Additional Practice Questions

1. **(J92/I/6)** The lines l_1 and l_2 have equations

$$\mathbf{r}=\begin{pmatrix}3\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\4\end{pmatrix},\ t\in\mathbb{R} \text{ and } \mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+s\begin{pmatrix}2\\1\\-1\end{pmatrix},\ s\in\mathbb{R}$$

Show that l_1 passes through the point $(2,-1,-4)$ but that l_2 does not pass through this point.

Find the acute angle between l_2 and the line joining the points $(1,-1,\ 1)$ and $(2,-1,-4)$, giving your answers correct to the nearest degree.

[Solution]

Since $\begin{pmatrix}2\\-1\\-4\end{pmatrix}=\begin{pmatrix}3\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\4\end{pmatrix}\Rightarrow t=-1$, l_1 passes through the point $(2,-1,-4)$.

Since we cannot find a value $s\in\mathbb{R}$ such that $\begin{pmatrix}2\\-1\\-4\end{pmatrix}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+s\begin{pmatrix}2\\1\\-1\end{pmatrix}$, thus l_2 does not pass through the point $(2,-1,-4)$.

Acute angle between l_2 and the line joining the points $(1,-1,\ 1)$ and $(2,-1,-4)$

$$=\cos^{-1}\left|\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\frac{1}{\sqrt{26}}\begin{pmatrix}1\\0\\-5\end{pmatrix}\right|$$

$$=\cos^{-1}\left|\frac{7}{\sqrt{6}\sqrt{26}}\right|=55.9^\circ\approx 56^\circ$$

2. **(N89/I/16)** The points A and B have position vectors $4\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ and $3\mathbf{i}+6\mathbf{j}+6\mathbf{k}$ respectively, relative to the origin O. The point P is such that $\overrightarrow{OP}=\frac{1}{2}\overrightarrow{OA}$ and the point Q is such that $\overrightarrow{OQ}=\frac{1}{3}\overrightarrow{OB}$. The point C is such that $OACB$ is a parallelogram.

(i) Find the length AB and the cosine of the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.
(ii) Find a vector equation, in parametric form, of the line PQ.
(iii) Find the position vector of the point of intersection of PQ and OC.

[Solution]

$\overrightarrow{OA}=\begin{pmatrix}4\\2\\4\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}3\\6\\6\end{pmatrix}$

(i) $\overrightarrow{AB}=\begin{pmatrix}-1\\4\\2\end{pmatrix}$. Thus $\left|\overrightarrow{AB}\right|=\sqrt{21}$.

$$\cos\measuredangle AOB=\frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|}$$

$$=\frac{48}{\sqrt{36}\sqrt{81}}=\frac{8}{9}$$

(ii) $\overrightarrow{OP}=\frac{1}{2}\begin{pmatrix}4\\2\\4\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}$, $\overrightarrow{OQ}=\frac{1}{3}\begin{pmatrix}3\\6\\6\end{pmatrix}=\begin{pmatrix}1\\2\\2\end{pmatrix}$

$\overrightarrow{PQ}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$

Vector equation of line PQ is $\mathbf{r}=\begin{pmatrix}2\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\lambda\in\mathbb{R}$

(iii) $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{OB}=\begin{pmatrix}7\\8\\10\end{pmatrix}$

Vector equation of line OC is $\mathbf{r}=\mu\begin{pmatrix}7\\8\\10\end{pmatrix}$, $\mu\in\mathbb{R}$

$$\begin{pmatrix}2\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\0\end{pmatrix}=\mu\begin{pmatrix}7\\8\\10\end{pmatrix}$$

$$\left.\begin{aligned}2-\lambda&=7\mu\\1+\lambda&=8\mu\\2&=10\mu\end{aligned}\right\}\Rightarrow\lambda=\frac{3}{5},\mu=\frac{1}{5}$$

$\therefore$ position vector of the point of intersection is $\frac{1}{5}\begin{pmatrix}7\\8\\10\end{pmatrix}$.

. Relative to an origin O, points A and B have position vectors $\mathbf{i}-2\mathbf{j}$ and $3\mathbf{j}+b\mathbf{k}$ respectively, where $b > 0$. Given that the angle between $\mathbf{a}$ and $\mathbf{b}$ is 120°, find the exact value of b.

