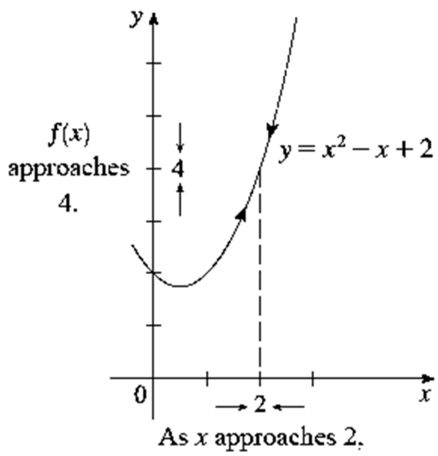
*H2 MATHEMATICS*
*TOPIC* *TECHNIQUES OF DIFFERENTIATION* *2022/JC1*

**Content Outline:**
- Differentiation of simple functions defined implicitly.
- Finding the approximate value of a derivative at a given point using a graphing calculator.

## 1 Limits

### 1.1 Limit of a Function

Consider the behaviour of a function $\mathrm{f}(x) = x^2 - x + 2$ for values of $x$ **close to** 2.




| $x$ | $\mathrm{f}(x)$ |
|---|---|
| 1.0 | 2.000000 |
| 1.5 | 2.750000 |
| 1.8 | 3.440000 |
| 1.9 | 3.710000 |
| 1.95 | 3.852500 |
| 1.99 | 3.970100 |
| 1.995 | 3.985025 |
| 1.999 | 3.997001 |
| $x$ values approaching 2 from the **left** | |

| $x$ | $\mathrm{f}(x)$ |
|---|---|
| 3.0 | 8.000000 |
| 2.5 | 5.750000 |
| 2.2 | 4.640000 |
| 2.1 | 4.310000 |
| 2.05 | 4.152500 |
| 2.01 | 4.030100 |
| 2.005 | 4.015025 |
| 2.001 | 4.003001 |
| $x$ values approaching 2 from the **right** | |

The above table gives the corresponding values of $\mathrm{f}(x)$ for values of $x$ **close to** 2 but not equal to 2.

From the graph and table of values, we observe that as $x$ **approaches** 2, $\mathrm{f}(x)$ gets closer and closer to 4.
We express this in words as "*the limit of the function* $\mathrm{f}(x)=x^2-x+2$ *as* $x$ *approaches 2 is 4*".
Mathematically, we write it as

$$\lim_{x\to 2}\mathrm{f}(x)=4\,.$$

In general, given a function $\mathrm{f}$, if $\mathrm{f}(x)$ approaches a fixed number $L$ as $x$ approaches $a$, then $L$ is the **limit** of $\mathrm{f}(x)$ as $x$ approaches $a$. We write it as

$$\lim_{x\to a}\mathrm{f}(x)=L$$

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🗎 **<u>Remark</u>:** To find the limit of a function $\mathrm{f}$, $\mathrm{f}(x)$ need not be defined at $x=a$.

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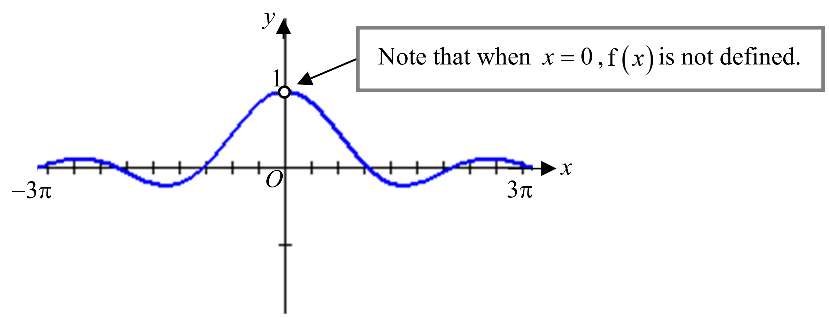
For example, consider the graph of $\mathrm{f}(x)=\dfrac{\sin x}{x}$ for $-3\pi\le x\le 3\pi$ as shown below.




We observe that even though $\mathrm{f}(0)$ does not exist, it is clear that the limit of the function $\mathrm{f}(x)=\dfrac{\sin x}{x}$ as $x$ approaches 0 is 1.

If $\mathrm{f}$ is defined at $x=a$ and $\lim_{x\to a}\mathrm{f}(x)=\mathrm{f}(a)$, we say $\mathrm{f}$ is **continuous** at $x=a$. If $\mathrm{f}$ is continuous at every point on an interval $(a,b)$, $\mathrm{f}$ is a continuous function on the interval $(a,b)$. Intuitively, continuous functions are the functions whose graphs can be drawn without lifting the pen off the paper. In the above example, $\mathrm{f}$ is not continuous at $x=0$ (hence it is excluded graphically with an empty circle).

In H2 Maths syllabus, the functions are all assumed to be continuous on the interval under consideration.

### 1.2 Properties of Limits (📖 Self-Reading)

If $\lim_{x \to a} \mathrm{f}(x)$ and $\lim_{x \to a} \mathrm{g}(x)$ both exist, then

| | Properties | Examples |
|---|---|---|
| **(a)** | $\lim_{x \to a} \left[k\mathrm{f}(x)\right] = k \lim_{x \to a} \mathrm{f}(x)$, where $k$ is a constant | $\lim_{x \to 4}\left(5x^3\right) = 5\lim_{x \to 4} x^3 = 5(4)^3 = 320$ |
| **(b)** | $\lim_{x \to a}\left[\mathrm{f}(x) \pm \mathrm{g}(x)\right] = \lim_{x \to a} \mathrm{f}(x) \pm \lim_{x \to a} \mathrm{g}(x)$ | $\lim_{x \to 3}\left(x^2 - x\right) = \lim_{x \to 3} x^2 - \lim_{x \to 3} x = 3^2 - 3 = 6$ |
| **(c)** | $\lim_{x \to a}\left[\mathrm{f}(x)\mathrm{g}(x)\right] = \left[\lim_{x \to a} \mathrm{f}(x)\right]\left[\lim_{x \to a} \mathrm{g}(x)\right]$ | $\lim_{x \to 2}\left[(\ln x)(x)\right] = \left[\lim_{x \to 2}(\ln x)\right]\left[\lim_{x \to 2}(x)\right] = (\ln 2)(2) = 2\ln 2 = \ln 4$ |
| **(d)** | $\lim_{x \to a}\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right] = \frac{\left[\lim_{x \to a} \mathrm{f}(x)\right]}{\left[\lim_{x \to a} \mathrm{g}(x)\right]}$, where $\lim_{x \to a} \mathrm{g}(x) \neq 0$ | $\lim_{x \to -1}\left(\frac{x^2+1}{2x+3}\right) = \frac{\lim_{x \to -1}\left(x^2+1\right)}{\lim_{x \to -1}\left(2x+3\right)} = \frac{(-1)^2+1}{2(-1)+3} = 2$ |

**Example 1 (📖 Self-Reading):**

Evaluate the following limits

**(a)** $\lim_{x \to 1}\left(\frac{x^2+4x-1}{x+2}\right)$, **(b)** $\lim_{x \to \infty}\left(\frac{1}{x}\right)$, **(c)** $\lim_{x \to \infty}\left(\frac{3x-1}{x+2}\right)$.

