2024 Nanyang Junior College Preliminary Examination H2 Mathematics Paper 2

Section A: Pure Mathematics [40 marks]

1 The area bounded by the loop of the curve $y^2=(1-x)^2(x+3)$ is given by $2\int_{\alpha}^{\beta}(1-x)\sqrt{x+3}\,\mathrm{d}x$.

(a) State the values of α and β. [1]

(b) Use the substitution $u=\sqrt{x+3}$ to find the exact area bounded by this loop. [4]

2 With reference to the origin O, the points A, B and R have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$ respectively. The points A and B are fixed and R varies.

(a) Interpret geometrically the vector equation $\mathbf{r}=\lambda\mathbf{a}$ where $0\le\lambda\le1$. [1]

(b) Interpret geometrically the vector equation $\mathbf{r}\times(\mathbf{a}-\mathbf{b})=\mathbf{0}$. [2]

(c) Interpret geometrically if $\mathbf{r}\cdot(\mathbf{r}-\mathbf{a})=0$. [2]

(d) Given that $(\mathbf{r}-\mathbf{a})\times(\mathbf{r}-\mathbf{b})=\mathbf{0}$. Find $\mathbf{r}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, showing your working clearly. [2]

3 **Do not use a calculator in answering the question.**

Three complex numbers are $z_1=-1+\sqrt{3}\mathrm{i}$, $z_2=-1-\mathrm{i}$ and $z_3=\sqrt{2}\left(\cos\frac{1}{12}\pi-\mathrm{i}\sin\frac{1}{12}\pi\right)$.

(a) Represent z_1, z_2 and z_3 on an Argand diagram. [2]

(b) Find $\dfrac{z_1}{z_2(z_3)^2}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [4]

A fourth complex number, z_4, is such that $\dfrac{z_1z_4}{z_2(z_3)^2}$ is purely real and $\left|\dfrac{z_1z_4}{z_2(z_3)^2}\right|=1$.

(c) Find the possible values of z_4 in the form $r(\cos\theta+\mathrm{i}\sin\theta)$, where $r>0$ and $-\pi<\theta\le\pi$. [3]

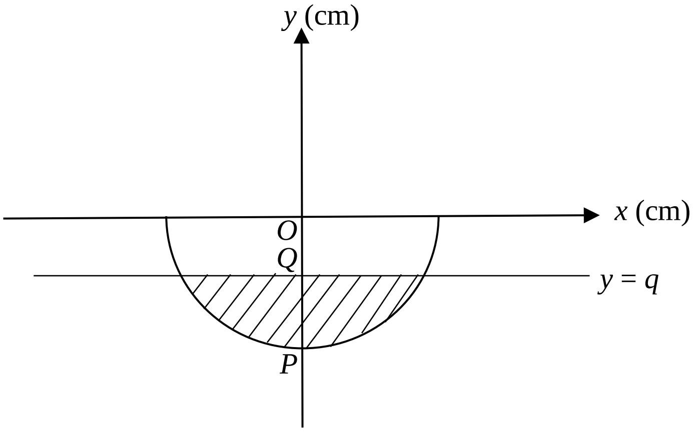
4 **(a)** Write $\dfrac{1}{4r^2-8r+3}$ in partial fractions. [2]

(b) Find an expression in terms of n for $\displaystyle\sum_{r=2}^{3n}\left(\frac{1}{4r^2-8r+3}\right)$. [3]

(c) Find $\displaystyle\sum_{r=2}^{\infty}\left(\frac{1}{4r^2-8r+3}\right)$. [1]

(d) Using the result in part **(b)**, find $\displaystyle\sum_{r=n+1}^{3n}\left(\frac{1}{4r^2-8r+3}\right)$. Write your answer as a simplified fraction in terms of n. [3]

5 In this question units are in centimetres (cm).



The diagram shows part of the circle $x^2 + y^2 = 256$ and the line $y = q$, where $q < 0$. The circle and the line cross the negative y-axis at points P and Q respectively.

(a) The shaded region between the circle and the line is rotated about the y-axis to form a wok of negligible thickness. The wok has depth PQ cm and a capacity of 3.3 litres. By finding the capacity of the wok in terms of q, calculate the depth of the wok giving your answer to the nearest integer.
[1 litre = 1000 cm^3] [4]

(b) A flat frying pan of negligible thickness with a capacity of 1.464π litres is formed by rotating the part of the circle between the lines $y = 0$ and $y = r$, $q < r < 0$, about the y-axis. Find a cubic equation satisfied by r and hence find the value of r. [3]

(c) Water is poured from a container to the wok in part **(a)** at a rate of $\dfrac{3}{55}t$ litres per second, where t is the time in seconds from when pouring begins. Find the time taken to fill this wok to its full capacity of 3.3 litres. [3]

Section B: Probability and Statistics [60 marks]

6 A carpark has a row of 10 parking lots. On one day the carpark has 9 cars of different makes and 1 empty lot. Of the 9 cars, 2 are blue, 2 are red and the remaining 5 are of other distinct colours.

(a) Find the number of different arrangements of the cars in the carpark. [1]

(b) Find the number of different arrangements that can be made with both the blue cars next to each other and both the red cars next to each other. [2]

(c) Find the number of different arrangements that can be made with no two adjacent cars the same colour. [3]

7 A computer randomly chooses 10 shapes from 18 squares and 12 triangles. The number of squares chosen is denoted by S.

(a) The most probable number of squares that the computer chooses is denoted by s. By using the fact that $\mathrm{P}(S = s) > \mathrm{P}(S = s+1)$, show that s satisfies the inequality

$$(s+1)(s+3) > (a-s)(b-s),$$

where a and b are constants to be determined. Hence find the value of s. [4]

In a game, the computer randomly chooses 6 squares and 4 triangles. The squares are numbered from 1 to 6 and the triangles are numbered from 1 to 4. A square and a triangle are randomly chosen. Let X be the score which is defined as the difference (taken as always positive) between the numbers of the chosen square and triangle.

