2019 GCE A Level H2 Further Maths 9649 Paper 1 Solutions

Section A: Pure Mathematics

Question 1

For a given initial value $u_1 = \alpha$, the sequence of real numbers $\{u_n\}$ is defined by the recurrence relation

$$u_{n+1} = \frac{1+u_n}{1-u_n}, \quad n \geq 1.$$

Prove that $n = 5$ gives the first value of n $(n > 1)$ for which $u_n = \alpha$, and that this is so for all but three values of α. State these three exceptional values of α . [6]

[Solution]

Method 1:

Given $u_1 = \alpha$

$$u_2 = \frac{1+u_1}{1-u_1} = \frac{1+\alpha}{1-\alpha}, \quad \alpha \neq 1$$

Let $\dfrac{1+\alpha}{1-\alpha} = \alpha$ [to check if $u_2 = \alpha$]

$\Rightarrow 1+\alpha = \alpha - \alpha^2 \Rightarrow \alpha^2 + 1 = 0$

Since there is no real solution for $\dfrac{1+\alpha}{1-\alpha} = \alpha$, $u_2 \neq \alpha$

$$u_3 = \frac{1+u_2}{1-u_2} = \frac{1+\frac{1+\alpha}{1-\alpha}}{1-\frac{1+\alpha}{1-\alpha}} = -\frac{1}{\alpha}, \quad \alpha \neq 0$$

Let $-\dfrac{1}{\alpha} = \alpha$ [to check if $u_3 = \alpha$]

$\Rightarrow \alpha^2 + 1 = 0 \Rightarrow$ No real solution

$\therefore u_3 \neq \alpha$

$$u_4 = \frac{1+u_3}{1-u_3} = \frac{1-\frac{1}{\alpha}}{1+\frac{1}{\alpha}} = \frac{\alpha-1}{\alpha+1}, \quad \alpha \neq -1$$

Let $\dfrac{\alpha-1}{\alpha+1} = \alpha$

$\Rightarrow \alpha - 1 = \alpha^2 + \alpha$

$\Rightarrow \alpha^2 + 1 = 0$

$\Rightarrow$ No real solution

$\therefore u_4 \neq \alpha$

$u_5 = \frac{1+u_4}{1-u_4} = \frac{1+\frac{\alpha-1}{\alpha+1}}{1-\frac{\alpha-1}{\alpha+1}} = \frac{2\alpha}{2} = \alpha$. Thus $u_1 = u_5$. Hence $n = 5$ is the first value for $u_n = \alpha$

This implies that the sequence is periodic with period 4.

Note: Need to check that $\frac{1+\alpha}{1-\alpha} = \alpha$, $-\frac{1}{\alpha} = \alpha$ and $\frac{\alpha-1}{\alpha+1} = \alpha$ has no real solution

Method 2:

$$u_{n+1} = \frac{1+u_n}{1-u_n} \Rightarrow u_{n+1} - u_{n+1}u_n = 1 + u_n \Rightarrow \qquad u_n = \frac{u_{n+1}-1}{u_{n+1}+1} = \frac{1}{-\left(\frac{u_{n+1}+1}{1-u_{n+1}}\right)} = -\frac{1}{u_{n+2}}$$

Thus $u_{n+2} = -\frac{1}{u_{n+4}} \Rightarrow u_{n+4} = -\frac{1}{u_{n+2}} = u_n$. The sequence $\{u_n\}$ is periodic with period = 4.

The first value of n is 5 for which $u_5 = \alpha$ and that this is so for all but three values of α which are 0 and ± 1

Question 2

(i) Evaluate the integral $\int_a^b x^3 \, dx$. [2]

(ii) Evaluate this integral using Simpson's rule with two strips. [3]

(iii) Deduce that Simpson's rule always gives the correct value for an integral of any cubic polynomial. [2]