For the following questions, take $b = 2$.

Write down a vector equation of the line passing through A and B.

(i) Find the position vector of the point P on the line AB such that $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{AB}$.

(ii) A point Q is on AB produced. Find the position vector of Q such that $AB = BQ$.

Solution:

$$\overrightarrow{OA}=\begin{pmatrix}-1\\2\\0\end{pmatrix},\quad \overrightarrow{OB}=\begin{pmatrix}0\\3\\b\end{pmatrix}$$

$$\begin{pmatrix}1\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}0\\3\\b\end{pmatrix}=\sqrt{1+4+0}\cdot\sqrt{0+9+b^2}\cos 120^\circ$$

$$-6=\sqrt{5}\cdot\sqrt{9+b^2}(-0.5)$$

$$12^2=5(9+b^2)$$

$$99=5b^2$$

$$b=\sqrt{\frac{99}{5}}\ \text{ since } b>0$$

$$\overrightarrow{OB}=\begin{pmatrix}0\\3\\2\end{pmatrix},\quad \overrightarrow{AB}=\begin{pmatrix}-1\\5\\2\end{pmatrix}$$

$$l_{AB}:\mathbf{r}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+t\begin{pmatrix}-1\\5\\2\end{pmatrix},\ t\in\mathbb{R}$$

(i)	Since P is on l_{AB}, $\overrightarrow{OP}=\begin{pmatrix}1-t\\-2+5t\\2t\end{pmatrix}$ for some t. $\overrightarrow{OP}\cdot\overrightarrow{AB}=0$ $\begin{pmatrix}1-t\\-2+5t\\2t\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\2\end{pmatrix}=0$ $-1+t-10+25t+4t=0$ $30t=11$ $t=\frac{11}{30}$

	$\therefore \overrightarrow{OP} = \begin{pmatrix} \frac{19}{30} \\ -\frac{1}{6} \\ \frac{11}{15} \end{pmatrix}$ or $\frac{1}{30}\begin{pmatrix} 19 \\ -5 \\ 22 \end{pmatrix}$
(ii)	Since Q is on l_{AB}, $\overrightarrow{OQ} = \begin{pmatrix} 1-t \\ -2+5t \\ 2t \end{pmatrix}$ for some t. $AB = BQ$: $\sqrt{1+25+4} = \left\| \begin{pmatrix} 1-t \\ -2+5t \\ 2t \end{pmatrix} \right\|$ $\sqrt{30} = \sqrt{(1-t)^2 + (-5+5t)^2 + (2t-2)^2}$ $30 = 1 - 2t + t^2 + 25 - 50t + 25t^2 + 4t^2 - 8t + 4$ $30t^2 - 60t = 0$ $t(t-2) = 0$ So, $t = 2$ or $t = 0$ (rej. as it gives point A) $\therefore \overrightarrow{OQ} = \begin{pmatrix} -1 \\ 8 \\ 4 \end{pmatrix}$

3. Relative to the origin O, the points A, B and C have position vectors $6\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}$, $-3\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $-4\mathbf{i} + 10\mathbf{j} + 6\mathbf{k}$, respectively.

(i) Find the Cartesian equation of the line AB.

(ii) Find the length of projection of $\overrightarrow{AC}$ onto the line AB.

(iii) Hence, or otherwise, find the perpendicular distance from C to the line AB, and the position vector of the foot N of the perpendicular from C to the line AB.

(iv) The point D lies on the line CN produced and is such that N is the mid-point of CD. Find the position vector of D.

Solution:

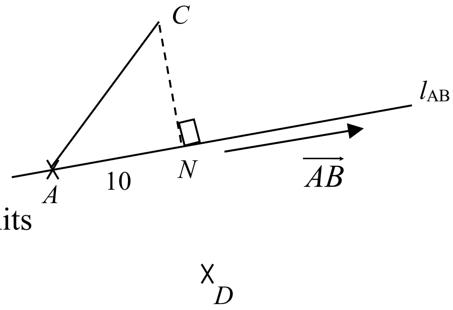
(i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix} = -3\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

Hence, a vector equation of l_{AB}: $\mathbf{r} = \begin{pmatrix} 6 \\ 5 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, $\lambda \in \mathbb{R}$

$\Rightarrow$ Cartesian equation of l_{AB}: $\dfrac{x-6}{3} = \dfrac{z-11}{4}, y = 5$.