**Solution:**

**(a)**
$$\lim_{x \to 1}\left(\frac{x^2+4x-1}{x+2}\right) = \frac{\lim_{x \to 1}\left(x^2+4x-1\right)}{\lim_{x \to 1}(x+2)}$$
$$= \frac{(1)^2+4(1)-1}{(1)+2}$$
$$= \frac{4}{3}$$

**(b)**
$$\lim_{x \to \infty}\left(\frac{1}{x}\right) = 0$$

**(c)** **Method ①:**

$$\lim_{x\to\infty}\left(\frac{3x-1}{x+2}\right)=\lim_{x\to\infty}\left(\frac{\frac{3x-1}{x}}{\frac{x+2}{x}}\right)$$

Divide throughout by the highest power of *x*

$$=\lim_{x\to\infty}\left(\frac{3-\frac{1}{x}}{1+\frac{2}{x}}\right)$$

$$=\frac{\lim_{x\to\infty}\left(3-\frac{1}{x}\right)}{\lim_{x\to\infty}\left(1+\frac{2}{x}\right)}$$

$$=\frac{3-0}{1+0}$$

$$=3$$

**Method ②:**

$$\lim_{x\to\infty}\left(\frac{3x-1}{x+2}\right)=\lim_{x\to\infty}\left(3-\frac{7}{x+2}\right)$$

Perform long division

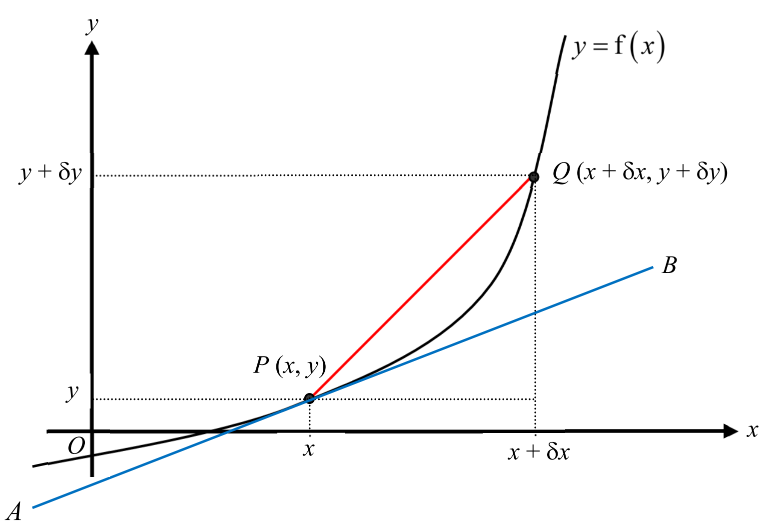
$$=3-\lim_{x\to\infty}\left(\frac{7}{x+2}\right)$$

$$=3-0$$

$$=3$$

## 2 The Derivative as a Limit (First Principles)

Consider a continuous function $y=\mathrm{f}(x)$. Let $P(x,y)$ and $Q(x+\delta x, y+\delta y)$ be two points on the curve $y=\mathrm{f}(x)$ where $\delta x$ is a small change in $x$ and $\delta y$ is a small change in $y$.




Therefore, gradient of $PQ=\dfrac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{(x+\delta x)-x}=\dfrac{y+\delta y-y}{\delta x}=\dfrac{\delta y}{\delta x}$.

As $Q$ moves closer to $P$, $\delta x\to 0$ and gradient of $PQ$ $\to$ gradient of $AB$.

Thus, the gradient of tangent to the curve at $P = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

We denote $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ as $\frac{dy}{dx}$. This is known as the **first derivative of f (with respect to $x$)** at $x$ and is also denoted by $f'(x)$ or $\frac{d}{dx}f(x)$. The process of finding the derivative, $f'(x)$, from $f(x)$ is called **differentiation**. The first derivative of $f(x)$ from **First Principles** is

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}\text{, provided the limit exists.}$$

**Example 2:**

Let $f : x \mapsto x^2,\ x \in \mathbb{R}$. Using the first principles of differentiation, show that $f'(x) = 2x$.

**Solution:**

$f(x) = x^2$ and $f(x+\delta x) = (x+\delta x)^2$

From first principles, $\frac{f(x+\delta x)-f(x)}{\delta x} = \frac{(x+\delta x)^2 - x^2}{\delta x}$

$$= \frac{x^2 + 2x(\delta x) + (\delta x)^2 - x^2}{\delta x}$$

Expand and simplify

$$= 2x + \delta x$$

$$\therefore f'(x) = \lim_{\delta x \to 0}(2x + \delta x)$$
$$= 2x \quad \text{(shown)}$$

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✪ Refer to **Section 5: Appendix** on **Page 17 - 20** (**Examples A to C**) for finding the derivatives of $\frac{1}{x}$, $\sin x$ and $e^x$ from the first principles.

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## 3 Rules of Differentiation

### 3.1 Basic Rules (📖 Self-Reading)

Given that $a$ and $n$ are real constants,

| |
|---|
| $\frac{\mathrm{d}}{\mathrm{d}x}(a)=0$ |
| $\frac{\mathrm{d}}{\mathrm{d}x}(x^n)=nx^{n-1}$ |
| $\frac{\mathrm{d}}{\mathrm{d}x}[a\mathrm{f}(x)]=a\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)]$ |
| $\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)\pm\mathrm{g}(x)]=\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)]\pm\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{g}(x)]$ |

**<u>Example 3:</u>** Differentiate $\frac{1}{x^3}-x^5+\sqrt{x^3}$ with respect to $x$.

**<u>Solution:</u>**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^3}-x^5+\sqrt{x^3}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-3}-x^5+x^{\frac{3}{2}}\right)$$

$$=-3x^{-4}-5x^4+\frac{3}{2}x^{\frac{1}{2}}$$

### 3.2 Chain Rule (📖 Self-Reading)

If $y$ is a function of $u$, where $u$ is a function of $x$, then $\boxed{\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}}$.

