2015 H2 Math Prelim Paper 1 Solutions

1

$$\frac{5x^2-x-14}{2x^2+x-3}\le 3$$

$$\frac{5x^2-x-14-3(2x^2+x-3)}{(2x+3)(x-1)}\le 0,\ \ x\ne -\frac{3}{2}, x\ne 1$$

$$\frac{5x^2-x-14-6x^2-3x+9}{(2x+3)(x-1)}\le 0$$

$$\frac{-x^2-4x-5}{(2x+3)(x-1)}\le 0$$

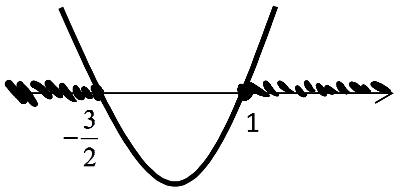
$$\frac{(x^2+4x)+5}{(2x+3)(x-1)}\ge 0$$

$$\frac{(x+2)^2-4+5}{(2x+3)(x-1)}\ge 0$$

$$\frac{(x+2)^2+1}{(2x+3)(x-1)}\ge 0$$

Since $(x+2)^2+1>0$ for all $x\in\mathbb{R}$

Therefore,
$(2x+3)(x-1)\ge 0$

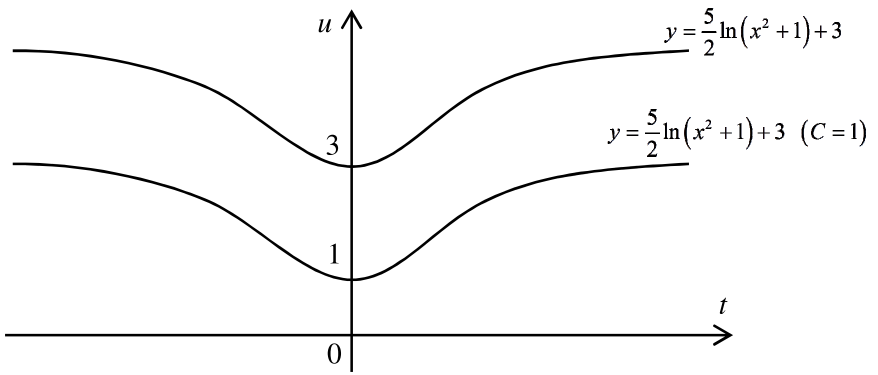


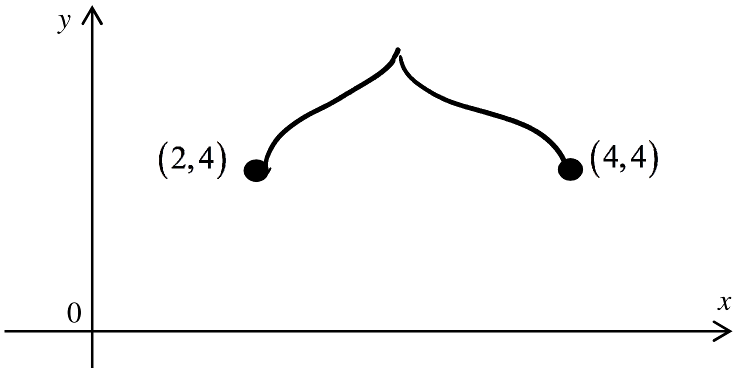
	Hence, $x \le -\frac{3}{2}$ or $x \ge 1$. Since $x \ne -\frac{3}{2}$ and $x \ne 1$, $x < -\frac{3}{2}$ or $x > 1$.
2	Let P_n be the statement $\sin x + \sin 11x + \sin 21x + ... + \sin(10n+1)x = \frac{\cos 4x - \cos(10n+6)x}{2\sin 5x}$ for $n = 0,1,2,3,...$ When $n = 0$, LHS $= \sin x$ $\text{RHS} = \frac{\cos 4x - \cos 6x}{2\sin 5x}$ $= \frac{-2\sin 5x \sin(-x)}{2\sin 5x} = \frac{2\sin 5x \sin x}{2\sin 5x}$ $= \sin x = \text{LHS}$ Hence P_0 is true. Assume P_k is true for some $k \in \{0,1,2,3,...\}$, i.e. $\sin x + \sin 11x + \sin 21x + ... + \sin(10k+1)x = \frac{\cos 4x - \cos(10k+6)x}{2\sin 5x}$. To prove P_{k+1} is true, i.e. $\sin x + \sin 11x + ... + \sin(10(k+1)+1)x = \frac{\cos 4x - \cos(10k+16)x}{2\sin 5x}$.

