JPJC J2 Preliminary Examination 2024
H3 Mathematics Solutions

Q1		
(a)i)	Let $\gcd(a,b)=d$ $\Rightarrow d \mid a$ and $d \mid b$ Since $d \mid a$ and $d \mid b$, $\Rightarrow k_1 d = a$ and $k_2 d = b$, where k_1 and k_2 are integers $b-a = k_2 d - k_1 d = (k_2 - k_1)d$ $\therefore d \mid b-a$ Since $d \mid a$ and $d \mid b-a$, $d \mid \gcd(a, b-a)$ Let $\gcd(a,b-a)=e$ $\Rightarrow e \mid a$ and $e \mid b-a$ $\Rightarrow k_3 e = a$ and $k_4 \mathrm{e} = b-a$, where k_3 and k_4 are integers $a+(b-a) = k_3 e + k_4 \mathrm{e} = (k_3+k_4)e$ $b=(k_3+k_4)e$ $\therefore e \mid b$ Since $e \mid a$ and $\mathrm{e} \mid b$, $e \mid \gcd(a,b)$ $d \mid e$ and $e \mid d$, then $d = e$ $\therefore \gcd(a,b-a) = \gcd(a,b)$	
(ii)	$\gcd(72,120) = \gcd(72,120-72) = \gcd(72,48)$ Similarly, $\gcd(48,72) = \gcd(48,24) = 24$	
(b)	Let $\gcd(a,\gcd(b,c)) = d_1$ and $\gcd(\gcd(a,b),c) = d_2$ Since $\gcd(a,\gcd(b,c)) = d_1$, $\Rightarrow d_1 \mid$ a and $d_1 \mid \gcd(b,c)$ $\Rightarrow d_1 \mid$ a and $d_1 \mid b$ and $d_1 \mid c$ $\Rightarrow d_1 \mid \gcd(a,b)$ and $d_1 \mid c$ $\Rightarrow d_1 \mid \gcd(\gcd(a,b),c)$ $\Rightarrow d_1 \mid d_2$	

	Similiarly, since $\gcd\left(\gcd\left(a,b\right),c\right)=d_2$, $\Rightarrow d_2 \mid \gcd\left(a,b\right)$ and $d_2 \mid c$ $\Rightarrow d_2 \mid a$ and $d_2 \mid b$ and $d_2 \mid c$ $\Rightarrow d_2 \mid a$ and $d_2 \mid \gcd\left(b,c\right)$ $\Rightarrow d_2 \mid \gcd\left(a,\gcd\left(b,c\right)\right)$ $\Rightarrow d_2 \mid d_1$ Since $d_2 \mid d_1$ and $d_1 \mid d_2$, $\Rightarrow$ $d_1 = d_2$	

2(i)(a)

Equivalent to $x_1 + x_2 + x_3 + x_4 + x_5 = 13$, $x_i \in \mathbb{Z}^+ \cup \{0\}$

Number of ways

$= \binom{13+4}{4}$

$= 2380$

(i)(b)

Equivalent to $x_1 + x_2 + x_3 + x_4 + x_5 = 13$, $x_1 \in \{1,2,3\}$, $x_i \in \mathbb{Z}^+$

Equivalent to $y_1 + y_2 + y_3 + y_4 + y_5 = 8$, $y_i \in \{0,1,2\}$, $y_i \in \mathbb{Z}^+ \cup \{0\}$ [so 1 coin of each type]

Number of ways $= \binom{8+4}{4}$

Complement is equivalent to $z_1 + z_2 + z_3 + z_4 + z_5 = 8$, $z_i \geq 3$, $z_i \in \mathbb{Z}^+ \cup \{0\}$ [at least 3 5 cent coins]

equivalent to $w_1 + w_2 + w_3 + w_4 + w_5 = 5$, $w_i \in \mathbb{Z}^+ \cup \{0\}$

Required number of ways $= \binom{5+4}{4}$

Required number of ways

$= \binom{8+4}{4} - \binom{5+4}{4}$

$= 369$

(ii)(a)

Number of ways $= 5 \times 4^{12} = 83886080$

(ii)(b)

