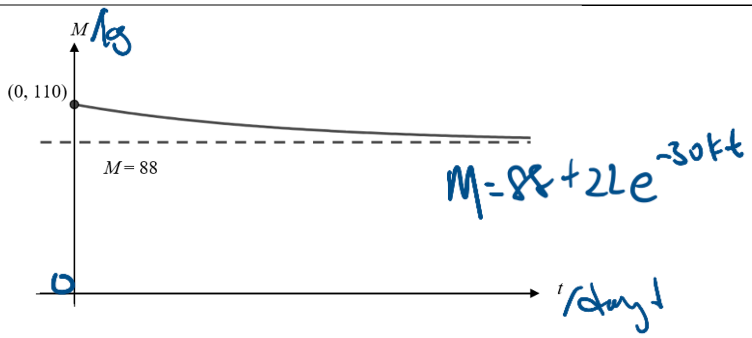
Andrew's mass after 75 days on the plan is 100 kg.

(d) Find the number of additional days required on the plan for Andrew's mass to fall below 96 kg. [4]

Solutions
When $t = 75, M = 100$, $100 = 88 + 22e^{-30k(75)}$ $e^{-30k} = \left(\frac{6}{11}\right)^{\frac{1}{75}}$ If $M < 96$ $88 + 22e^{-30kt} < 96$ $\left(e^{-30k}\right)^t < \frac{4}{11}$ $\left[\left(\frac{6}{11}\right)^{\frac{1}{75}}\right]^t < \frac{4}{11}$ $t > \dfrac{\ln\left(\frac{4}{11}\right)}{\ln\left[\left(\frac{6}{11}\right)^{\frac{1}{75}}\right]}$ $t > 125.17$ The number of additional days = 126 – 75 = 51 days.

(e) (i) Sketch a graph of Andrew's mass while on this plan. Explain why Andrew cannot achieve a mass of 80 kg using this plan. [2]

Solutions



As $t \to \infty$, $e^{-30kt} \to 0$, $M \to 88$. Andrew's mass will decrease with time and approach (but not go below) 88 kg in the long run.

Thus, it is not possible for Andrew to achieve a mass of 80 kg using this plan.

(ii) State the range of possible values of energy intake for which Andrew could achieve a mass of 80 kg. [1]

Solutions
By observation, the horizontal asymptote is $M = \frac{C}{30}$.   In order for Andrew to achieve a mass of 80 kg, $\therefore 0 < \frac{C}{30} < 80$ $0 < C < 2400$

11 Wei is saving for a property that she intends to buy in the future. Wei needs to save a minimum of \$50 000. She saves regularly in an account which offers no interest. She makes an initial deposit on 31 January 2021 of \$$a$. Each subsequent month, she deposits \$50 more than she deposited in the previous month. Her final deposit is made on 31 December 2023.

(a) Find, to the nearest cent, the smallest value of a so that she saves at least \$50 000. [2]

Solutions

nth month	Month/Year	Amount Deposited on the last day of the *nth* month
1	Jan 2021	a
2	Feb 2021	$a + 50$
3	March 2021	$a + 2(50)$
36	Dec 2023	$a + 35(50)$

Given that she made her final deposit on 31 December 2023, where $n = 36$, to have at least $50 000,

$S_{36} \geq 50\ 000$

$\frac{36}{2}\left(a + a + 35(50)\right) \geq 50\ 000$

$36a + 31500 \geq 50\ 000$

$a \geq 513.8889$

Smallest value of a is 513.89.

Wei has arranged to purchase a property on 1 January 2024 with the aid of a loan of $400 000. The terms of the loan are that interest of 0.1% is added to the amount owing at the start of every month, with the first interest amount added on 1 January 2024. Wei makes a monthly repayment of $$x$ at the end of every month, with the first repayment on 31 January 2024.

(b) Show that the amount, in dollars, that Wei owes at the end of n months is

$$400\ 000 \times 1.001^n - 1000x(1.001^n - 1).$$ [3]

Solutions

n	Mth	Amount owing at the start of month	Amount owing at the end of month
1	Jan 2024	400 000(1.001)	400 000(1.001) – x
2	Feb 2024	400 000$(1.001)^2$ – x (1.001)	400 000$(1.001)^2$ – x $(1.001)^2$ – x
3	Mar 2024	400 000$(1.001)^3$ – x $(1.001)^2$ – x(1.001)	400 000$(1.001)^3$ – x $(1.001)^2$ – x(1.001) – x
n		$400\ 000(1.001)^n - x(1.001)^{n-1} - \dots - x(1.001)$	$400\ 000(1.001)^n - x(1.001)^{n-1} - \dots - x(1.001) - x$

The amount, in dollars, that Wei owes at the end of n months

$$= 400\ 000(1.001)^n - x(1.001)^{n-1} - \dots - x(1.001) - x$$

$$= 400\ 000(1.001)^n - x\left[1 + 1.001 + 1.001^2 + \dots + (1.001)^{n-1}\right]$$

$$= 400\ 000(1.001)^n - \frac{x(1)(1.001^n - 1)}{1.001 - 1}$$

$$= 400\ 000(1.001)^n - 1000x(1.001^n - 1) \text{ (Shown)}$$

(c) (i) Suppose that the loan is repaid in 360 monthly repayments. Find the value of the monthly repayment to the nearest cent. [2]

Solutions
Since the loan is repaid in exactly 360 repayments, $400\ 000(1.001)^{360} - 1000x(1.001^{360} - 1) = 0$ $x = \dfrac{400\ 000(1.001)^{360}}{1000(1.001^{360} - 1)}$ $x = 1323.63$

(ii) Use your answer in part **(c)(i)** calculate the total interest paid on the loan in this case. [1]

Solutions
Total interest = 360(1323.63) – 400 000 = 76 506.80

(d) (i) Now suppose Wei decides to repay the loan at the rate of \$1600 per month for k months, plus a final monthly repayment of \$$y$, where $y < 1600$. Find k and y. [4]

Solutions
Amount owed at the end of n month

$= 400\,000 \times 1.001^n - 1000x(1.001^n - 1)$.

<u>Method 1:</u>
From GC,

n	Amount owed
286	2917.30
287	1320.22
288	–278.46

$k = 287$ and $y = 1320.22(1.001) = 1321.54$

<u>Method 2</u>:

$400\,000 \times 1.001^k - 1000 \times 1600\,(1.001^k - 1) < 1600$
$1.001^k\,(1\,600\,000 - 400\,000) > 1\,598\,400$
$k \lg(1.001) > \lg(1.332)$
$k > 286.82$
$\therefore k = 287$

$y = 400000 \times 1.001^{287} - 1000 \times 1600(1.001^{287} - 1) \times 1.001$
$= 1321.54$

(ii) Hence find the total saving to Wei by making the repayments in part **(d)(i)** rather than the repayments in part **(c)(i)**, giving your answer to 4 significant figures. [1]

Solutions

Total Interest paid
$= 287(1600) + 1321.54 - 400\,000$
$= 60\,521.54$

Total saving
$= 76\,506.80 - 60\,521.54$
$= 15985.26 = 15990$ (4 sig fig)