(ii) length of projection of $\overrightarrow{AC}$ onto the line $AB = \left|\overrightarrow{AC} \bullet \widehat{\mathbf{m}}\right|$

$$= \left| \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} \bullet \frac{1}{\sqrt{9+0+16}} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right| = \left| \frac{1}{5}(-30+0-20) \right| = 10 \text{ units}$$



(iii) $AC = \sqrt{100+25+25} = \sqrt{150}$

ACN is a right angled triangle.

$CN^2 + AN^2 = AC^2$

$CN^2 + 10^2 = 150$

$\therefore$ perpendicular dist.(CN) = $\sqrt{150-10^2} = \sqrt{50} = 5\sqrt{2}$ units

$$\overrightarrow{AN} = \left(\overrightarrow{AC} \bullet \widehat{\mathbf{m}}\right)\widehat{\mathbf{m}}$$

$$\Rightarrow \overrightarrow{ON} - \overrightarrow{OA} = (-10)\widehat{\mathbf{m}}$$

$$\Rightarrow \overrightarrow{ON} = \begin{pmatrix} 6 \\ 5 \\ 11 \end{pmatrix} - 10 \cdot \left[\frac{1}{5} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}$$

and $N(0, 5, 3)$.

<u>Otherwise method</u>

Since N is a point on the line l_{AB},

So $\overrightarrow{ON} = \begin{pmatrix} 6 \\ 5 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ for <u>**some**</u> value of $\lambda \in \mathbb{R}$ to be determined.

Then $\overrightarrow{NC} = \begin{pmatrix} -4 \\ 10 \\ 6 \end{pmatrix} - \left[\begin{pmatrix} 6 \\ 5 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right] = \begin{pmatrix} -10-3\lambda \\ 5 \\ -5-4\lambda \end{pmatrix}$

and since $\overrightarrow{NC}$ and $\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ are perpendicular, so $\overrightarrow{NC} \bullet \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -10-3\lambda \\ 5 \\ -5-4\lambda \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 0$

$$\Rightarrow \quad -50 - 25\lambda = 0 \quad \Rightarrow \quad \lambda = -2$$

Thus $\overrightarrow{ON} = \begin{pmatrix} 6 \\ 5 \\ 11 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}$ and $N(0, 5, 3)$.

$\therefore$ perpendicular dist.(CN) = $\left|\overrightarrow{NC}\right| = \left|\begin{pmatrix}-4\\5\\3\end{pmatrix}\right| = \sqrt{50} = 5\sqrt{2}$ units.

(iv) Take note that point D is the reflection of point C about l_{AB}.

$$\overrightarrow{OD} = \overrightarrow{OC} + 2\overrightarrow{CN}$$

$$= \begin{pmatrix}-4\\10\\6\end{pmatrix} + 2\left[\begin{pmatrix}0\\5\\3\end{pmatrix} - \begin{pmatrix}-4\\10\\6\end{pmatrix}\right] = \begin{pmatrix}4\\0\\0\end{pmatrix}$$

. **ACJC/2017/Prelim Paper2/Q4(a)**

(i) The unit vector **d** makes angles of $60°$ with both the x- and y-axes, and θ with the z-axis, where $0° \le \theta \le 90°$. Show that **d** is parallel to $\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$.

(ii) The line m is parallel to **d** and passes through the point with coordinates $(2, -1, 0)$. Find the coordinates of the point on m that is closest to the point with coordinates $(3, 2, 0)$.

Solution:

(i)

$$\mathbf{d} = \cos 60°\mathbf{i} + \cos 60°\mathbf{j} + \cos\gamma\mathbf{k}$$

$$\cos^2 60° + \cos^2 60° + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2\gamma = 1 - \tfrac{1}{4} - \tfrac{1}{4} = \tfrac{1}{2}$$

$$\Rightarrow \cos\gamma = \tfrac{1}{\sqrt{2}} \left(\because \gamma \text{ is acute}\right)$$

$$\mathbf{d} = \tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j} + \tfrac{1}{\sqrt{2}}\mathbf{k} \,//\, \mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$$