Number of ways

$= 5^{13} - \binom{5}{1} \times 4^{13} + \binom{5}{2} \times 3^{13} - \binom{5}{3} \times 2^{13} + \binom{5}{4} \times 1^{13}$

$= 901020120$

Q3		
3(i)	$(x+y)^p = x^p + y^p + \sum_{i=1}^{p-1} \binom{p}{i} x^{p-i} y^i$ Note that $\binom{p}{i} = \dfrac{p(p-1)\cdots(p-i+1)}{i!}$ For $1 \le i \le p-1$, since $i < p$ and p is prime, thus $i! \mid (p-1)\cdots(p-i+1)$ and p is a factor of $\binom{p}{i}$. Accordingly, $(x+y)^p \equiv x^p + y^p \pmod{p}$	
3(ii)	Let P_a be the proposition that $a^p \equiv a \pmod{p}$ for all positive integers a. Clearly, $1^p = 1$. Thus P_1 is true. Suppose P_k is true for some $k \in \mathbb{Z}^+$. Consider P_{k+1}. $(k+1)^p \equiv k^p + 1 \pmod{p}$ $\equiv k+1 \pmod{p}$ (by induction hypothesis) Thus P_{k+1} is true. Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, $a^p \equiv a \pmod{p}$ for all positive integers a.	
3(iii)	Since n is not a multiple of 4, we must have $n = 4k + r$ for some $k \in \mathbb{Z}^+$ and $r = 1, 2, 3$. Using (ii), for $a < p$, we must have $a \cdot a^{p-1} \equiv a \pmod{p} \Rightarrow a^{p-1} \equiv 1 \pmod{p}$ Now for $i = 1, 2, 3, 4$, $i^n = i^{4k+r} = i^{4k} i^r \equiv 1^k i^r = i^r \pmod{5}$ Thus $\sum_{i=1}^{4} i^n \equiv \sum_{i=1}^{4} i^r \pmod{5}$ For $r = 1, 2, 3,$, $\sum_{i=1}^{4} i^r = 10, 30, 100$ respectively. Thus $\sum_{i=1}^{4} i^n \equiv 0 \pmod{5}$.	

4(a)
$\left(\sqrt{a}\cdot\sqrt{p}+\sqrt{b}\cdot\sqrt{q}+\sqrt{c}\cdot\sqrt{r}\right)^2 \le (a+b+c)(p+q+r)$ $\left(\sqrt{a}\cdot\sqrt{p}+\sqrt{b}\cdot\sqrt{q}+\sqrt{c}\cdot\sqrt{r}\right) \le \sqrt{(a+b+c)(p+q+r)}$ $\sqrt{ap}+\sqrt{bq}+\sqrt{cr} \le \sqrt{(a+b+c)(p+q+r)}$
(b)
$x+y \ge 2\sqrt{xy}$ $\frac{1}{x+y} \le \frac{1}{2\sqrt{xy}}$ $\frac{x}{x+y} \le \frac{x}{2\sqrt{xy}} = \frac{1}{2}\sqrt{\frac{x}{y}} - (1)$
Similarly,
$\frac{y}{y+z} \le \frac{1}{2}\sqrt{\frac{y}{z}} - (2)$
$\frac{z}{z+x} \le \frac{1}{2}\sqrt{\frac{z}{x}} - (3)$
$(1)\times(2)\times(3)$
$\frac{x}{(x+y)}\cdot\frac{y}{(y+z)}\cdot\frac{z}{(x+z)} \le \frac{1}{2}\sqrt{\frac{x}{y}}\cdot\frac{1}{2}\sqrt{\frac{y}{z}}\cdot\frac{1}{2}\sqrt{\frac{z}{x}}$ $\frac{xyz}{(x+y)(y+z)(x+z)} \le \frac{1}{8}\sqrt{\frac{x}{y}}\cdot\sqrt{\frac{y}{z}}\cdot\sqrt{\frac{z}{x}}$ $\frac{xyz}{(x+y)(y+z)(x+z)} \le \frac{1}{8}$
(c)