(b) Find the probability distribution of X. [3]

Three independent games are played. The total score of the games is 3.

(c) Find the probability that the score of the first game is 2. [3]

8 **In this question, you should state the parameters of any distributions you use.**

A baby quilt is made by stitching rectangular cotton fabric together. The length, in centimetres (cm), of each rectangular cotton fabric follows the distribution $\mathrm{N}\left(24, 1.5^2\right)$ and the breadth follows the distribution $\mathrm{N}\left(20, 1.2^2\right)$. Opposite sides of the rectangular cotton fabric are cut to the exact same measurement.

(a) Find the probability that the length of a randomly chosen rectangular cotton fabric is less than 23.5 cm. [1]

(b) Find the probability that the perimeter of a randomly chosen rectangular cotton fabric is more than 90 cm. [3]

(c) State a necessary assumption for your calculations to hold in part **(b)**. [1]

(d) A baby quilt is made by stitching 48 pieces of rectangular cotton fabric together. Calculate the expected number of pieces that have length between 23 and 25 cm. [2]

(e) Rectangular cotton fabrics can be paired together if their lengths differ from each other by at most k cm. Find the least value of k if 2 randomly chosen rectangular cotton fabrics have at least 90% chance of being paired together. [3]

9 A restaurant serves a signature dish. The probability that a diner orders the dish is 0.7. It may be assumed that each diner orders at most one portion of the dish and the number of diners who order the dish follows a binomial distribution.

(a) Find the probability that 10 diners order at least 3 portions of the signature dish. [2]

(b) Find the probability that 10 diners order between 3 and 8 portions of the signature dish. [2]

(c) If the restaurant has enough ingredients to prepare 80 portions of the signature dish, find the maximum possible number of diners if the restaurant has at least 90% chance of meeting the number of orders for the dish. [2]

On a particular day, the restaurant caters for a function with 40 tables of 10 diners each.

(d) Find the probability that each table orders at least 3 portions of the signature dish. [1]

(e) Find an estimate of the probability that the average number of portions of the signature dish served per table is at least 6.9. [3]

10 A bus company is contracted to ferry students at a particular school on request. Following feedback from teachers and students that the buses usually arrive, on average, more than 15 minutes **after the scheduled pick-up time**, the school administration manager conducts a study of the arrival times of the bus after the scheduled pick-up time.

The administration manager collects a random sample of 30 bus arrival times after the scheduled pick-up time. Summary data for the arrival times after the scheduled pick-up time, t minutes, of these buses is as follows.

$$n = 30 \qquad \sum t = 543 \qquad \sum t^2 = 12722$$

(a) Calculate unbiased estimates of the population mean and variance for the bus arrival time after the scheduled pick-up time. [2]

(b) State hypotheses that can be used to test if the administration manager should agree with the feedback from teachers and students, defining any symbols you use. Work out the test statistic in this case, and use it to carry out the test at the 5% level of significance, giving your answer in the context of the question. [5]

The bus company makes changes to its operations and claims that their buses will arrive, on average, 15 minutes after the scheduled pick-up time. The administration manager then takes another sample of 40 bus arrival times after the scheduled pick-up time. He carries out a 2-tail test and finds no significant evidence to reject the company's claim at the 10% level of significance.

(c) Use an algebraic method to evaluate the range of possible values of the sample mean bus arrival times after the scheduled pick-up time, $\overline{t}$. [3]

(d) State two necessary assumptions for your calculations in part **(c)**. [2]

11 Various studies suggest that plants exposed to music tend to grow taller compared to those not exposed to music. A junior biologist wants to investigate the effects of music on the growth of different plants. In one experiment, vines of a particular species were exposed to classical music. He recorded the average length of the vines, h centimetres, at age t days, in the table below.

t	7	10	13	16	19	22	25	28
h	19.2	25.4	36.5	41.4	45	k	51.9	53.8

(a) Using the data, the junior biologist calculated the least squares regression line of h on t to be $h = 1.6393t + 11.475$. Show that the value of k is 48.1, correct to 1 decimal place. [2]

The junior biologist realised that the relationship between h and t is not linear and can be modelled by either

$$h = a\sqrt[3]{t} + b \quad \text{or} \quad h = c\sqrt{t} + d,$$

where a, b, c and d are constants.

(b) By calculating the relevant product moment correlation coefficients, explain why $h = a\sqrt[3]{t} + b$ is a better model than $h = c\sqrt{t} + d$. Find the equation of the better model. [4]

(c) An estimate for the average length of the vines when they are 2 months old is obtained using the better model. Explain whether the estimate is reliable. [1]

(d) Sketch a scatter diagram for h against $\sqrt[3]{t}$. Draw the line of best fit on your scatter diagram. [2]

For a line of best fit $y = \mathrm{f}(x)$, the residual is the difference between the observed and predicted value. The difference between an observed value (p, q) plotted on the scatter diagram and the predicted value $(p, \mathrm{f}(p))$ is defined as $q - \mathrm{f}(p)$.

(e) Find the value of the residual corresponding to $t = 28$, for the line of best fit found in part **(b)**. [1]

(f) Explain why the line of best fit is referred to as the 'least squares' regression line in relation to the residuals. [2]