**<u>Example 4:</u>** Differentiate $\frac{1}{(6x^2+4x+1)^2}$ with respect to $x$.

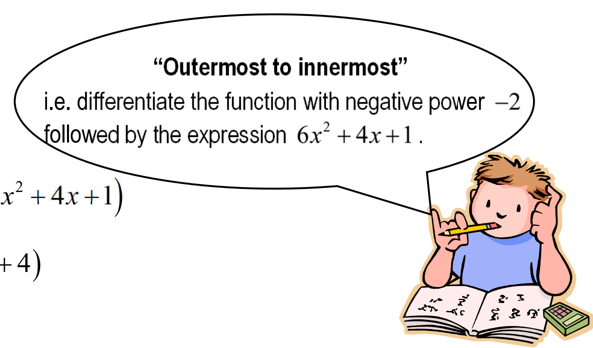
**<u>Solution:</u>**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{(6x^2+4x+1)^2}\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[(6x^2+4x+1)^{-2}\right]$$

$$=-2(6x^2+4x+1)^{-3}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(6x^2+4x+1)$$

$$=-2(6x^2+4x+1)^{-3}\cdot(12x+4)$$

$$=-\frac{8(3x+1)}{(6x^2+4x+1)^3}$$



### 3.3 Product Rule (📖 Self-Reading)

Let $y = uv$ where $u$ and $v$ are both functions of $x$.

Then $\boxed{\dfrac{dy}{dx} = \dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}}$.

**Example 5:** Differentiate $(3x^2+1)(2x+7)^{\frac{1}{2}}$ with respect to $x$.

**Solution:**

$$\begin{aligned}\frac{d}{dx}\left[(3x^2+1)(2x+7)^{\frac{1}{2}}\right] &= (3x^2+1)\frac{d}{dx}\left[(2x+7)^{\frac{1}{2}}\right] + (2x+7)^{\frac{1}{2}}\frac{d}{dx}(3x^2+1) \\ &= (3x^2+1)\cdot\frac{1}{2}(2x+7)^{-\frac{1}{2}}(2) + (2x+7)^{\frac{1}{2}}\cdot(6x) \\ &= (3x^2+1)(2x+7)^{-\frac{1}{2}} + 6(x)(2x+7)^{\frac{1}{2}} \\ &= (2x+7)^{-\frac{1}{2}}\left[(3x^2+1)+6x(2x+7)\right] \\ &= (2x+7)^{-\frac{1}{2}}(3x^2+1+12x^2+42x) \\ &= (2x+7)^{-\frac{1}{2}}(15x^2+42x+1)\end{aligned}$$

Factorise (take out smaller power)

### 3.4 Quotient Rule (📖 Self-Reading)

Let $y = \dfrac{u}{v}$ where $u$ and $v$ are both functions of $x$.

Then $\boxed{\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}}$.

**Example 6:** Differentiate $\dfrac{2x^3+3}{x^4+2}$ with respect to $x$.

**Solution:**

$$\begin{aligned}\frac{d}{dx}\left(\frac{2x^3+3}{x^4+2}\right) &= \frac{(x^4+2)\frac{d}{dx}(2x^3+3) - (2x^3+3)\frac{d}{dx}(x^4+2)}{(x^4+2)^2} \\ &= \frac{(x^4+2)(6x^2)-(2x^3+3)(4x^3)}{(x^4+2)^2} \\ &= \frac{2x^2\left[(3x^4+6)-(4x^4+6x)\right]}{(x^4+2)^2} \\ &= \frac{2x^2(-x^4-6x+6)}{(x^4+2)^2}\end{aligned}$$

Factorise $2x^2$ to simplify quickly

**Self-Practice 1:**

Differentiate the following with respect to $x$:

**(a)** $\dfrac{2}{\sqrt{x}}+\dfrac{6}{\sqrt[3]{x}}$ **(b)** $\left(x^2+3\right)^4\left(2x^3-5\right)^3$.

**Answers:**

**(a)** $-x^{-\frac{3}{2}}-2x^{-\frac{4}{3}}$ **(b)** $2x\left(x^2+3\right)^3\left(2x^3-5\right)^2\left(17x^3+27x-20\right)$

### 3.5 Derivatives of Trigonometric Functions

For the following results, the **angles $x$ and $\mathbf{f}(x)$** are measured in **radians**.

| | Basic | General |
|---|---|---|
| **(a)** | $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\cos x$ | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sin \mathrm{f}(x)\right]=\left[\cos \mathrm{f}(x)\right]\cdot\dfrac{\mathrm{df}(x)}{\mathrm{d}x}$ |
| **(b)** | $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos x)=-\sin x$ | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\cos \mathrm{f}(x)\right]=-\left[\sin \mathrm{f}(x)\right]\cdot\dfrac{\mathrm{df}(x)}{\mathrm{d}x}$ |
| **(c)** | $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan x)=\sec^2 x$ | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\tan \mathrm{f}(x)\right]=\left[\sec^2 \mathrm{f}(x)\right]\cdot\dfrac{\mathrm{df}(x)}{\mathrm{d}x}$ |
| **(d)** | $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec x)=\sec x\tan x$ | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sec \mathrm{f}(x)\right]=\left[\sec \mathrm{f}(x)\tan \mathrm{f}(x)\right]\cdot\dfrac{\mathrm{df}(x)}{\mathrm{d}x}$ |
| **(e)** | $\dfrac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$ | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{cosec} \mathrm{f}(x)\right]=-\left[\operatorname{cosec} \mathrm{f}(x)\cot \mathrm{f}(x)\right]\cdot\dfrac{\mathrm{df}(x)}{\mathrm{d}x}$ |
| **(f)** | $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cot x)=-\operatorname{cosec}^2 x$ | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\cot \mathrm{f}(x)\right]=-\left[\operatorname{cosec}^2 \mathrm{f}(x)\right]\cdot\dfrac{\mathrm{df}(x)}{\mathrm{d}x}$ |

**[In MF26]**

**Example 7:** Differentiate the following with respect to $x$

**(a)** $x^2\sin x$, **(b)** $\dfrac{1+\tan x}{\sec x}$, **(c)** $\sqrt{\cos 2x}$, **(d)** $\cot x^2+\sec^2 x$.

**Solution:**

**(a)**
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\sin x\right)=x^2(\cos x)+(\sin x)(2x)$$
$$=x(x\cos x+2\sin x)$$

**(b)** $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+\tan x}{\sec x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sec x}+\frac{\tan x}{\sec x}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x+\frac{\sin x}{\cos x}\cdot\cos x\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x+\sin x\right)$$
$$=-\sin x+\cos x$$

**(c)** $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{\cos 2x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\cos 2x\right)^{\frac{1}{2}}\right]$$
$$=\frac{1}{2}\left(\cos 2x\right)^{-\frac{1}{2}}\left(-\sin 2x\right)\left(2\right)$$
$$=-\frac{\sin 2x}{\sqrt{\cos 2x}}$$

**(d)** $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot x^2+\sec^2 x\right)=\left[-\operatorname{cosec}^2\left(x^2\right)\right]\cdot\left(2x\right)+2\left(\sec x\right)\left(\sec x\tan x\right)$$
$$=-2x\operatorname{cosec}^2\left(x^2\right)+2\sec^2 x\tan x$$

**<u>Example 8:</u>** Prove $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot x\right)=-\operatorname{cosec}^2 x$.