	$\text{LHS} = \sin x + \sin 11x + ... + \sin(10k+1)x + \sin(10k+11)x$ $= \dfrac{\cos 4x - \cos(10k+6)x}{2\sin 5x} + \sin(10k+11)x$ $= \dfrac{\cos 4x - \cos(10k+6)x + 2\sin(10k+11)x\sin 5x}{2\sin 5x}$ $= \dfrac{\cos 4x - \cos(10k+6)x + \cos(10k+6)x - \cos(10k+16)x}{2\sin 5x}$ $= \dfrac{\cos 4x - \cos(10k+16)x}{2\sin 5x} = RHS$ Hence P_k is true implies P_{k+1} is true. Since P_0 is true, and P_k is true implies P_{k+1} is true, by Mathematical induction, P_n is true for all $n \in \{0,1,2,3,..\}$
3(i)	$\text{f}(x) = \dfrac{3x-5}{x-2} = 3 + \dfrac{1}{x-2}$. $\text{f}'(x) = -\dfrac{1}{(x-2)^2}$ $\text{f}'(x) < 0$ for all $x \in \mathbb{R}, x \neq 2$ since $(x-2)^2 > 0$ for all $x \in \mathbb{R}, x \neq 2$. Hence, f is decreasing on any interval in the domain.
3(ii)	From graph of $y = \text{f}(x)$, $\text{D}_{\text{f}^{-1}} = \text{R}_\text{f} = \mathbb{R} \setminus \{3\}$. Let $y = \text{f}(x)$

	$y-3=\dfrac{1}{x-2}$ $x-2=\dfrac{1}{y-3}$ $x=2+\dfrac{1}{y-3}=\dfrac{2y-5}{y-3}$ Hence, $f^{-1}:x\mapsto\dfrac{2x-5}{x-3}$, for $x\in\mathbb{R}, x\neq 3$.
4(i)	$x^3y^2+x^2y^3=1$ Differentiate with respect to x: $x^3\left(2y\dfrac{dy}{dx}\right)+y^2\left(3x^2\right)+x^2\left(3y^2\dfrac{dy}{dx}\right)+y^3\left(2x\right)=0$ $\dfrac{dy}{dx}\left(2x^3y+3x^2y^2\right)=-\left(2y^3x+3x^2y^2\right)$ $\dfrac{dy}{dx}=-\dfrac{xy^2\left(2y+3x\right)}{x^2y\left(2x+3y\right)}$ For stationary point, $\dfrac{dy}{dx}=0$. Since $x\neq 0, y\neq 0$: $2y+3x=0\Rightarrow y=-\dfrac{3}{2}x$ or $x=-\dfrac{2}{3}y$ Substitute back into equation of curve: $x^3\left(-\dfrac{3}{2}x\right)^2+x^2\left(-\dfrac{3}{2}x\right)^3=1$ $\dfrac{9}{4}x^5-\dfrac{27}{8}x^5=1$ $-\dfrac{9}{8}x^5=1$