$$
\begin{aligned}
\sqrt{\frac{2x}{x+y}}+\sqrt{\frac{2y}{y+z}}+\sqrt{\frac{2z}{z+x}} &= \sqrt{\frac{2x(y+z)(z+x)}{(x+y)(y+z)(z+x)}}+\sqrt{\frac{2y(x+z)(x+y)}{(y+z)(x+y)(z+x)}}+\sqrt{\frac{2z(x+y)(y+z)}{(z+x)(x+y)(y+z)}} \\
&= \frac{\sqrt{2x(y+z)(z+x)}+\sqrt{2y(x+z)(x+y)}+\sqrt{2z(x+y)(y+z)}}{\sqrt{(x+y)(y+z)(z+x)}} \\
&\le \frac{\sqrt{\{2x(y+z)+2y(x+z)+2z(x+y)\}\{(z+x)+(x+y)+(y+z)\}}}{\sqrt{(x+y)(y+z)(z+x)}} \\
&= \frac{\sqrt{2}\sqrt{2}\sqrt{2(xy+yz+xz)(x+y+z)}}{\sqrt{(x+y)(y+z)(z+x)}} \\
&= \frac{2\sqrt{2}\sqrt{(xy+yz+xz)(x+y+z)}}{\sqrt{(x+y)(y+z)(z+x)}} \\
&= \frac{2\sqrt{2}\sqrt{(x+y)(y+z)(z+x)+xyz}}{\sqrt{(x+y)(y+z)(z+x)}} \\
&= 2\sqrt{2}\sqrt{\frac{(x+y)(y+z)(z+x)+xyz}{(x+y)(y+z)(z+x)}} \\
&= 2\sqrt{2}\sqrt{1+\frac{xyz}{(x+y)(y+z)(z+x)}} \\
&\le 2\sqrt{2}\sqrt{1+\frac{1}{8}} \\
&= 2\sqrt{2}\,\frac{3}{2\sqrt{2}} \\
&= 3
\end{aligned}
$$

5(i)

Eugene does not do a threshold run on two consecutive days and he does not do a recovery run for more than two consecutive days. Call this condition (*).

For a_{n+1}, Day 1 is a threshold run.

Day 2 cannot be a threshold run.

Days 2 to $n+1$ is a sequence of n runs satisfying (*) where Day 2 is a tempo run or recovery run.

By Addition Principle, $a_{n+1} = b_n + c_n$

For b_{n+1}, Day 1 is a tempo run.

Days 2 to $n+1$ is a sequence of n runs satisfying (*) where Day 2 can be any run.

By Addition Principle, $b_{n+1} = a_n + b_n + c_n$

For c_{n+2}, Day 1 is a recovery run.

Case 1: Day 2 is a threshold run
Days 2 to $n+2$ is a sequence of $n+1$ runs satisfying (*) where Day 2 is a threshold run.

Case 2: Day 2 is a tempo run
Days 2 to $n+2$ is a sequence of $n+1$ runs satisfying (*) where Day 2 is a tempo run.

Case 3: Day 2 is a recovery run
Day 3 cannot be a recovery run.

Case 3A: Day 3 is a threshold run

Days 3 to $n+2$ is a sequence of *n* runs satisfying (*) where Day 3 is a threshold run.

Case 3B: Day 3 is a tempo run

Days 3 to $n+2$ is a sequence of *n* runs satisfying (*) where Day 3 is a tempo run.

By Addition Principle,
$c_{n+2} = a_{n+1} + b_{n+1} + \left(a_n + b_n\right)$

(ii)
Doing a replacement yields
$a_{n+4} = b_{n+3} + c_{n+3} \ldots (1)$
$b_{n+4} = a_{n+3} + b_{n+3} + c_{n+3} \ldots (2)$
$c_{n+3} = a_{n+2} + b_{n+2} + a_{n+1} + b_{n+1} \ldots (3)$

Sub (1) into (2), $b_{n+4} = a_{n+3} + a_{n+4} \ldots (4)$
Doing a replacement yields
$b_{n+1} = a_n + a_{n+1} \ldots (5)$
$b_{n+2} = a_{n+1} + a_{n+2} \ldots (6)$
$b_{n+3} = a_{n+2} + a_{n+3} \ldots (7)$