(i)	$\int_a^b x^3 \, dx = \left[\frac{x^4}{4}\right]_a^b = \frac{1}{4}\left(b^4 - a^4\right)$
(ii)	Using Simpson's Rule, $\int_a^b x^3 \, dx = \frac{1}{3}\left(\frac{b-a}{2}\right)\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right]$ $= \frac{1}{3}\left(\frac{b-a}{2}\right)\left[a^3 + \frac{1}{2}\left(a^3 + 3a^2b + 3ab^2 + b^3\right) + b^3\right]$ $= \left(\frac{b-a}{12}\right)\left[3a^3 + 3a^2b + 3ab^2 + 3b^3\right]$ $= \frac{1}{12}\left(3a^3b + 3a^2b^2 + 3ab^3 + 3b^4 - 3a^4 - 3a^3b - 3a^2b^2 - 3ab^3\right)$
(iii)	Let $f(x) = px^3 + qx^2 + rx + s$ $\int_a^b px^3 + qx^2 + rx + s \, dx = \int_a^b px^3 \, dx + \int_a^b qx^2 + rx + s \, dx$ $= p\int_a^b x^3 \, dx + \int_a^b qx^2 + rx + s \, dx$ From part (i) and (ii), using Simpson's rule with 2 strips on $\int_a^b x^3 \, dx$ gives the same exact value. Simpson's rule also gives an exact value to $\int_a^b qx^2 + rx + s \, dx$ since this is the integral of any

	quadratic curve and Simpson's rule model a curve by a quadratic approximation. Hence Simpson's rule always gives the correct value for an integral of any cubic polynomial.

Question 3

The curve C is such that $\frac{dy}{dx} = \sin(xy)$ and $y = 1.5$ when $x = 1$.

(i) Use the Euler method with steps of size $h = \frac{1}{3}$ to find an approximation to the value of y when $x = 2$. Give all intermediate values of y correct to 4 decimal places. [4]

(ii) **(a)** What feature of C do these values of y suggest occurs at some point P between $x = 1$ and $x = 2$? [1]

(b) Use the results of part (i) and the differential equation $\frac{dy}{dx} = \sin(xy)$ to estimate the x-coordinate of P. [2]

(i)	$\frac{dy}{dx} = \sin(xy)$ $y_1 = 1.5 + \frac{1}{3}\sin(1 \times 1.5) \approx 1.832498 \approx 1.8325$ $y_2 = 1.832498 + \frac{1}{3}\sin\left(\frac{4}{3} \times 1.832498\right) \approx 2.046794 \approx 2.0468$ $y_3 = 2.046794 + \frac{1}{3}\sin\left(\frac{5}{3} \times 2.046794\right) \approx 1.957970 \approx 1.9580$ Hence $y \approx 1.9580$ when $x = 2$
(ii)(a)	The y values increases and decreases suggesting that at some point P between $x = 1$ and $x = 2$, there is a maximum turning point.
*(b)	Let $\frac{dy}{dx} = \sin(xy) = 0$ $\sin(xy) = 0$ $xy = \pi$ When $y \approx 2.046794$, $x \approx \frac{\pi}{2.046794} \approx 1.53$

Question 4

(i) Determine the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 6 & 5 \end{pmatrix}$. [5]

(ii) By expressing $\mathbf{M}$ in the diagonalised form $\mathbf{QDQ}^{-1}$, find $\mathbf{M}^n$ for positive integers n. [4]

[Solution]

(i) det ($\mathbf{M} - \lambda\mathbf{I}$) = 0

$$\det \begin{pmatrix} 2-\lambda & 3 \\ 6 & 5-\lambda \end{pmatrix} = 0 \Rightarrow (2-\lambda)(5-\lambda) - 18 = 0$$

$$\lambda^2 - 7\lambda - 8 = 0$$

$(\lambda - 8)(\lambda + 1) = 0$. So $\lambda = 8$ or -1 are the eigenvalues

When $\lambda = 8$, $\begin{pmatrix} -6 & 3 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -6x + 3y = 0 \Rightarrow y = 2x$

A corresponding eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

When $\lambda = -1$, $\begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3x + 3y = 0 \Rightarrow x = -y$

A corresponding eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(ii) Let $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ then its corresponding diagonal matrix is $\mathbf{D} = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$