(a)(ii)

$$m : \mathbf{r} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}$$

$$\left(\begin{pmatrix}2\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix} - \begin{pmatrix}3\\2\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix} = 0 \Rightarrow \left(\begin{pmatrix}-1\\-3\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix} = 0$$

$$\therefore (-1-3) + \lambda(1^2 + 1^2 + \sqrt{2}^2) = 0 \Rightarrow \lambda = 1$$

Therefore position vector of point is $\begin{pmatrix}2\\-1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix} = \begin{pmatrix}3\\0\\\sqrt{2}\end{pmatrix}$

Coordinates = $\left(3, 0, \sqrt{2}\right)$

<u>OR</u>

$$\overrightarrow{AN} = \left(\overrightarrow{AP} \cdot \hat{\mathbf{d}}\right)\hat{\mathbf{d}} = \frac{\left(\begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}\right) \cdot \begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}}{\sqrt{1+1+2}} \frac{\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}}{\sqrt{1+1+2}} = \begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}$$

$$\therefore \overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix} = \begin{pmatrix}3\\0\\\sqrt{2}\end{pmatrix}$$

*4. **(modified from IJC/2011/II/4)** Relative to origin O, the position vectors of two points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. The vector $\mathbf{a}$ is a unit vector which is perpendicular to $\mathbf{a} + 3\mathbf{b}$. The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{2\pi}{3}$.

(i) Show that $|\mathbf{b}| = \frac{2}{3}$.

(ii) By expanding $(\mathbf{b} - 2\mathbf{a}) \cdot (\mathbf{b} - 2\mathbf{a})$, find the exact value of $|\mathbf{b} - 2\mathbf{a}|$.

(iii) The point P divides the line AB in the ratio $\lambda : 1 - \lambda$. Find the area of triangle OAP in terms of λ.

(iv) Find $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, where N is the foot of perpendicular of the origin O to the line AB.

Solution:

(i)

$\mathbf{a} \cdot (\mathbf{a} + 3\mathbf{b}) = 0$

$|\mathbf{a}|^2 + 3\mathbf{a} \cdot \mathbf{b} = 0$

$1 + 3\mathbf{a} \cdot \mathbf{b} = 0$ since $\mathbf{a}$ is a unit vector

$\mathbf{a} \cdot \mathbf{b} = -\frac{1}{3}$

Since angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{2\pi}{3}$,

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow -\frac{1}{3} = |\mathbf{b}|\left(-\frac{1}{2}\right)$

$|\mathbf{b}| = \frac{2}{3}$

(ii)

$(\mathbf{b} - 2\mathbf{a}) \cdot (\mathbf{b} - 2\mathbf{a}) = |\mathbf{b} - 2\mathbf{a}|^2$

$|\mathbf{b}|^2 + 4|\mathbf{a}|^2 - 4\mathbf{a} \cdot \mathbf{b} = |\mathbf{b} - 2\mathbf{a}|^2$

$\left(\frac{2}{3}\right)^2 + 4 - 4\left(-\frac{1}{3}\right) = |\mathbf{b} - 2\mathbf{a}|^2$

$|\mathbf{b} - 2\mathbf{a}|^2 = \frac{52}{9}$

$\left|\mathbf{b}-2\mathbf{a}\right|=\sqrt{\frac{52}{9}}=\frac{2\sqrt{13}}{3}$

(iii)

By Ratio Theorem, $\overrightarrow{OP}=\lambda\mathbf{b}+(1-\lambda)\mathbf{a}$

Area of triangle OAP

$=\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OP}\right|=\frac{1}{2}\left|\mathbf{a}\times\left(\lambda\mathbf{b}+(1-\lambda)\mathbf{a}\right)\right|$

$=\frac{1}{2}\left|\lambda\mathbf{a}\times\mathbf{b}+(1-\lambda)\mathbf{a}\times\mathbf{a}\right|$

$=\frac{1}{2}\left|\lambda\mathbf{a}\times\mathbf{b}\right|$

$=\frac{\lambda}{2}\left|\mathbf{a}\times\mathbf{b}\right|$

$=\frac{\lambda}{2}\left|\mathbf{a}\right|\left|\mathbf{b}\right|\sin\left(\frac{2\pi}{3}\right)$

$=\frac{\lambda}{2}\left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right)=\frac{\lambda\sqrt{3}}{6}$

(iv)