Sub (5) and (6) into (3),
$$c_{n+3} = a_{n+2} + \left(a_{n+1} + a_{n+2}\right) + a_{n+1} + \left(a_n + a_{n+1}\right)$$
$$= 2a_{n+2} + 3a_{n+1} + a_n \ldots (8)$$

Sub (7) and (8) back into (1),
$$a_{n+4}$$
$$= \left(a_{n+2} + a_{n+3}\right) + \left(2a_{n+2} + 3a_{n+1} + a_n\right)$$
$$= a_{n+3} + 3a_{n+2} + 3a_{n+1} + a_n$$

(iii)

Method 1: Recurrence

$a_1 = 1$

$a_2 = 1 \times 2 = 2$

$a_3 = 1 \times 2 \times 3 = 6$

$$a_4 = \underbrace{2+3+3}_{\substack{\text{1st day THR}\\ \text{2nd day TEM}}} + \underbrace{2+3+2}_{\substack{\text{1st day THR}\\ \text{2nd day REC}}} = 15$$

$a_5 = 15 + 3(6+2) + 1$

$= 40$ (shown)

Method 2:

For a_5, Day 1 is a threshold run

Day 2 can be only be a tempo run or recovery run.

Case 1: Day 2 is tempo run

Case 1A: Day 3 is threshold run, Day 4 is tempo or recovery run, Day 5 is any run

No. of ways $= 2 \times 3 = 6$

Case 1B: Day 3 is tempo run, Day 4 is threshold run, Day 5 is tempo or recovery run

No. of ways $= 2$

Case 1C: Day 3 is tempo run, Day 4 is tempo or recovery run, Day 5 is any run

No. of ways $= 2 \times 3 = 6$

Case 1D: Day 3 is recovery run, Day 4 is threshold run, Day 5 is tempo or recovery run

No. of ways $= 2$

Case 1E: Day 3 is recovery run, Day 4 is tempo run, Day 5 is any run

No. of ways $= 3$

Case 1F: Day 3 is recovery run, Day 4 is recovery run, Day 5 is threshold or tempo run

No. of ways $= 2$

Case 2: Day 2 is recovery run

Case 2A: Day 3 is threshold run, Day 4 is tempo or recovery run, Day 5 is any run

No. of ways $= 2 \times 3 = 6$

Case 2B: Day 3 is tempo run, Day 4 is threshold run, Day 5 is tempo or recovery run

No. of ways $= 2$

Case 2C: Day 3 is tempo run, Day 4 is tempo or recovery run, Day 5 is any run

No. of ways $= 2 \times 3 = 6$

Case 2D: Day 3 is recovery run, Day 4 is threshold run, Day 5 is tempo or recovery run
No. of ways $= 2$

Case 2E: Day 3 is recovery run, Day 4 is tempo run, Day 5 is any run
No. of ways $= 3$

By Addition Principle, no. of ways $= 40$

Method 3:
For a_5, Day 1 is a threshold run
Day 2 can be only be a tempo run or recovery run.

Total no. of ways for Days 2 to 5 without restriction, without restrictions for Days 3 to 5
$= 2 \times 3^3$

No. of ways where there are at least 3 consecutive days of recovery runs
$= \underbrace{3}_{\substack{\text{D2 to D4: All REC} \\ \text{D5: Any}}} + \underbrace{1}_{\substack{\text{D2: TEM} \\ \text{D3 to D5: All REC}}}$
$= 4$

No. of ways where there are at least 2 consecutive days of threshold runs
$= \underbrace{2}_{\text{D2}}\left(\underbrace{1}_{\text{D3 to D5: All THR}} + \underbrace{2 \times 2}_{\text{D3 to D5: 2 THR}}\right)$
$= 10$