Then $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}^{-1}$

$$\mathbf{M}^n = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}^n\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 8^n & 0 \\ 0 & (-1)^n \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Recall:
$\mathbf{M} = \mathbf{QDQ}^{-1} \Rightarrow \mathbf{M}^n = \mathbf{QD}^n\mathbf{Q}^{-1}$

$$= \frac{1}{3}\begin{pmatrix} 8^n & (-1)^n \\ 2(8^n) & (-1)^{n+1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 8^n & (-1)^n \\ 2(8^n) & (-1)^{n+1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 8^n + 2(-1)^n & 8^n + (-1)^{n+1} \\ 2(8^n) + 2(-1)^{n+1} & 2(8^n) + (-1)^{n+2} \end{pmatrix}$$

Question 5

The sequence $\{P_n\}$ is given by $P_0 = 5,\ P_1 = 10$ and $P_{n+1} = \frac{1}{2}\left(P_n + P_{n-1}\right) + 3$ for $n \geq 1$.

(i) By considering the sequence $\{Q_n\}$, where $P_n = Q_n + 2n$ for $n = 0, 1, 2, \ldots$, find an expression for P_n as a function of *n*. [9]

(ii) Describe the long-term behaviour of P_n. [1]

[Solution]

(i) $P_{n+1} = \frac{1}{2}\left(P_n + P_{n-1}\right) + 3$ for $n \geq 1$.

$P_n = Q_n + 2n \Rightarrow P_{n-1} = Q_{n-1} + 2(n-1)$ and $P_{n+1} = Q_{n+1} + 2(n+1)$

$$P_{n+1} = \frac{1}{2}\left(P_n + P_{n-1}\right) + 3 \Rightarrow Q_{n+1} + 2(n+1) = \frac{1}{2}\ Q_n + 2n + Q_{n-1} + 2n - 2\ + 3$$

$$\Rightarrow 2Q_{n+1} + 4n + 4 = Q_n + 2n + Q_{n-1} + 2n - 2 + 6$$

$$\Rightarrow 2Q_{n+1} - Q_n - Q_{n-1} = 0$$

Characteristic equation: $2m^2 - m - 1 = 0$

$(2m + 1)(m - 1) = 0$. So $m = 1$ or $-\frac{1}{2}$

General solution is $Q_n = A(1)^n + B(-\frac{1}{2})^n = A + \frac{B(-1)^n}{2^n}$

$P_0 = 5,\ P_1 = 10 \Rightarrow Q_0 = 5$ and $Q_1 = 8$

So $5 = A + B$ ---- (1) and $8 = A - \frac{1}{2}B$ ---- (2)

Solving (1) and (2): $A = 7$ and $B = -2$

Thus $P_n = Q_n + 2n = 7 - 2(-\frac{1}{2})^n + 2n$

(ii) As $n \to \infty, \left(-\frac{1}{2}\right)^n \to 0$, $P_n \to 2n + 7$ (a arithmetic sequence) and thus $P_n \to \infty$ in terms of its value. (Do not just state that $P_n \to \infty$)

Question 6

The function f is given by $\mathrm{f}(x)=\sqrt{1-x^2}+\cos x-1$ for $0\le x\le 1$. It is known, from graphical work, that the equation $\mathrm{f}(x)=0$ has a single root $x=\alpha$.

(i) Express $\mathrm{g}(x)$ in terms of x, where $\mathrm{g}(x)=x-\dfrac{\mathrm{f}(x)}{\mathrm{f}'(x)}$. [2]

A student attempts to use the Newton-Raphson method, based on the form $x_{n+1}=\mathrm{g}(x_n)$, to calculate the value of α correct to 3 decimal places.

(ii) **(a)** The student first uses an initial approximation to α of $x_1=0$. Explain why this will be unsuccessful in finding a value for α. [1]

(b) The student next uses an initial approximation to α of $x_1=1$. Explain why this will also be unsuccessful in finding a value for α. [1]

(c) The student then uses an initial approximation to α of $x_1=0.5$. Investigate what happens in this case. [1]

(d) By choosing a suitable value for x_1, use the Newton-Raphson method, based on the form $x_{n+1}=\mathrm{g}(x_n)$ to determine α correct to 3 decimal places.