**<u>Solution:</u>** L.H.S. $$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot x\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\tan x}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\tan x\right)^{-1}\right]$$
$$=-\left(\tan x\right)^{-2}\left(\sec^2 x\right)$$
$$=-\frac{\sec^2 x}{\tan^2 x}$$
$$=-\frac{1}{\cos^2 x}\cdot\frac{\cos^2 x}{\sin^2 x}$$
$$=-\frac{1}{\sin^2 x}$$
$$=-\operatorname{cosec}^2 x=\text{R.H.S. (proved)}$$

**<u>Self-Practice 2:</u>**

Given that $y=\frac{3}{2}x+2\sin x+\frac{1}{2}\sin x\cos x$, show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\left(1+\cos x\right)^2$.

### 3.6 Derivatives of Exponential and Logarithmic Functions

| | Basic | General |
|---|---|---|
| **(a)** | $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x}\right)=\mathrm{e}^{x}$ | $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{\mathrm{f}(x)}\right]=\mathrm{e}^{\mathrm{f}(x)}\cdot\frac{\mathrm{df}(x)}{\mathrm{d}x}$ |
| **(b)** | $\frac{\mathrm{d}}{\mathrm{d}x}\left(a^{x}\right)=a^{x}\ln a$ | $\frac{\mathrm{d}}{\mathrm{d}x}\left[a^{\mathrm{f}(x)}\right]=a^{\mathrm{f}(x)}\ln a\cdot\frac{\mathrm{df}(x)}{\mathrm{d}x}$ |
| **(c)** | $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)=\frac{1}{x}$ | $\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln \mathrm{f}(x)\right]=\frac{1}{\mathrm{f}(x)}\cdot\frac{\mathrm{df}(x)}{\mathrm{d}x}$ |
| **(d)** | $\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln x}{\ln a}\right)=\frac{1}{x\ln a}$ | $\frac{\mathrm{d}}{\mathrm{d}x}\left[\log_a \mathrm{f}(x)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\ln \mathrm{f}(x)}{\ln a}\right]=\frac{1}{\mathrm{f}(x)\ln a}\cdot\frac{\mathrm{df}(x)}{\mathrm{d}x}$ |

**Example 9:** Differentiate the following with respect to $x$

**(a)** $4\mathrm{e}^{\left(x^2+1\right)}$, **(b)** $3^{\tan x}$, **(c)** $\ln\left(x+x^3\right)$, **(d)** $\log_3\left(x^2+4\right)$.

**Solution:**

**(a)**
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[4\mathrm{e}^{\left(x^2+1\right)}\right]=4\mathrm{e}^{\left(x^2+1\right)}\cdot 2x$$
$$=8x\mathrm{e}^{\left(x^2+1\right)}$$

**(b)**
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(3^{\tan x}\right)=3^{\tan x}\left(\ln 3\right)\cdot\sec^2 x$$

**(c)**
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(x+x^3\right)\right]=\frac{1}{x+x^3}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x+x^3\right)$$
$$=\frac{1+3x^2}{x+x^3}$$

**(d)**
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log_3\left(x^2+4\right)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\ln\left(x^2+4\right)}{\ln 3}\right]$$
$$=\frac{1}{\ln 3}\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(x^2+4\right)\right]$$
$$=\frac{1}{\ln 3}\cdot\frac{2x}{x^2+4}$$
$$=\frac{2x}{(\ln 3)\left(x^2+4\right)}$$

**Example 10:** Differentiate $\ln\sqrt{\dfrac{e^x}{2+3^{x^2}}}$ with respect to $x$.

**Solution:**

$$\begin{aligned}\frac{d}{dx}\left(\ln\sqrt{\frac{e^x}{2+3^{x^2}}}\right) &= \frac{1}{2}\frac{d}{dx}\left[\ln\left(\frac{e^x}{2+3^{x^2}}\right)\right] \\ &= \frac{1}{2}\frac{d}{dx}\left[\ln\left(e^x\right)-\ln\left(2+3^{x^2}\right)\right] \\ &= \frac{1}{2}\frac{d}{dx}\left[x-\ln\left(2+3^{x^2}\right)\right] \\ &= \frac{1}{2}\left[1-\frac{3^{x^2}(\ln 3)(2x)}{2+3^{x^2}}\right] \\ &= \frac{1}{2}-\frac{3^{x^2}(\ln 3)x}{2+3^{x^2}}\end{aligned}$$

**Self-Practice 3:**
Differentiate the following with respect to $x$

**(a)** $x\sin x$, **(b)** $e^{1+\cos x}$, **(c)** $\ln\sqrt{1+\cos x}$.

**Answers:**

**(a)** $x\cos x+\sin x$ **(b)** $-\sin x \cdot e^{1+\cos x}$ **(c)** $-\dfrac{\sin x}{2(1+\cos x)}$

### 3.7 Implicit Differentiation

Let us first understand the term **explicit**. An explicit function is an equation where $y$ is defined entirely in terms of $x$. Some examples of explicit functions are $y=\dfrac{2}{x}+3x^2$ and $y=\dfrac{4x}{x-1}$.

On the other hand, an **implicit** function is an equation involving $x$ and $y$, where $y$ is not in terms of $x$. Some examples of implicit functions are $y=\dfrac{2}{x}+3y^2$, $xy=y^2-10$ and $\sqrt{x+1}=y^x+e^y$.

When you are differentiating terms involving $y$ with respect to $x$, we need to apply the **Chain Rule** since we are assuming that $y$ is defined implicitly as a function of $x$.