Required no. of ways $= 2 \times 3^3 - 4 - 10 = 40$

(iv)
$b_5 = a_4 + a_5 = 55$
$c_5 = 2a_4 + 3a_3 + a_2 = 50$

Required no. of ways $= 55 + 50 + 40 = 145$ A1

Q6		
(i)	Substitute $x=\sin\theta$, then $\dfrac{dx}{d\theta}=\cos\theta$ and $\theta=\dfrac{\pi}{2}$, when $x=1$ and $\theta=0$, when $x=0$. $\int_0^1 (1-x^2)^n dx = \int_0^{\frac{\pi}{2}} (\cos^2\theta)^n (\cos\theta)\, d\theta$ $= \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+1}\, d\theta$ $= I_{2n+1}$ Substitute $x=\tan\theta$, then $\dfrac{dx}{d\theta}=\sec^2\theta$ and $\theta=\dfrac{\pi}{4}$, when $x=1$ and $\theta=0$, when $x=0$. $\int_0^1 (1+x^2)^{-n} dx = \int_0^{\frac{\pi}{4}} (\sec^2\theta)^{-n} (\sec^2\theta)\, d\theta$ $= \int_0^{\frac{\pi}{4}} (\cos\theta)^{2n} \left(\dfrac{1}{\cos^2\theta}\right) d\theta$ $= \int_0^{\frac{\pi}{4}} (\cos\theta)^{2n-2}\, d\theta$ $< \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n-2}\, d\theta = I_{2n-2}$	
(ii)	From MF26, $e^{x^2} = 1+x^2+\dfrac{x^4}{2}+\dfrac{x^6}{6}+\dots$ and $(1-x^2)^{-1} = 1+x^2+x^4+x^6+\dots$ hence, $1+x^2 \le e^{x^2} \le (1-x^2)^{-1}$ taking reciprocal, $1-x^2 \le e^{-x^2} \le (1+x^2)^{-1}$ (shown) Raising to n power and integrate, $\int_0^1 (1-x^2)^n dx \le \int_0^1 e^{-nx^2} dx \le \int_0^1 (1+x^2)^{-n} dx$ Substitute $y=\sqrt{n}x$, then $\dfrac{dy}{dx}=\sqrt{n}$ and $y=\sqrt{n}$, when $x=1$ and $y=0$, when $x=0$. $\int_0^1 e^{-nx^2} dx = \dfrac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} dy$ Using (i), $I_{2n+1} = \int_0^1 (1-x^2)^n dx \le \dfrac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2}\, dy \le \int_0^1 (1+x^2)^{-n} dx < I_{2n-2}$	

	$\sqrt{n}I_{2n+1} \le \int_0^{\sqrt{n}} e^{-y^2}\ dy < \sqrt{n}I_{2n-2}$ (shown)	
(iii)	As $n \to \infty$, $\sqrt{n}I_{2n+1} = \frac{\sqrt{n}}{\sqrt{2n+1}}\sqrt{2n+1}\ I_{2n+1} = \sqrt{\frac{1}{2+\frac{1}{n}}}\sqrt{2n+1}\ I_{2n+1} \to \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2}$ $\sqrt{n}I_{2n-2} = \frac{\sqrt{n}}{\sqrt{2n-2}}\sqrt{2n-2}\ I_{2n-2} = \sqrt{\frac{1}{2-\frac{2}{n}}}\sqrt{2n-2}\ I_{2n-2} \to \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2}$ hence, $\frac{\sqrt{\pi}}{2} \le \int_0^{\sqrt{n}} e^{-y^2}\ dy < \frac{\sqrt{\pi}}{2}$ as $n \to \infty$ therefore, $\int_0^{\infty} e^{-y^2}\ dy = \frac{\sqrt{\pi}}{2}$	
(iv)	Consider $\frac{d}{dx}\left(x^{2n-1}e^{-x^2}\right) = (2n-1)x^{2n-2}e^{-x^2} - 2x^{2n}e^{-x^2}$ $\int_0^{\infty} \frac{d}{dx}\left(x^{2n-1}e^{-x^2}\right) dx = (2n-1)\int_0^{\infty} x^{2n-2}e^{-x^2}dx - 2\int_0^{\infty} x^{2n}e^{-x^2}dx$ $\left(x^{2n-1}e^{-x^2}\right)_0^{\infty} = (2n-1)U_{n-1} - 2U_n$ $0 = (2n-1)U_{n-1} - 2U_n$ since $x^{2n-1}e^{-x^2} \to 0$ as $x \to \infty$ $U_n = \frac{2n-1}{2}U_{n-1}$ (shown) $U_1 = \frac{1}{2}U_0$ $U_2 = \frac{3}{2}U_1 = \frac{1\times 3}{2^2}U_0$ $U_3 = \frac{5}{2}U_2 = \frac{1\times 3\times 5}{2^3}U_0$ $\vdots$ $U_n = \frac{1\times 3\times 5\times \ldots \times (2n-1)}{2^n}U_0$ $= \frac{(2n)!}{2^n(2\times 4 \ldots \times 2n)}U_0$ $= \frac{(2n)!}{2^{2n}(n!)}\left(\frac{\sqrt{\pi}}{2}\right)$ by (i) $U_0 = \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ $\int_0^{\infty} x^{2n}e^{-x^2}dx = \frac{(2n)!\sqrt{\pi}}{2^{2n+1}n!}$	