[Solution]

(i) $\mathrm{f}(x)=\sqrt{1-x^2}+\cos x-1$

f'(x) = $\dfrac{-2x}{2\sqrt{1-x^2}}-\sin x=\dfrac{-x}{\sqrt{1-x^2}}-\sin x$

$$\mathrm{g}(x)=x-\frac{\mathrm{f}(x)}{\mathrm{f}'(x)}=x-\frac{\sqrt{1-x^2}+\cos x-1}{\dfrac{-x}{\sqrt{1-x^2}}-\sin x}=x+\frac{\sqrt{1-x^2}+\cos x-1}{\dfrac{x}{\sqrt{1-x^2}}+\sin x}\quad\text{or}\quad x+\frac{(1-x^2)+\sqrt{1-x^2}(\cos x-1)}{x+\sqrt{1-x^2}\sin x}$$

(ii) (a)Given f(x) = 0 has a root α.

$x_1 = 0$, f'(0) = 0, so g(0) is not defined. The iterative formula $x_{n+1}=\mathrm{g}(x_n)$ is not defined at $x_1 = 0$.

(b) $x_1 = 1$, f'(1) is undefined as it has a vertical asymptote at x = 1.

(c) $x_1=0.5$, x_2 = g(0.5) = $0.5+\dfrac{(1-0.5^2)+\sqrt{1-0.5^2}(\cos 0.5-1)}{0.5+\sqrt{1-0.5^2}\sin 0.5}=0.5+\dfrac{0.6439833887}{0.9151946957}=1.204>1$

which falls outside of the domain of f , $0\le x\le 1$ so g(1.204) is not defined

(d) Let x_1 = 0.7 and using a GC [Note: using he ANS key]

x_2 = g(x_1) = 0.994867

x_3 = 0.9616706

x_4 = 0.9261648

x_5 = 0.9194135

x_6 = 0.9192791

So, correct to 3 dps, α = 0.919

Question 7

Unsupported answers from a graphing calculator are not allowed in this question.

In a manufacturing process, a company wishes to maximise the profit function

$$P = 2x + 3y + 2z + 3w,$$

where x, y, z and w represent the numbers of units of each of four different types of product that the company makes per week.

However, there are constraints on the numbers of these products that arise from the costs of the raw materials, the time taken to make each product and the transport costs to get the finished products out to their sales outlets. These are, in suitable units, as follows.

$$\begin{aligned} 3x + y + 2z + 10w &= 200 \\ x + 2y + 3z + w &= 100 \\ x + y + z + w &= 66 \end{aligned}$$

(i) Determine each of x, y and z in terms of w. [7]

(ii) Hence find the maximum value of P and state the values of x, y, z and w that give it. [6]

(iii) Suggest a course of action that the company might take as a result of the answers found in part **(ii)**. [1]

[Solution]

(i)A matrix equation for

$$\begin{aligned} 3x + y + 2z + 10w &= 200 \\ x + 2y + 3z + w &= 100 \\ x + y + z + w &= 66 \end{aligned} \quad \text{is : } \begin{pmatrix} 3 & 1 & 2 & 10 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 200 \\ 100 \\ 66 \end{pmatrix}$$

The augmented matrix is $\left(\begin{array}{cccc|c} 3 & 1 & 2 & 10 & 200 \\ 1 & 2 & 3 & 1 & 100 \\ 1 & 1 & 1 & 1 & 66 \end{array}\right)$

By carrying out the row operations (workings must be shown clearly),

the equation is equivalent to: $\begin{pmatrix} 1 & 0 & 0 & \frac{10}{3} & \frac{166}{3} \\ 0 & 1 & 0 & -\frac{14}{3} & -\frac{38}{3} \\ 0 & 0 & 1 & \frac{7}{3} & \frac{70}{3} \end{pmatrix}$

$$x + \tfrac{10}{3}w = \tfrac{166}{3} \Rightarrow x = \frac{1}{3}(166 - 10w)$$

$$y - \tfrac{14}{3}w = -\tfrac{38}{3} \Rightarrow y = \frac{1}{3}(14w - 38) \quad \text{and} \quad z + \tfrac{7}{3}w = \tfrac{70}{3} \Rightarrow z = \frac{1}{3}(70 - 7w)$$