Equation of line AB can be written as $l:\mathbf{r}=\mathbf{a}+\mu(\mathbf{b}-\mathbf{a}),\ \mu\in\mathbb{R}$

Since N lies on the line AB, $\overrightarrow{ON}=\mathbf{a}+\mu(\mathbf{b}-\mathbf{a})$, for some μ

$\overrightarrow{ON}=\mathbf{a}+\mu(\mathbf{b}-\mathbf{a})$

$\quad=\mu\mathbf{b}+(1-\mu)\mathbf{a}$

Since N is the foot of perpendicular of O to line AB,

$\overrightarrow{ON}\cdot\overrightarrow{AB}=0$

$\left(\mu\mathbf{b}+(1-\mu)\mathbf{a}\right)\cdot(\mathbf{b}-\mathbf{a})=0$

$\mu\mathbf{b}\cdot\mathbf{b}-\mu\mathbf{b}\cdot\mathbf{a}+(1-\mu)\mathbf{a}\cdot\mathbf{b}-(1-\mu)\mathbf{a}\cdot\mathbf{a}=0$

$\mu\left|\mathbf{b}\right|^2+(1-2\mu)\mathbf{a}\cdot\mathbf{b}-(1-\mu)\left|\mathbf{a}\right|^2=0$

$\mu\left(\frac{2}{3}\right)^2+(1-2\mu)\left(-\frac{1}{3}\right)-(1-\mu)=0$

$-\frac{4}{3}+\frac{19}{9}\mu=0$

$\mu=\frac{12}{19}$

Thus $\overrightarrow{ON}=\frac{7}{19}\mathbf{a}+\frac{12}{19}\mathbf{b}$

5. **AJC/2018 Promo/Q10(modified)**

With reference to a fixed origin O, the position vectors of points A and B are $\frac{3}{2}\mathbf{j}-5\mathbf{k}$ and $-6\mathbf{i}+8\mathbf{j}+b\mathbf{k}$ respectively. The line l_1 contains the point A and is parallel to $3\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and line l_2 is of the equation $\frac{x}{12}=\frac{y}{\alpha}=\frac{z}{4}$.

(i) Show that l_1 and l_2 do not intersect and find values that α can take if l_1 and l_2 are skew lines.

(ii) For the case that $\alpha=2$, find the angle between l_1 and l_2.

(iii) For the case that B lies on l_2, find the coordinates of the foot of perpendicular from B to l_1.

(i)

l_1: $\mathbf{r}=\begin{pmatrix}0\\1.5\\-5\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\1\end{pmatrix}$ and l_2: $\mathbf{r}=\mu\begin{pmatrix}12\\\alpha\\4\end{pmatrix}$ where $\lambda,\mu\in\mathbb{R}$

If lines intersect, $\begin{pmatrix}3\lambda\\1.5-2\lambda\\-5+\lambda\end{pmatrix}=\begin{pmatrix}12\mu\\\alpha\mu\\4\mu\end{pmatrix}\begin{matrix}-(1)\\-(2)\\-(3)\end{matrix}$

From (1) $\lambda=4\mu$, sub into (3),

LHS$=-5+\lambda=-5+4\mu$ and RHS = $4\mu \Rightarrow$ no solution.

Therefore, the system is inconsistent and the lines do not intersect. (shown)

For values of α such that lines are skew,

lines cannot be parallel,

i.e. $\begin{pmatrix}3\\-2\\1\end{pmatrix}\neq k\begin{pmatrix}12\\\alpha\\4\end{pmatrix}$

so $-2\neq\frac{1}{4}\alpha$, i.e. $\alpha\neq-8$

Hence, α can take all real values except –8.

(ii)

For $\alpha=2$, l_2: $\mathbf{r}=\mu\begin{pmatrix}12\\2\\4\end{pmatrix}$ where $\mu\in\mathbb{R}$

Angle between l_1 and l_2 $=\cos^{-1}\left|\widehat{\begin{pmatrix}3\\-2\\1\end{pmatrix}}\cdot\widehat{\begin{pmatrix}12\\2\\4\end{pmatrix}}\right|=\cos^{-1}\frac{36}{\sqrt{14}\sqrt{164}}=41.3°$ (to 1 dp)