[Turn over

Q7		
	$(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are collinear points $\Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$ $(x_3 - x_2)y_2 - (x_3 - x_2)y_1 = (x_2 - x_1)y_3 - (x_2 - x_1)y_2$ $(x_3 - x_1)y_2 - (x_3 - x_2)y_1 - (x_2 - x_1)y_3 = 0$ $-(x_3 - x_2)y_1 - (x_1 - x_3)y_2 - (x_2 - x_1)y_3 = 0$ $(x_3 - x_2)y_1 + (x_1 - x_3)y_2 + (x_2 - x_1)y_3 = 0$	
	Sub $y = -2x$, $\mathrm{g}(\mathrm{f}(x - 2x)) = \mathrm{f}(x) + (2x - 2x)\mathrm{g}(-2x)$ $\mathrm{g}(\mathrm{f}(-x)) = \mathrm{f}(x)$ Sub $x = -x - y$, $\mathrm{g}(\mathrm{f}(-x - y + y)) = \mathrm{f}(-x - y) + (2(-x - y) + y)\mathrm{g}(y)$ $\mathrm{g}(\mathrm{f}(-x)) = \mathrm{f}(-x - y) - (2x + y)\mathrm{g}(y)$ but $\mathrm{g}(\mathrm{f}(-x)) = \mathrm{f}(x)$, so $\mathrm{f}(x) = \mathrm{f}(-x - y) - (2x + y)\mathrm{g}(y)$ $\mathrm{f}(-x - y) = \mathrm{f}(x) + (2x + y)\mathrm{g}(y)$	
	Let $x = -b$ and $y = a + b$ $\mathrm{f}(b - a - b) = \mathrm{f}(-b) + (-2b + a + b)\mathrm{g}(a + b)$ $\mathrm{f}(-a) = \mathrm{f}(-b) + (a - b)\mathrm{g}(a + b)$ WLOG, we have $\mathrm{f}(-b) = \mathrm{f}(-c) + (b - c)\mathrm{g}(b + c)$ $\mathrm{f}(-c) = \mathrm{f}(-a) + (c - a)\mathrm{g}(c + a)$	
	Adding the three equations in (iii), $\mathrm{f}(-a) + \mathrm{f}(-b) + \mathrm{f}(-c) = \mathrm{f}(-a) + \mathrm{f}(-b) + \mathrm{f}(-c)$ $+ (a - b)\mathrm{g}(a + b) + (b - c)\mathrm{g}(b + c) + (c - a)\mathrm{g}(c + a)$ we have, $(a - b)\mathrm{g}(a + b) + (b - c)\mathrm{g}(b + c) + (c - a)\mathrm{g}(c + a) = 0$ Let $\mathrm{g}(a + b) = y_1$ then $x_1 = a + b$ $\mathrm{g}(b + c) = y_2$ then $x_2 = b + c$ $\mathrm{g}(c + a) = y_3$ then $x_3 = c + a$ and $[(a + c) - (b + c)]\mathrm{g}(a + b) + [(a + b) - (c + a)]\mathrm{g}(b + c) + [(b + c) - (a + b)]\mathrm{g}(c + a) = 0$ by (i), $(a + b, \mathrm{g}(a + b)), (b + c, \mathrm{g}(b + c))$ and $(c + a, \mathrm{g}(c + a))$ are collinear points on g(x), thus g(x) is linear. Let $\mathrm{g}(x) = Ax + B$ where A and B are constants.	