To differentiate a function $g(y)$ with respect to $x$, we have the following

$$\frac{d}{dx}g(y)=\left[\frac{d}{dy}g(y)\right]\frac{dy}{dx}$$

**Example 11:** Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$ and $y$ when

**(a)** $x^2 + y^3 = xy^2$ **(b)** $\cos x = 1 + y\cos y$

**Solution:**

**(a)**

$$x^2 + y^3 = xy^2$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + y^3\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(xy^2\right)$$

$$2x + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = x \cdot 2y\frac{\mathrm{d}y}{\mathrm{d}x} + y^2$$

$$\left(3y^2 - 2xy\right)\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 2x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2x}{3y^2 - 2xy}$$

**(b)**

$$\cos x = 1 + y\cos y$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(1 + y\cos y\right)$$

$$-\sin x = y \cdot \left(-\sin y\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \cos y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left(y\sin y - \cos y\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin x}{y\sin y - \cos y}$$

**Self-Practice 4:**

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given that $x + xy - 2y = 5$.

**Answers:**

$\frac{1+y}{2-x}$

## 3.8 Logarithmic Differentiation

This method is used for differentiating

- expressions of the form $u^v$, where $u$ and $v$ are functions of $x$, e.g. $x^x$
- complicated products and quotients

**Procedure**

**Step 1:** Take natural logarithm on both sides.

**Step 2:** Simplify the logarithms, where possible.

**Step 3:** Perform implicit differentiation with respect to $x$.

**Example 12:** Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

**(a)** $y = x^{\sin x}$ for $x,\ y > 0$, **(b)** $x^y = y^x$ for $x,\ y > 0$ .

**Solution:**

**(a)**

$$y = x^{\sin x}$$
$$\ln y = \ln\left(x^{\sin x}\right)$$
$$\ln y = (\sin x)(\ln x)$$
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \cdot \frac{1}{x} + \ln x \cdot \cos x$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\left[\frac{\sin x}{x} + (\ln x)(\cos x)\right]$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^{\sin x}\left[\frac{\sin x}{x} + (\ln x)(\cos x)\right]$$

**(b)**

$$x^y = y^x$$
$$\ln x^y = \ln y^x$$
$$y\ln x = x\ln y$$
$$y\cdot\frac{1}{x} + \ln x\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = x\cdot\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \ln y$$
$$\left(\ln x - \frac{x}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \ln y - \frac{y}{x}$$
$$\left(\frac{y\ln x - x}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\ln y - y}{x}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(x\ln y - y)}{x(y\ln x - x)}$$

**Self-Practice 5:**

Prove by differentiation that $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^x\right) = x^x(1+\ln x)$.

## 3.9 Derivatives of Inverse Trigonometric Functions

### 3.9.1 Definitions of Inverse Trigonometric Functions

Recall that for trigonometric functions, the inverses are defined for the respective principal range. We have

| Inverse Trigonometric Function | Principal Domain | Principal Range |
|---|---|---|
| $y=\sin^{-1}x$ | $[-1,1]$ | $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ |
| $y=\cos^{-1}x$ | $[-1,1]$ | $[0,\pi]$ |
| $y=\tan^{-1}x$ | $\mathbb{R}$ | $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ |

---

**Remark**: $\sin^{-1}x\neq\dfrac{1}{\sin x}$.

$\sin^{-1}x$ is the inverse function and is not the same as $(\sin x)^{-1}$, which is the reciprocal function. Similarly, $\cos^{-1}x\neq\dfrac{1}{\cos x}$ and $\tan^{-1}x\neq\dfrac{1}{\tan x}$.

---

### 3.9.2 Important Results

| | Basic | General |
|---|---|---|
| **(a)** | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}x\right)=\dfrac{1}{\sqrt{1-x^2}}$, $\quad -1\le x\le 1$ | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sin^{-1}\mathrm{f}(x)\right]=\dfrac{1}{\sqrt{1-[\mathrm{f}(x)]^2}}\cdot\dfrac{\mathrm{df}(x)}{\mathrm{d}x}$, $\quad -1\le \mathrm{f}(x)\le 1$ |
| **(b)** | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}x\right)=-\dfrac{1}{\sqrt{1-x^2}}$, $\quad -1\le x\le 1$ | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\cos^{-1}\mathrm{f}(x)\right]=-\dfrac{1}{\sqrt{1-[\mathrm{f}(x)]^2}}\cdot\dfrac{\mathrm{df}(x)}{\mathrm{d}x}$, $\quad -1\le \mathrm{f}(x)\le 1$ |
| **(c)** | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}x\right)=\dfrac{1}{1+x^2}$, $\quad x\in\mathbb{R}$ | $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\mathrm{f}(x)\right]=\dfrac{1}{1+[\mathrm{f}(x)]^2}\cdot\dfrac{\mathrm{df}(x)}{\mathrm{d}x}$, $\quad \mathrm{f}(x)\in\mathbb{R}$ |

**[In MF26]**

To prove that $\dfrac{d}{dx}\left(\sin^{-1}x\right)=\dfrac{1}{\sqrt{1-x^2}}$:

Let $y=\sin^{-1}x$, then $\sin y = x$.

Differentiate both sides with respect to $x$, we have $\cos y\cdot\dfrac{dy}{dx}=1$

$$\frac{dy}{dx}=\frac{1}{\cos y}$$

But $\sin^2 y+\cos^2 y=1 \Rightarrow \cos y=\sqrt{1-\sin^2 y}$

$$=\sqrt{1-x^2} \qquad \left[-\frac{\pi}{2}\le y\le\frac{\pi}{2} \Rightarrow \cos y\ge 0\right]$$

$\therefore$
$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$$

---

✪ Refer to **Section 5: Appendix** on **Page 20** (**Examples D and E**) for the proofs of

$$\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}} \quad\text{and}\quad \frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}.$$

---

**Example 13:** Differentiate the following with respect to $x$

**(a)** $\cos^{-1}(2x)$, **(b)** $\tan^{-1}(x^2-1)$, **(c)** $\sin^{-1}(ax+b)$.

**Solution:**

**(a)** $$\frac{d}{dx}\left[\cos^{-1}(2x)\right]=-\frac{1}{\sqrt{1-(2x)^2}}\cdot 2$$
$$=-\frac{2}{\sqrt{1-4x^2}}$$

**(b)** $$\frac{d}{dx}\left[\tan^{-1}\left(x^2-1\right)\right]=\frac{1}{1+\left(x^2-1\right)^2}\cdot 2x$$
$$=\frac{2x}{1+\left(x^2-1\right)^2}$$

**(c)** $$\frac{d}{dx}\left[\sin^{-1}(ax+b)\right]=\frac{1}{\sqrt{1-(ax+b)^2}}\cdot a$$
$$=\frac{a}{\sqrt{1-(ax+b)^2}}$$

**Self-Practice 6:**

Differentiate the following with respect to $x$ **(a)** $\cos^{-1}\sqrt{x}$, **(b)** $\tan^{-1}\left(x^2\right)$.