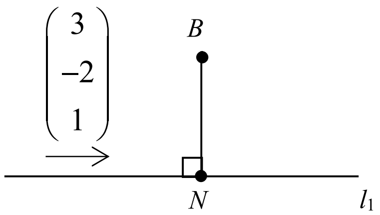
(iii)

B lies on l_2: let $\mu\begin{pmatrix}12\\ \alpha\\ 4\end{pmatrix}=\begin{pmatrix}-6\\ 8\\ b\end{pmatrix}$ $\Rightarrow$ $\mu=-0.5$, $\alpha=-16$ and $b=-2$

$\therefore \overrightarrow{OB}=\begin{pmatrix}-6\\ 8\\ -2\end{pmatrix}$

Let the foot of perpendicular from B to l_1 be N.

Since N is on l_1, $\overrightarrow{ON}=\begin{pmatrix}3\lambda\\ 1.5-2\lambda\\ -5+\lambda\end{pmatrix}$ for **<u>some</u>** $\lambda\in\mathbb{R}$



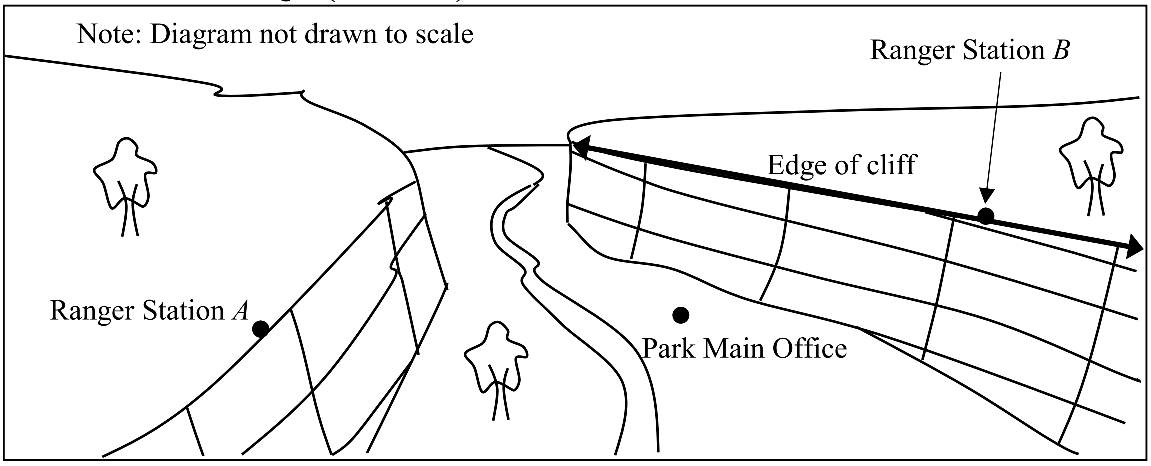
$$\overrightarrow{BN}\bullet\begin{pmatrix}3\\ -2\\ 1\end{pmatrix}=0\Rightarrow\begin{pmatrix}3\lambda+6\\ -6.5-2\lambda\\ -3+\lambda\end{pmatrix}\bullet\begin{pmatrix}3\\ -2\\ 1\end{pmatrix}=0$$

$$\Rightarrow 9\lambda+18+13+4\lambda-3+\lambda=0$$

$$\Rightarrow \lambda=-2$$

$\therefore \overrightarrow{ON}=\begin{pmatrix}-6\\ 5.5\\ -7\end{pmatrix}$, i.e. $N=(-6,\ 5.5,-7)$

6. **EJC/2018 Promo/Q13(modified)**



A national park sits across a canyon with two steep cliff walls (see diagram). The park main office sits at the origin $(0,0,0)$, which is located within the canyon between the cliff walls. Points (x, y, z) are defined with respect to this origin.

Ranger Station A is located at $(-6, 15, 26)$ on the edge of one cliff. Ranger Station B lies on the edge of the other cliff, which follows the line described by the vector equation $\mathbf{r} = \begin{pmatrix} -10 \\ -4 \\ 35 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$ where $\lambda \in \mathbb{R}$. At the location of Ranger Station B, $\lambda = 1$.