From (ii), sub $x = 0$ and $y = -y$

$$\mathrm{f}(y) = \mathrm{f}(0) + (-y)(-Ay + B)$$

$$= Ay^2 - By + C \quad \text{where } C = \mathrm{f}(0)$$

For all x, $\mathrm{g}(\mathrm{f}(-x)) = \mathrm{f}(x) \Rightarrow \quad \mathrm{g}(Ax^2 + Bx + C) = Ax^2 - Bx + C$

$$A^2x^2 + ABx + AC + B = Ax^2 - Bx + C \quad ---(1)$$

Comparing coefficient, $A^2 = A \Rightarrow A = 0 \text{ or } 1$

When $A = 0$, then (1) becomes $B = -Bx + C$ then $B = C = 0$ hence $\mathrm{f}(x) = \mathrm{g}(x) = 0$

When $A = 1$, then (1) becomes

$$x^2 + Bx + C + B = x^2 - Bx + C$$

$$2Bx + B = 0$$

$$B = 0$$

then $\mathrm{f}(x) = x^2 + C$ and $\mathrm{g}(x) = x$

Q8		
(i)	For $k \geq 2,\ m^m \geq 2^m$ $\dfrac{1}{m^m} \leq \dfrac{1}{2^m}$ $x_n = \sum_{m=1}^{n} \dfrac{1}{m^m}$ $= 1 + \sum_{m=2}^{n} \dfrac{1}{m^m}$ $\leq 1 + \sum_{m=2}^{n} \dfrac{1}{2^m}$ for $m \geq 2$ $\leq 1 + \dfrac{\frac{1}{2^2}\left(1 - \frac{1}{2^{n-1}}\right)}{1 - \frac{1}{2}}$ $\leq 1 + \dfrac{\frac{1}{2^2}}{1 - \frac{1}{2}} = \dfrac{3}{2}$ $\therefore x_n \leq \dfrac{3}{2}$ for all $n \geq 1$ (shown)	
(ii)	$x_n - x_{n-1} = \dfrac{1}{n^n} > 0,$ hence the sequence $\{x_n\}$ is strictly increasing By Monotone Convergence Theorem, since the sequence is bounded above by $\dfrac{3}{2}$ and is strictly increasing, the sequence $\{x_n\}$ is convergent.	
(b)(i)	**Show base case *r*=1 is true** $F_1 \leq 2^0 F_1$ $F_1 \leq F_1$ $\therefore F_r \leq 2^{r-1} F_1$ is true when $r=1$ **Assume *case r=k* is true** Assume $F_k \leq 2^{k-1} F_1$ is true for all $k \geq 1$ **To show *case r=k*+1 is true:** $F_{k+1} \leq 2^k F_1$ $F_{k+1} = F_k + F_{k-1}$ $\leq 2F_k$ since $F_{k-1} \leq F_k$ $\leq 2\left(2^{k-1} F_1\right)$ $\therefore F_{k+1} \leq 2^k F_1$	