**Answers:**

**(a)** $-\dfrac{1}{2\sqrt{x(1-x)}}$ **(b)** $\dfrac{2x}{1+x^4}$

## 4 Evaluating Derivatives using a Graphing Calculator (G.C.) (Self-Reading)

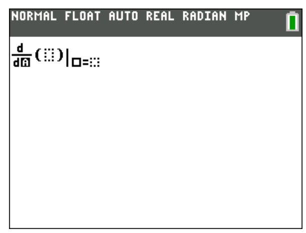
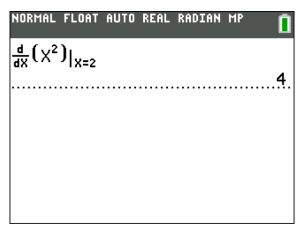
**Example 14:** Evaluate the first derivative of the following functions at the given value of $x$

**(a)** $y = x^2$ when $x = 2$,

**(b)** $y = \ln\left(2x^3\right)$ when $x = 5$.

**Solution:**

**(a)** To find the first derivative of $y = x^2$ when $x = 2$ using G.C.:

| | Press | Screen Display |
|---|---|---|
| 1. | Press [math] followed by **8:nDeriv(**. <br><br> or <br><br> Press [alpha] [window] followed by **3:nDeriv(** . |   |
| 2. | Key in the given function as shown and press [enter] . |   |

∴ first derivative of $y = x^2$ when $x = 2$ is $4$.

**(b)** Using G.C., the first derivative of $y = \ln\left(2x^3\right)$ when $x = 5$ is $0.600$ (to 3 s.f.).

## 5 Appendix

**Example A:** Let $f : x \mapsto \frac{1}{x}$. Using the first principles of differentiation, show that $f'(x) = -\frac{1}{x^2}$.

**Solution:** $f(x) = \frac{1}{x}$ and $f(x+\delta x) = \frac{1}{x+\delta x}$

From first principles,
$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{\frac{1}{x+\delta x} - \frac{1}{x}}{\delta x}$$
$$= \frac{x - (x+\delta x)}{(\delta x)(x)(x+\delta x)}$$
$$= -\frac{1}{x^2 + x(\delta x)}$$

$$\therefore f'(x) = \lim_{\delta x \to 0}\left[-\frac{1}{x^2 + x(\delta x)}\right] = -\frac{1}{x^2} \text{ (shown)}$$

---

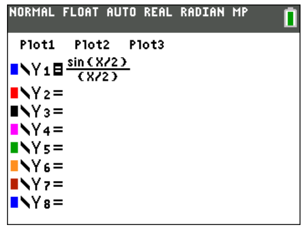
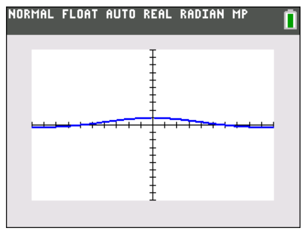
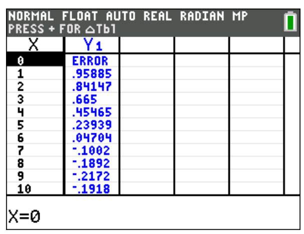
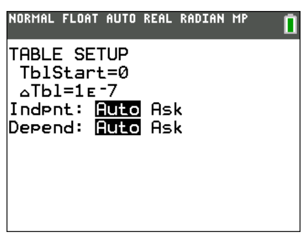
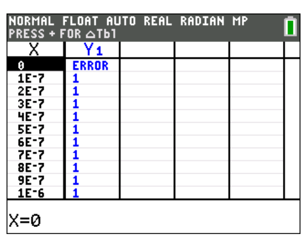
**Example B:** Let f: $x \mapsto \sin x$. Using the first principles of differentiation, show that $f'(x) = \cos x$.

**Solution:** $f(x) = \sin x$ and $f(x+\delta x) = \sin(x+\delta x)$

From first principles,
$$\frac{f(x+\delta x) - f(x)}{\delta x}$$
$$= \frac{\sin(x+\delta x) - \sin x}{\delta x}$$
$$= \frac{2\cos\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right)}{\delta x}$$
$$= \frac{2\sin\left(\frac{\delta x}{2}\right)\cos\left(x+\frac{\delta x}{2}\right)}{\delta x}$$
$$= \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}\cos\left(x+\frac{\delta x}{2}\right)$$

**"Sum to Product" Formula:**
$\sin P - \sin Q = 2\cos \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$

Use G.C. to determine the value of $\lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$ :

<table>
<tr><th></th><th>Press</th><th>Screen Display</th><th>Remarks</th></tr>
<tr><td>1.</td><td>Press [y=] and key in the given function.</td><td>
</td><td>Enter the equation $y=\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}$ .</td></tr>
<tr><td>2.</td><td>Press [graph] to view the graph of $Y_1$ in the viewing window.</td><td>
</td><td>Note that the graph is discontinuous at $x=0$ which is not clearly shown on the graph.</td></tr>
<tr><td>3.</td><td>Press [2nd] [graph] to view the table of values.</td><td>
</td><td>The table shows that at $x=0$ , $\frac{\sin(0)}{0}$ gives an error.</td></tr>
<tr><td>4.</td><td>Press [2nd] [window] to view table setup. Key in as shown in the screen on the right.</td><td>
</td><td>Set the start value of the table to 0 and the change in the $x$ value to $1 \times 10^{-7}$ .<br>The small change allows us to observe how $\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}$ behaves as $x$ approaches 0.</td></tr>
<tr><td>5.</td><td>Press [2nd] [graph] to view the table of values.</td><td>
</td><td></td></tr>
</table>

From G.C., it is observed that $\lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} = 1$ .

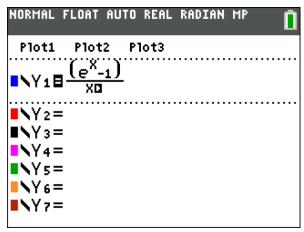
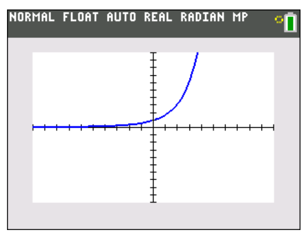
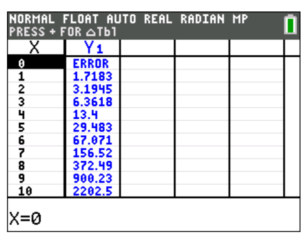
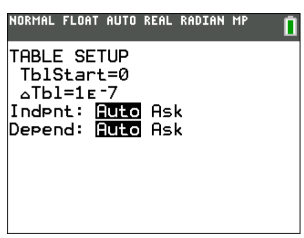
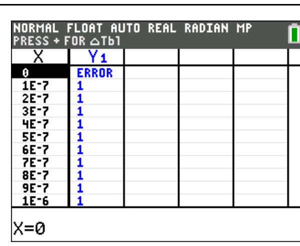
$$\therefore f'(x) = \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \cdot \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) = 1 \cdot \cos x = \cos x \quad \text{(shown)}$$