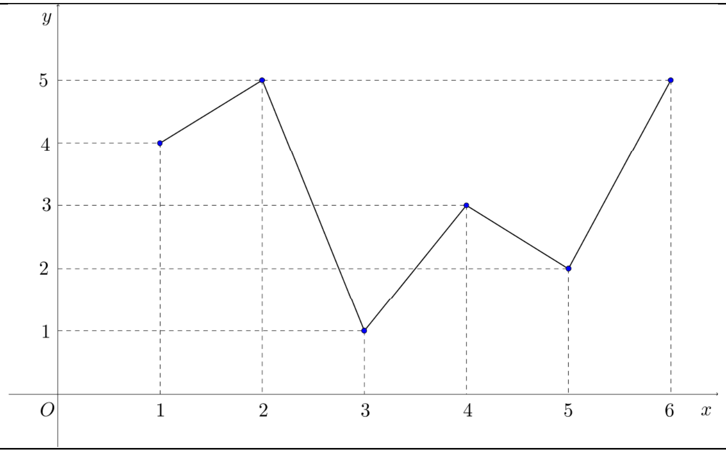
	$x=-\sqrt[5]{\frac{8}{9}}$ $y=-\frac{3}{2}\left(-\sqrt[5]{\frac{8}{9}}\right)=\frac{3}{2}\sqrt[5]{\frac{8}{9}}$ Hence, the coordinates of A is $\left(-\sqrt[5]{\frac{8}{9}},\frac{3}{2}\sqrt[5]{\frac{8}{9}}\right)$ or $\left(-\frac{2}{3}\sqrt[5]{\frac{27}{4}},\sqrt[5]{\frac{27}{4}}\right)$
(ii)	Since B is the reflection of A in $y=x$, the coordinates of B is $\left(\frac{3}{2}\sqrt[5]{\frac{8}{9}},-\sqrt[5]{\frac{8}{9}}\right)$
5(i)	Transformation 1: stretch with scale factor *k* parallel to *x*-axis Transformation 2: m units in positive *x*-direction Transformation 3: n units in negative *y*-direction $C_1:\frac{x^2}{6^2}+\frac{y^2}{3^2}=1\xrightarrow{\text{Trans 1}}\frac{\left(\frac{x}{k}\right)^2}{6^2}+\frac{y^2}{3^2}=1$ $\xrightarrow{\text{Trans 2}}\frac{(x-m)^2}{(6k)^2}+\frac{y^2}{3^2}=1\xrightarrow{\text{Trans 3}}\frac{(x-m)^2}{(6k)^2}+\frac{(y+n)^2}{3^2}=1$ Final equation: $C_2:\frac{(x-m)^2}{(6k)^2}+\frac{(y+n)^2}{3^2}=1$
5(ii)	If C_2 is a circle with centre (4, −7), then $\frac{(x-m)^2}{(6k)^2}+\frac{(y+n)^2}{3^2}=1$ to $\frac{(x-4)^2}{(6k)^2}+\frac{(y+7)^2}{3^2}=1$ means $m=4,n=7$

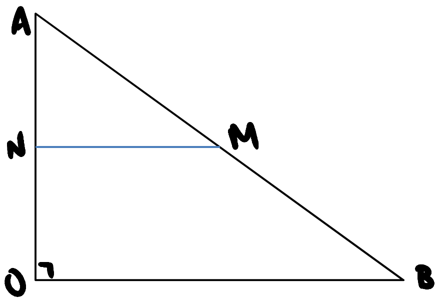
	and $6k = 3 \Rightarrow k = \frac{1}{2}$
6 (i)	$\frac{du}{dt} = \frac{5t}{t^2+1}$ Integrating both sides with respect to t, $u = \frac{5}{2}\int \frac{2t}{t^2+1}\, dt$ $= \frac{5}{2}\ln\left(t^2+1\right) + C,$ since $\left(t^2+1\right) > 0,$ where C is arbitrary constant. Substitute values $t = 0$ and $u = 3$: $C = 3$ Particular solution is $u = \frac{5}{2}\ln\left(t^2+1\right) + 3$.
6 (ii)	As $t \to \pm\infty$, $\frac{du}{dt} = \frac{5t}{t^2+1} \to 0$. The gradient of every solution curve tends towards zero as $t \to \pm\infty$.
6 (iii)	