(i) It is proposed to build a straight bridge across the canyon to join Ranger Station A and Ranger Station B. Assume that the width and thickness of the bridge are negligible. Find the length of the bridge, and the vector equation of the line representing the bridge. [4]

(ii) The bridge proposed in part **(i)** is found to be too steep, so an alternative bridge is proposed. This new bridge will join Ranger Station A with a point C on the edge of the opposite cliff, where point C is chosen such that the length of the bridge is minimised.
Find the length of the new proposed bridge and the position vector of point C.[5]

(i) $\overrightarrow{OB} = \begin{pmatrix} -10 \\ -4 \\ 35 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -7 \\ -3 \\ 32 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} -7 \\ -3 \\ 32 \end{pmatrix} - \begin{pmatrix} -6 \\ 15 \\ 26 \end{pmatrix} = \begin{pmatrix} -1 \\ -18 \\ 6 \end{pmatrix}$

$\left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + (-18)^2 + 6^2} = \sqrt{361} = 19$

Equation of bridge is $\mathbf{r} = \begin{pmatrix} -6 \\ 15 \\ 26 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -18 \\ 6 \end{pmatrix}, \mu \in \mathbb{R}, 0 \le \mu \le 1$ **or**

$\mathbf{r} = \begin{pmatrix} -7 \\ -3 \\ 32 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -18 \\ 6 \end{pmatrix}, \mu \in \mathbb{R}, -1 \le \mu \le 0$

(ii) $\overrightarrow{OC} = \begin{pmatrix} -10+3\lambda \\ -4+\lambda \\ 35-3\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} -4+3\lambda \\ -19+\lambda \\ 9-3\lambda \end{pmatrix}$

Minimum length of $\left|\overrightarrow{AC}\right|$ occurs when

$\overrightarrow{AC} \cdot \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix} = 0$ i.e. $\begin{pmatrix} -4+3\lambda \\ -19+\lambda \\ 9-3\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} = 0$

$19\lambda - 58 = 0$

$\lambda = \frac{58}{19}$ or $3\frac{1}{9}$

Substituting $\lambda = \frac{58}{19}$ into $\overrightarrow{OC}$ gives

$\overrightarrow{OC} = \begin{pmatrix} -16/19 \\ -18/19 \\ 491/19 \end{pmatrix}$ **or** $\overrightarrow{OC} = \begin{pmatrix} -0.842 \\ -0.947 \\ 25.8 \end{pmatrix}$ (to 3 s.f.)

$\overrightarrow{AC} = \begin{pmatrix} 98/19 \\ -303/19 \\ -3/19 \end{pmatrix}$

Minimum length $= \sqrt{\frac{5338}{19}} = \frac{\sqrt{101422}}{19}$

≈ 16.8 (to 3 s.f.)

ALTERNATIVELY

$$\left|\overrightarrow{AC}\right| = \sqrt{(-4+3\lambda)^2 + (-19+\lambda)^2 + (9-3\lambda)^2}$$

$$= \sqrt{19\lambda^2 - 116\lambda + 458}$$

$$= \sqrt{19\left(\lambda - \frac{58}{19}\right)^2 + \frac{5338}{19}}$$

Minimum length $= \sqrt{\frac{5338}{19}} = \frac{\sqrt{101422}}{19}$

≈ 16.8 (to 3 s.f.)

Minimum occurs when $\lambda = \frac{58}{19}$ or $3\frac{1}{9}$

Substituting $\lambda = \frac{58}{19}$ into $\overrightarrow{OC}$ gives

$\overrightarrow{OC} = \begin{pmatrix} -16/19 \\ -18/19 \\ 491/19 \end{pmatrix}$ **or** $\overrightarrow{OC} = \begin{pmatrix} -0.842 \\ -0.947 \\ 25.8 \end{pmatrix}$ (to 3 s.f.)

7. 2021 DHS Prelim/ P1/ Q9 (part of)

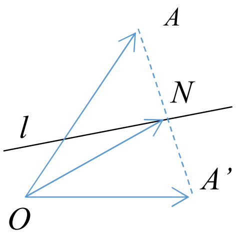
Points A and B have position vectors $\begin{pmatrix}1\\q\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ respectively, where q is a positive constant. The equation of the line l is $\frac{4x-15}{-2}=y,\ z=\frac{5}{2}$.