**Example C:** Let f: $x \mapsto e^x$. Using the first principles of differentiation, show that $f'(x) = e^x$.

**Solution:** $f(x) = e^x$ and $f(x + \delta x) = e^{x+\delta x}$

From first principles, $\dfrac{f(x+\delta x) - f(x)}{\delta x} = \dfrac{e^{x+\delta x} - e^x}{\delta x} = \dfrac{e^x\left(e^{\delta x} - 1\right)}{\delta x}$

Use G.C. to determine the value of $\lim\limits_{\delta x \to 0} \dfrac{\left(e^{\delta x} - 1\right)}{\delta x}$ :

| | Press | Screen Display | Remarks |
|---|---|---|---|
| 1. | Press [y=] and key in the given function. |  | Enter the equation $y = \dfrac{\left(e^x - 1\right)}{x}$. |
| 2. | Press [graph] to view the graph of $Y_1$ in the viewing window. |  | Note that the graph is discontinuous at $x$ = 0 which is not clearly shown on the graph. |
| 3. | Press [2nd] [graph] to view the table of values. |  | The table shows that at $x$ = 0, $\dfrac{\left(e^0 - 1\right)}{0}$ gives an error. |
| 4. | Press [2nd] [window] to view table setup. Key in as shown in the screen on the right. |  | Set the start value of the table to 0 and the change in the $x$ value to $1 \times 10^{-7}$.<br>The small change allows us to observe how $\dfrac{\left(e^x - 1\right)}{x}$ behaves as $x$ approaches 0. |
| 5. | Press [2nd] [graph] to view the table of values. |  | |

From G.C., it is observed that $\lim_{\delta x \to 0} \frac{\left(e^{\delta x} - 1\right)}{\delta x} = 1$

$$\therefore f'(x) = \lim_{\delta x \to 0} \frac{e^{x}\left(e^{\delta x} - 1\right)}{\delta x} = e^{x} \lim_{\delta x \to 0} \frac{\left(e^{\delta x} - 1\right)}{\delta x} = e^{x} \quad \text{(shown)}$$

---

**Example D:** Prove that $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$.

**Solution:** Let $y = \cos^{-1} x$, then $\cos y = x$.

Differentiate with respect to $x$, we have $-\sin y \frac{dy}{dx} = 1$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

But $\sin^2 y + \cos^2 y = 1 \Rightarrow \sin y = \sqrt{1-\cos^2 y}$

$$= \sqrt{1-x^2} \qquad \left[0 \le y \le \pi \Rightarrow \sin y \ge 0\right]$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

---

**Example E:** Prove that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

**Solution:** Let $y = \tan^{-1} x$, then $\tan y = x$.

Differentiate with respect to $x$, we have $\sec^2 y \frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

But $1 + \tan^2 y = \sec^2 y \Rightarrow \sec^2 y = 1 + x^2$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

## MASTERY LEARNING OBJECTIVES | Techniques of Differentiation

At the end of this chapter, I should be able to:

| | At the end of lecture… | At the end of tutorial… |
|---|---|---|
| • Find the limit of a given function | | |
| • Find the derivative of a given function using the first principles | | |
| • Apply the following rules of differentiation: | | |
| ➢ Chain Rule | | |
| ➢ Product Rule | | |
| ➢ Quotient Rule | | |
| • Use the following methods to differentiate: | | |
| ➢ Implicit differentiation | | |
| ➢ Logarithmic differentiation | | |
| • Differentiate functions involving: | | |
| ➢ Powers of $x$ | | |
| ➢ Trigonometric | | |
| ➢ Exponential | | |
| ➢ Logarithmic | | |
| ➢ Inverse Trigonometric | | |
| • Find the approximate value of a derivative at a given point using a graphing calculator | | |

***H2 MATHEMATICS TUTORIAL***
***TOPIC*** ***TECHNIQUES OF DIFFERENTIATION*** ***2022/JC1***

**DISCUSSION**

**1** Evaluate $\left.\frac{\mathrm{d}}{\mathrm{d}x}2^{3x^2+\sin^2\left(\mathrm{e}^{x^2}\right)}\right|_{x=1}$, correct to 5 significant figures.

**2** Find the derivatives of the following

**(a)** $\frac{\pi\sqrt{8100+x^2}}{x}$, **(b)** $\sec^3(2x)$, **(c)** $\sin\sqrt{x^2+1}$,

**(d)** $\log_{10}5x$, **(e)** $\ln\left[\left(x^3+2\right)\left(x^2+3\right)\right]$, **(f)** $x^2\mathrm{e}^{x\sin 2x}$,

**(g)** $\left(\frac{1}{\sin^{-1}3x}\right)^4$, **(h)** $\sin^{-1}(2\sin x)$, **(i)** $\tan^{-1}\left(\frac{1+x}{1-x}\right)$.

**3** **[2008/MJC/Promo/10]**

A curve has equation $xy^2+3\mathrm{e}^y=4x$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$ and $y$.

**4** **[2009/CJC/Promo/10(b)]**

A curve has equation $\sin^{-1}y+x\mathrm{e}^y=3x$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$ and $y$.

**5** Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given that $y=x^{\sqrt{x}}$ where $x>0$.

**6** **[N99/I/12]**

Find $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^{-1}\sqrt{\left(1-x^2\right)}\right]$.

**7** **[2015/MJC/Promo/7]**

**(a)** Differentiate the following with respect to $x$, giving your answers as single fractions.

**(i)** $\ln\left(\frac{x}{\sqrt{(1-2x)}}\right)$,

**(ii)** $\frac{1}{\cos^{-1}\left(x^2\right)}$.

**(b)** The variables $x$ and $y$ are related by

$$\mathrm{e}^{xy^2}=y\left(x^2+2\mathrm{e}^x\right).$$

Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x=0$ and $y=\frac{1}{2}$.