7 (i)	
7 (ii)	Area $= \int_2^4 y \, dx - 4(4-2)$ $= \int_0^\pi (4+\sin\theta)(3\cos^2\theta\sin\theta) \, d\theta - 8$ $= 12\int_0^\pi \cos^2\theta\sin\theta \, d\theta + 3\int_0^\pi \cos^2\theta\sin^2\theta \, d\theta - 8$ $= -12\int_0^\pi \cos^2\theta(-\sin\theta) \, d\theta + \frac{3}{4}\int_0^\pi \sin^2 2\theta \, d\theta - 8$ $= -12\left[\frac{\cos^3\theta}{3}\right]_0^\pi + \frac{3}{8}\int_0^\pi (1-\cos 4\theta) \, d\theta - 8$ $= -4(-1-1) + \frac{3}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^\pi - 8$ $= \frac{3\pi}{8}$ units^2

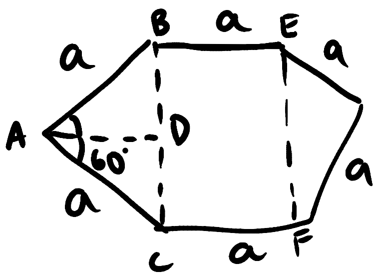
8 (i)	$\frac{2}{r}-\frac{3}{r+1}+\frac{1}{r+2}\equiv\frac{2(r+1)(r+2)-3r(r+2)+r(r+1)}{r(r+1)(r+2)}$ $\equiv\frac{2r^2+6r+4-3r^2-6r+r^2+r}{r(r+1)(r+2)}$ $\equiv\frac{r+4}{r(r+1)(r+2)}$
(ii)	$S_n=\sum_{r=1}^{n}\frac{r+4}{r(r+1)(r+2)}$ $=\sum_{r=1}^{n}\left(\frac{2}{r}-\frac{3}{r+1}+\frac{1}{r+2}\right)$ $\begin{array}{llll} =(& \frac{2}{1} & -\frac{3}{2} & +\cancel{\frac{1}{3}} \\ & +\frac{2}{2} & -\cancel{\frac{3}{3}} & +\cancel{\frac{1}{4}} \\ & +\cancel{\frac{2}{3}} & -\cancel{\frac{3}{4}} & +\cancel{\frac{1}{5}} \\ & \vdots & \vdots & \vdots \\ & +\cancel{\frac{2}{n-2}} & -\cancel{\frac{3}{n-1}} & +\cancel{\frac{1}{n}} \\ & \cancel{+\frac{2}{n-1}} & -\cancel{\frac{3}{n}} & +\frac{1}{n+1} \\ & +\cancel{\frac{2}{n}} & -\frac{3}{n+1} & +\frac{1}{n+2}) \end{array}$ $=\left(\frac{2}{1}-\frac{3}{2}+\frac{2}{2}+\frac{1}{n+1}-\frac{3}{n+1}+\frac{1}{n+2}\right)$ $=\frac{3}{2}-\frac{2}{n+1}+\frac{1}{n+2}$

(iii)	$\sum_{r=2}^{n}\frac{r^2+3r-4}{r\left(r^2-1\right)(r+2)}=\sum_{r=2}^{n}\frac{(r+4)(r-1)}{r\left(r-1\right)\left(r+1\right)(r+2)}$ $=\sum_{r=2}^{n}\frac{r+4}{r\left(r+1\right)\left(r+2\right)}$ $=\sum_{r=1}^{n}\frac{r+4}{r\left(r+1\right)\left(r+2\right)}-\frac{5}{6}$ $=\frac{3}{2}-\frac{2}{n+1}+\frac{1}{n+2}-\frac{5}{6}$ $=\frac{2}{3}-\frac{2}{n+1}+\frac{1}{n+2}$
9(i)	Since $\mathrm{g}\left(2\right)=\mathrm{g}\left(6\right)=5$, the function g is not one-to-one and hence does not have an inverse function.
(ii)	$\mathrm{g}^3\left(3\right)=\mathrm{ggg}\left(3\right)=\mathrm{gg}\left(1\right)=\mathrm{g}\left(4\right)=3$. Since $\mathrm{g}^2\left(3\right)=4$ and $\mathrm{g}^3\left(3\right)=3$, n can be $2,5,8,...$ The set of values of n is $\left\{3k-1:k\in\mathbb{Z}^+\right\}$ (Also accept answers such as $\left\{2,5,8,11,...\right\}$, $\left\{3k+2:k=0,1,2,3,...\right\}$ etc.)
(iii)	$\mathrm{g}\left(x\right)=\mathrm{g}\left(1\right)+\left(\mathrm{g}\left(2\right)-\mathrm{g}\left(1\right)\right)\left(x-1\right)$, for $1<x<2$ $\mathrm{g}\left(1.5\right)=4+\left(5-4\right)\left(1.5-1\right)$ $=4+1\left(0.5\right)$ $=4.5$. $\mathrm{g}\left(x\right)=\mathrm{g}\left(2\right)+\left(\mathrm{g}\left(3\right)-\mathrm{g}\left(2\right)\right)\left(x-1\right)$, for $2<x<3$ $\mathrm{g}\left(2.7\right)=5+\left(1-5\right)\left(2.7-2\right)$ $=5-4\left(0.7\right)$ $=2.2$.