	By Mathematical Induction, since $F_r \le 2^{r-1}F_1$ is true when r=1, and by assuming $F_k \le 2^{k-1}F_1$ is true for all $k \ge 1$, and $F_{k+1} \le 2^k F_1$ is true when r=k+1. Therefore $F_r \le 2^{r-1}F_1$ for all $r \ge 1$.	
(ii)	$LHS = 81\sum_{r=1}^{n}\frac{F_{r+1}}{9^{r+1}} - 9\sum_{r=1}^{n}\frac{F_r}{9^r} - \sum_{r=1}^{n}\frac{F_{r-1}}{9^{r-1}}$ replace r with r -1 replace r with $r+1$ $= 81\sum_{r=2}^{n+1}\frac{F_r}{9^r} - 9\sum_{r=1}^{n}\frac{F_r}{9^r} - \sum_{r=0}^{n-1}\frac{F_r}{9^r}$ $= 81\left(\sum_{r=1}^{n+1}\frac{F_r}{9^r} - \frac{F_1}{9}\right) - 9\sum_{r=1}^{n}\frac{F_r}{9^r} - \left(\sum_{r=1}^{n-1}\frac{F_r}{9^r} + \frac{F_0}{9^0}\right)$ $= 81S_{n+1} - 9F_1 - 9S_n - S_{n-1} - F_0$ $= 81\left(S_n + u_{n+1}\right) - 9F_1 - 9S_n - \left(S_n - u_n\right) - F_0$ where u_n is the n^{th} term $= (81-9-1)S_n - 9F_1 - F_0 + 81\frac{F_{n+1}}{9^{n+1}} + \frac{F_n}{9^n}$ $= 71S_n - 9F_1 - F_0 + \frac{F_n}{9^n} + \frac{F_{n+1}}{9^{n-1}}$ =RHS (shown)	
(iii)	$\sum_{r=1}^{\infty}\frac{F_r}{9^r} = S_\infty = \lim_{n\to\infty} S_n$ From (b), $\sum_{r=1}^{n}\frac{1}{9^{r-1}}\left(F_{r+1} - F_r - F_{r-1}\right) = 71S_n - 9F_1 - F_0 + \frac{F_n}{9^n} + \frac{F_{n+1}}{9^{n-1}}$ $0 = 71S_n - 9F_1 - F_0 + \frac{F_n}{9^n} + \frac{F_{n+1}}{9^{n-1}}$ since $F_{r+1} = F_r + F_{r-1}$ for $r \ge 1$ $S_n = \frac{1}{71}\left(9F_1 + F_0 - \frac{F_n}{9^n} - \frac{F_{n+1}}{9^{n-1}}\right)$ $= \frac{1}{71}\left(9 - \frac{F_n}{9^n} - \frac{F_{n+1}}{9^{n-1}}\right)$ $\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{71}\left(9 - \frac{F_n}{9^n} - \frac{F_{n+1}}{9^{n-1}}\right)$	

	From (a), $F_r \le 2^{r-1}F_1$ $F_n \le 2^{n-1}F_1$ $\le 2^{n-1}$ $\le 2^n\left(\frac{1}{2}\right)$ $\therefore \frac{F_n}{9^n} \le \frac{2^n}{9^n}\left(\frac{1}{2}\right)$ and $\frac{F_{n+1}}{9^{n-1}} \le \frac{2^{n+1}}{9^{n-1}}\left(\frac{1}{2}\right)$ As $n \to \infty, \left(\frac{2}{9}\right)^n\left(\frac{1}{2}\right) \to 0$ and $\left(\frac{2}{9}\right)^{n+1}(81)\left(\frac{1}{2}\right) \to 0,$ $\therefore \frac{F_n}{9^n} \to 0$ and $\frac{F_{n+1}}{9^{n-1}} \to 0$ $\sum_{r=1}^{\infty}\frac{F_r}{9^r} = S_\infty$ $= \lim_{n\to\infty}\frac{1}{71}\left(9 - \frac{F_n}{9^n} - \frac{F_{n+1}}{9^{n-1}}\right)$ $= \frac{9}{71}$ (shown)	
(iv)	$\frac{9}{71} = \sum_{r=1}^{\infty}\frac{F_r}{9^r}$ $= \frac{F_1}{9^1}+\frac{F_2}{9^2}+\frac{F_3}{9^3}+\frac{F_4}{9^4}+\frac{F_5}{9^5}+\frac{F_6}{9^6}+\sum_{r=7}^{\infty}\frac{F_r}{9^r}$ $= \frac{1}{9^1}+\frac{1}{9^2}+\frac{2}{9^3}+\frac{3}{9^4}+\frac{5}{9^5}+\frac{8}{9^6}+\sum_{r=7}^{\infty}\frac{F_r}{9^r}$ $= 0.1267572506+\sum_{r=7}^{\infty}\frac{F_r}{9^r}$ $\frac{1}{71} = 0.014084139+\frac{1}{9}\sum_{r=7}^{\infty}\frac{F_r}{9^r}$ where $\frac{1}{9}\sum_{r=7}^{\infty}\frac{F_r}{9^r} < \frac{1}{9}(2\times10^{-6}) = 2.\dot{2}\times10^{-7} = 0.000000\dot{2}$ Hence the first 6 digits of the decimal expansion of $\frac{1}{71}$ are 0.014084.	