(i) Given that the length of projection of vector $\overrightarrow{AB}$ onto line l is $\frac{8}{\sqrt{5}}$, show that $q=5$. [3]

(ii) Find the coordinates of the point of A', where A' is the image of point A reflected in the line l. [4]

(i)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-2\\-q\\-1\end{pmatrix}$ $l: r=\begin{pmatrix}\frac{15}{4}\\0\\\frac{5}{2}\end{pmatrix}+\lambda\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix},\lambda\in\mathbb{R}$ $\frac{\left\|\overrightarrow{AB}\bullet\mathbf{d}_l\right\|}{\left\|\mathbf{d}_l\right\|}=\frac{8}{\sqrt{5}}$ $\frac{\left\|\begin{pmatrix}-2\\-q\\-1\end{pmatrix}\bullet\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}\right\|}{\left\|\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}\right\|}=\frac{8}{\sqrt{5}}$ $\|1-q\|=\frac{8}{\sqrt{5}}\left(\frac{\sqrt{5}}{2}\right)$ $1-q=4$ or $1-q=-4$ $q=-3$ (reject $\because q>0$) or $q=5$

(ii)



$$\overrightarrow{ON}=\begin{pmatrix}\frac{15}{4}-\frac{1}{2}\lambda\\ \lambda\\ \frac{5}{2}\end{pmatrix}\text{, for some } \lambda\in\mathbb{R}$$

$$\overrightarrow{AN}=\overrightarrow{ON}-\overrightarrow{OA}=\begin{pmatrix}\frac{15}{4}-\frac{1}{2}\lambda\\ \lambda\\ \frac{5}{2}\end{pmatrix}-\begin{pmatrix}1\\5\\2\end{pmatrix}=\begin{pmatrix}\frac{11}{4}-\frac{1}{2}\lambda\\ \lambda-5\\ \frac{1}{2}\end{pmatrix}$$

$$\overrightarrow{AN}\bullet\mathbf{d}_l=0$$

$$\begin{pmatrix}\frac{11}{4}-\frac{1}{2}\lambda\\ \lambda-5\\ \frac{1}{2}\end{pmatrix}\bullet\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}=0$$

$$\lambda=\frac{51}{10}$$

$$\therefore\overrightarrow{ON}=\begin{pmatrix}\frac{15}{4}-\frac{1}{2}\left(\frac{51}{10}\right)\\ \frac{51}{10}\\ \frac{5}{2}\end{pmatrix}=\begin{pmatrix}\frac{12}{10}\\ \frac{51}{10}\\ \frac{5}{2}\end{pmatrix}=\frac{1}{10}\begin{pmatrix}12\\51\\25\end{pmatrix}$$

Using ratio theorem,

$$\overrightarrow{ON}=\frac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$$

$$\overrightarrow{OA'}=2\overrightarrow{ON}-\overrightarrow{OA}=\frac{2}{10}\begin{pmatrix}12\\51\\25\end{pmatrix}-\begin{pmatrix}1\\5\\2\end{pmatrix}=\frac{1}{10}\begin{pmatrix}14\\52\\30\end{pmatrix}=\frac{1}{5}\begin{pmatrix}7\\26\\15\end{pmatrix}$$

Coordinates of A' $=\left(\frac{7}{5},\frac{26}{5},3\right)$

8 2022 AJC Promo Q7(a)

Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non parallel vectors. Point C lies on OA, between O and A, such that $OC : CA = 2 : 1$. Point D lies on OB produced such that $BD = 5OB$.

Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point E where the lines AB and CD meet.

	$\overrightarrow{OE} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$ $= \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$ $= (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$ $\overrightarrow{OE} = \overrightarrow{OC} + \mu\overrightarrow{CD}$ $= \frac{2}{3}\mathbf{a} + \mu\left(\mathbf{6b} - \frac{2}{3}\mathbf{a}\right)$ $= \frac{2}{3}(1-\mu)\mathbf{a} + 6\mu\mathbf{b}$ Comparing the coefficients of $\mathbf{a}$ and $\mathbf{b}$, $(1-\lambda) = \frac{2}{3}(1-\mu)$ $\lambda = 6\mu$ $\lambda = \frac{6}{16}, \ \mu = \frac{1}{16}$ $\overrightarrow{OE} = \left(1 - \frac{6}{16}\right)\mathbf{a} + \frac{6}{16}\mathbf{b} = \frac{5}{8}\mathbf{a} + \frac{\mathbf{3}}{\mathbf{8}}\mathbf{b}$	