(iv)	
(v)	When $g(x)=k$ has four real distinct roots, the graph of $y=k$ intersects the graph of $y=g(x)$ at four distinct points. From (iv), $2<k<3$.
10(i)	$\dfrac{\overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|}=\dfrac{\mathbf{b}-\mathbf{a}}{\|\mathbf{b}-\mathbf{a}\|}$
10 (ii)	By Sine Rule, $\dfrac{AM}{\sin\frac{\pi}{6}}=\dfrac{\left\|\overrightarrow{OA}\right\|}{\sin\frac{2\pi}{3}}$ $AM=\dfrac{\|\mathbf{a}\|}{\left(\frac{\sqrt{3}}{2}\right)}\left(\dfrac{1}{2}\right)$ $=\dfrac{1}{\sqrt{3}}\|\mathbf{a}\|$ Hence,

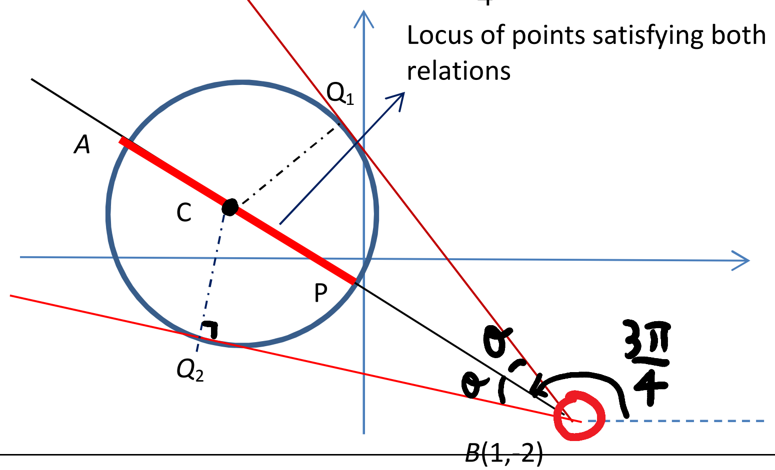
	$\overrightarrow{AM} = \frac{1}{\sqrt{3}}\lvert\mathbf{a}\rvert\left(\frac{\mathbf{b}-\mathbf{a}}{\lvert\mathbf{b}-\mathbf{a}\rvert}\right)$ $= \frac{\lvert\mathbf{a}\rvert}{\sqrt{3}\lvert\mathbf{b}-\mathbf{a}\rvert}(\mathbf{b}-\mathbf{a})$
10 (iii)	 Shortest distance from M to line OA $= \left\lvert\overrightarrow{AM} \times \frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}\right\rvert$ $= \left\lvert\frac{\lvert\mathbf{a}\rvert}{\sqrt{3}\lvert\mathbf{b}-\mathbf{a}\rvert}(\mathbf{b}-\mathbf{a}) \times \frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}\right\rvert$ $= \frac{1}{\sqrt{3}\lvert\mathbf{b}-\mathbf{a}\rvert}\lvert(\mathbf{b}-\mathbf{a})\times\mathbf{a}\rvert$ $= \frac{1}{\sqrt{3}\lvert\mathbf{b}-\mathbf{a}\rvert}\lvert\mathbf{b}\times\mathbf{a}-\mathbf{a}\times\mathbf{a}\rvert$ $= \frac{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}{\sqrt{3}\lvert\mathbf{b}-\mathbf{a}\rvert}$, since $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ and $\lvert\mathbf{a}\times\mathbf{b}\rvert = \left\lvert\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\left(\frac{\pi}{2}\right)\right\rvert$ **<u>Alternative Method</u>**