**Answers:**

**1** 12.017

**2** **(a)** $-\dfrac{8100\pi}{x^2\sqrt{8100+x^2}}$ **(b)** $6\sec^3(2x)\tan(2x)$ **(c)** $\dfrac{x\cos\sqrt{x^2+1}}{\sqrt{x^2+1}}$

**(d)** $\dfrac{1}{x\ln 10}$ **(e)** $\dfrac{3x^2}{x^3+2}+\dfrac{2x}{x^2+3}$ **(f)** $xe^{x\sin 2x}\left(2+x\sin 2x+2x^2\cos 2x\right)$

**(g)** $-\dfrac{12}{\left(\sin^{-1}3x\right)^5\sqrt{1-9x^2}}$ **(h)** $\dfrac{2\cos x}{\sqrt{1-4\sin^2 x}}$ **(i)** $\dfrac{1}{1+x^2}$

**3** $\dfrac{4-y^2}{2xy+3e^y}$

**4** $\dfrac{\left(3-e^y\right)\sqrt{1-y^2}}{1+xe^y\sqrt{1-y^2}}$

**5** $\dfrac{x^{\sqrt{x}}\left(2+\ln x\right)}{2\sqrt{x}}$

**6** $\dfrac{x}{|x|\sqrt{1-x^2}}$

**7** **(a)(i)** $\dfrac{1-x}{x(1-2x)}$ **(ii)** $\dfrac{2x}{\left[\cos^{-1}\left(x^2\right)\right]^2\sqrt{1-x^4}}$ **(b)** $-\dfrac{3}{8}$

## ESSENTIAL PRACTICE

**1** **[2014/PJC/Promo/1]**

Differentiate each of the following with respect to $x$.

**(i)** $e^{\sec x}$, [1]

**(ii)** $\tan^{-1}\left(x^2\right)$, [2]

**(iii)** $\cos^4(2x)$. [2]

**2** **[2013/NYJC/Promo/2]**

Differentiate the following expressions with respect to $x$, simplifying your answers as far as possible:

**(a)** $\tan^{-1}\left(\frac{2}{x}\right)$, [3]

**(b)** $\ln\sqrt{\frac{1+x}{1-x}}$. [3]

**3** **[2013/ACJC/Promo/3]**

Differentiate the following with respect to $x$.

**(i)** $\sqrt{\cos^{-1}\left(\frac{x}{2}\right)}$, [2]

**(ii)** $\ln\sqrt{\frac{(x+1)^3}{x^2-1}}$ . [2]

**4** **[2016/PJC/Promo/1]**

Differentiate with respect to $x$, giving your answers as a single fraction,

**(a)** $\ln\sqrt{a^2-x^2}$, where $a$ is a constant, [2]

**(b)** $\tan^{-1}\left(\frac{1}{2x}\right)$. [2]

**5** **[2015/CJC/Promo/1]**

Differentiate the following expressions with respect to $x$, simplifying your answers whenever possible.

**(a)** $\tan^{-1}\left(e^{3x}\right)$, [2]

**(b)** $5^{2x}$, [2]

**(c)** $\frac{ax}{\sqrt{x^3+1}}$, where $a$ is a constant . [3]

**6** **[2010/ACJC/Promo/5]**

Differentiate the following with respect to $x$, leaving your answers in terms of $x$.

**(a)** $e^{ex}\cot\left(\frac{x}{2}\right)$, [3]

**(b)** $(2x)^{(1/x)}$. [4]

**7** **[2012/ACJC/Promo/2]**

Differentiate the following with respect to $x$.

**(i)** $\cos^{-1}(\sin x)$, where $\frac{\pi}{2} < x < \frac{3\pi}{2}$. [2]

**(ii)** $\ln\sqrt{\frac{e^x+1}{1-e^{-x}}}$. [3]

**8** **[2016/CJC/Promo/1]**

Given that $y^3 + e^{\tan y} = \cos\left(2x+\frac{\pi}{3}\right)$, find $\frac{dy}{dx}$ in terms of $x$ and $y$. [4]

**9** **[2014/CJC/Promo/2]**

The variables $x$ and $y$ are related by $3^y = xy - \sec x$. Find $\frac{dy}{dx}$ in terms of $x$ and $y$. [4]

**10** **[2012/CJC/Promo/11(b)]**

Given that $y = (2x+1)^x$ for $x > 0$, find $\frac{dy}{dx}$. [3]

**11** **[2013/YJC/Promo/4a]**

It is given that $y^x = e^y x^2$. Find $\frac{dy}{dx}$ in terms of $x$ and $y$, simplifying your answer. [4]

**12** **[2014/MJC/Promo/5]**

**(a)** Differentiate $\frac{x-2x^3}{\ln x}$ with respect to $x$. [2]

**(b)** Given that $0 < x < \frac{\pi}{2}$, show that $\frac{d}{dx}\left[\sin^{-1}(\cos x)\right] = k$, where $k$ is a real constant to be determined. [3]

**(c)** Given that $e^{xy} = (1+y^2)^2$, find $\frac{dy}{dx}$ in terms of $x$ and $y$, simplifying your answer. [4]

**Answers:**

**1** **(i)** $(\sec x \tan x)e^{\sec x}$ **(ii)** $\frac{2x}{1+x^4}$ **(iii)** $-8\cos^3(2x)\sin(2x)$

**2** **(a)** $\frac{-2}{x^2+4}$ **(b)** $\frac{1}{(1-x)(1+x)}$

**3** **(i)** $-\frac{1}{2\sqrt{(4-x^2)}\cos^{-1}\left(\frac{x}{2}\right)}$ **(ii)** $\frac{1}{x+1}-\frac{1}{2(x-1)}$

**4** **(a)** $-\frac{x}{a^2-x^2}$ **(b)** $-\frac{2}{4x^2+1}$

**5** **(a)** $\frac{3e^{3x}}{1+e^{6x}}$ **(b)** $(2\ln 5)(5^{2x})$ **(c)** $a\left(x^3+1\right)^{-\frac{1}{2}}-\frac{3}{2}ax^3\left(x^3+1\right)^{-\frac{3}{2}}$

**6** **(a)** $e^{ex}\left[-\frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right)+e\cot\left(\frac{x}{2}\right)\right]$ **(b)** $\frac{(2x)^{\left(\frac{1}{x}\right)}}{x^2}\left[1-\ln(2x)\right]$

**7** **(i)** 1 **(ii)** $\frac{1}{2}\left(\frac{e^x}{e^x+1}-\frac{1}{e^x-1}\right)$

**8** $-\frac{2\sin\left(2x+\frac{\pi}{3}\right)}{3y^2+\sec^2 y\cdot e^{\tan y}}$

**9** $\frac{y-\sec x\tan x}{3^y\ln 3-x}$

**10** $(2x+1)^x\left[\frac{2x}{2x+1}+\ln(2x+1)\right]$

**11** $\frac{y(2-x\ln y)}{x(x-y)}$

**12** **(a)** $\frac{(1-6x^2)\ln x+2x^2-1}{(\ln x)^2}$ **(b)** $k=-1$ **(c)** $\frac{ye^{xy}}{4y(1+y^2)-xe^{xy}}$ or $\frac{y(1+y^2)}{4y-x(1+y^2)}$