	Shortest distance from M to line OA $= \text{projection of } \overrightarrow{\text{AM}} \text{ onto } \overrightarrow{\text{OB}}$ $= \left\| \overrightarrow{AM} \bullet \frac{\mathbf{b}}{\|\mathbf{b}\|} \right\|$ $= \left\| \frac{\|\mathbf{a}\|}{\sqrt{3}\|\mathbf{b}-\mathbf{a}\|}(\mathbf{b}-\mathbf{a}) \bullet \frac{\mathbf{b}}{\|\mathbf{b}\|} \right\|$ $= \frac{\|\mathbf{a}\|}{\sqrt{3}\|\mathbf{b}\|\|\mathbf{b}-\mathbf{a}\|} \|(\mathbf{b}-\mathbf{a}) \bullet \mathbf{b}\|$ $= \frac{\|\mathbf{a}\|}{\sqrt{3}\|\mathbf{b}\|\|\mathbf{b}-\mathbf{a}\|} \|\mathbf{b} \bullet \mathbf{b} - \mathbf{a} \bullet \mathbf{b}\|$ $= \frac{\|\mathbf{a}\|\|\mathbf{b}\|}{\sqrt{3}\|\mathbf{b}\|\|\mathbf{b}-\mathbf{a}\|}$ (Since $\mathbf{b} \bullet \mathbf{b} = \|\mathbf{b}\|^2$ and $\mathbf{a} \bullet \mathbf{b} = 0$) $= \frac{\|\mathbf{a}\|}{\sqrt{3}\|\mathbf{b}-\mathbf{a}\|}$
11 (i)	$\ln(1+y) = \tan^{-1} x$ Differentiate w.r.t. x $\frac{1}{1+y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$ $(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 1+y$ (Shown)
(ii)	Differentiate w.r.t. x: $(1+x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}$

	$\left(1+x^2\right)\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2x\frac{d^2y}{dx^2}=\frac{d^2y}{dx^2}$ When $x=0$, $y=e^{\tan^{-1}x}-1=0, \frac{dy}{dx}=1, \frac{d^2y}{dx^2}=1, \frac{d^3y}{dx^3}=-1$ $\therefore \quad y=x+\frac{x^2}{2}-\frac{x^3}{6}+....$
(iii) (a) (b)	$\int_0^{\frac{1}{2}}\left(e^{\tan^{-1}x}-1\right)dx \approx \int_0^{\frac{1}{2}}\left(x+\frac{x^2}{2}-\frac{x^3}{6}\right)dx$ $\approx 0.14323 \approx 0.143$ (to 3sf) Using GC, $\int_0^{\frac{1}{2}}\left(e^{\tan^{-1}x}-1\right)dx = 0.141709578 \approx 0.142$ (3 s.f) The approximation will be better if more terms in the Maclaurin's series are included in the integral.
12(i)	 For a regular hexagon, each internal angle is $\frac{(6-2)\times180^0}{6}=120^0$. Consider the triangle ADC:

	$\sin 60^0 = \frac{DC}{a}$ and $\cos 60^0 = \frac{AD}{a}$ $DC = \frac{a\sqrt{3}}{2}$ and $AD = \frac{1}{2}a$ $= 2(\text{Area of triangle ABC}) + \text{Area of rect BCFE}$ Area of the hexagon $= 2\left(\frac{1}{2}\times\frac{1}{2}a\times a\sqrt{3}\right) + a^2\sqrt{3}$ $= \frac{a^2\sqrt{3}}{2} + a^2\sqrt{3} = \frac{3\sqrt{3}}{2}a^2$
(ii)	Given that the volume is 100, $V = \frac{3\sqrt{3}}{2}a^2 h = 100$ Thus, $h = \frac{100(2)}{3\sqrt{3}a^2} = \frac{200}{3\sqrt{3}a^2}$ Surface Area, A $= 6ah + 6kah + 3\sqrt{3}a^2$ $= 6ah(k+1) + 3\sqrt{3}a^2$ $= \frac{6a(k+1)200}{3\sqrt{3}a^2} + 3\sqrt{3}a^2$ $= \frac{400(k+1)}{\sqrt{3}a} + 3\sqrt{3}a^2$ $\frac{dA}{da} = -\frac{400(k+1)}{\sqrt{3}a^2} + 6\sqrt{3}a$ For stationary points, $\frac{dA}{da} = 0$

	$\frac{400(k+1)}{\sqrt{3}a^2}=6\sqrt{3}a$ $400(k+1)=18a^3$ $a^3=\frac{400(k+1)}{18}=\frac{200(k+1)}{9}$ $a=\sqrt[3]{\frac{200(k+1)}{9}}$ $\frac{dA}{da}=-\frac{400(k+1)}{\sqrt{3}a^2}+6\sqrt{3}a$ $\Rightarrow \frac{d^2A}{da^2}=\frac{800(k+1)}{\sqrt{3}a^3}+6\sqrt{3}>0$ Thus, $a=\sqrt[3]{\frac{200(k+1)}{9}}$ gives a minimum surface area.
(iii)	$\frac{h}{a}=\frac{200}{3\sqrt{3}a^3}=\frac{200}{3\sqrt{3}\left(\frac{200(k+1)}{9}\right)}=\frac{3}{\sqrt{3}(k+1)}=\frac{\sqrt{3}}{(k+1)}$
(iv)	$0<k\le 1$ $1<k+1\le 2$ $\frac{1}{2}\le\frac{1}{k+1}<1$ $\frac{\sqrt{3}}{2}\le\frac{\sqrt{3}}{k+1}<\sqrt{3}\Rightarrow\frac{\sqrt{3}}{2}\le\frac{h}{a}<\sqrt{3}$

13(a)	$w=\dfrac{z-2i}{z+4}$, where $z\neq -4$, Let $z = x+iy$, $w=\dfrac{(x+iy)-2i}{(x+iy)+4}\bullet\dfrac{(x+4)-iy}{(x+4)-iy}$ $=\dfrac{(x^2+4x+y(y-2))+i(-xy+x(y-2)+4(y-2)}{(x+4)^2+y^2}$ If Re(w) = 0, then $\dfrac{x^2+4x+y^2-2y}{(x+4)^2+y^2}=0$, $\Rightarrow x^2+4x+y^2-2y=0$ $\Rightarrow(x+2)^2 - 4+ (y-1)^2- 1= 0$ $\Rightarrow(x+2)^2 + (y-1)^2= (\sqrt{5})^2$ $\therefore$ The locus of P is a circle with centre at (-2,1) and radius $\sqrt{5}$ units (Shown)
13(b)	$\lvert z+2-i\rvert\leq\sqrt{5}$ and $\arg(z-1+2i)=\dfrac{3\pi}{4}$ 

(ii)	Minimum $\lvert z-1+2i \rvert = PB = BC - CP = \sqrt{18}-\sqrt{5}$ units Maximum $\lvert z-1+2i \rvert = AB = BC + AC = \sqrt{18}+\sqrt{5}$ units
(iii)	$\sin\theta = \dfrac{\sqrt{5}}{\sqrt{18}}$ Minimum arg(w-1+2i) $= \dfrac{3\pi}{4} - \theta$ $= \dfrac{3\pi}{4} - \sin^{-1}(\dfrac{\sqrt{5}}{\sqrt{18}})$ $= \dfrac{3\pi}{4} - 0.55512$ $= 1.80$ rad Maximum arg(w-1+2i) $= \dfrac{3\pi}{4} + \theta$ $= \dfrac{3\pi}{4} + \sin^{-1}(\dfrac{\sqrt{5}}{\sqrt{18}})$ = 2.91 rad