(ii) $P = 2x + 3y + 2z + 3w$

$$= \frac{2}{3}(166 - 10w) + (14w - 38) + \frac{2}{3}(70 - 7w) + 3w$$

$$= \frac{17}{3}w + \frac{358}{3}$$

Since x, y, z and $w \geq 0 \Rightarrow 166 - 10w \geq 0 \Rightarrow w \leq \frac{166}{10}$

$$14w - 38 \geq 0 \Rightarrow w \geq \frac{19}{7} \text{ and } 70 - 7w \geq 0 \Rightarrow w \leq 10$$

Taking intersection, $\frac{19}{7} \leq w \leq 10$

So max $P = \frac{17}{3}w + \frac{358}{3} = \frac{170+358}{3} = 176$ when $w = 10$, $z = 0$, $x = 22$ and $y = 34$

(iii) Product represented by z is not worth producing as it does not contribute to profit. However, if there is a requirement of all the products must be produced, the model (represented by the system of equations) needs to be modified.

Question 8

For $n>2$, the complex numbers $z_1, z_2, \ldots, z_n$ are the n roots of the equation $z^n - 1 = 0$, with

$$0 < \arg(z_i) < \arg(z_j) \le 2\pi \quad \text{for } 1 \le i < j \le n$$

so that $z_n = 1$. The points $P_1, P_2, \ldots, P_n$ in the Argand diagram correspond to $z_1, z_2, \ldots, z_n$ respectively.

(i) Write down the coordinates of P_r in terms of r $(r = 1, 2, \ldots, n)$. [1]

(ii) By considering $\mathrm{Re}\left(e^{i\frac{2r\pi}{n}}\right)$, show that $\sum_{r=1}^{n} \cos\frac{2r\pi}{n} = 0$.

Find also the value of $\sum_{r=1}^{n} \cos\frac{4r\pi}{n}$, justifying your answer. [6]

(iii) The point A in the Argand diagram has coordinates $\left(\frac{1}{2}, 0\right)$. The distance from A to P_r is d_r.

Evaluate $\sum_{r=1}^{n} (d_r)^4$, giving your answer in terms of n. [6]

[Solution]

(i) $z^n - 1 = 0 \Rightarrow z^n = 1 = \cos 2r\pi + i \sin 2r\pi$ for $r = 1, 2, \ldots, n$

$$\Rightarrow z_r = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$$

Given P_r represents z_r

Thus $z_r = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$. So coordinates of P_r are $\left(\left(\cos\frac{2r\pi}{n}, \sin\frac{2r\pi}{n}\right)\right.$.

(ii) The roots of the equation $z^n - 1 = 0$ are $\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ where $k = 1, 2, \ldots, n$.

Method 1: Using sum of roots

As coefficient of z^{n-1} is 0, thus sum of roots = $-\frac{0}{1} = 0 \Rightarrow \sum_{r=1}^{n}\left(\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}\right) = 0$

Comparing the real part: $\sum_{r=1}^{n} \cos\frac{2r\pi}{n} = 0$

Method 2: Using Sum of GP

$$\sum_{r=1}^{n} e^{i\frac{2r\pi}{n}} = \frac{e^{i\frac{2\pi}{n}}\left(1 - e^{\left(i\frac{2\pi}{n}\right)n}\right)}{1 - e^{\left(i\frac{2\pi}{n}\right)}} = \frac{e^{i\frac{2\pi}{n}}\left(1 - e^{i2\pi}\right)}{1 - e^{\left(i\frac{2\pi}{n}\right)}} = \frac{e^{i\frac{2\pi}{n}}(1-1)}{1 - e^{\left(i\frac{2\pi}{n}\right)}} = 0$$

Hence $\sum_{r=1}^{n} \cos\frac{2r\pi}{n} = \mathrm{Re}\sum_{r=1}^{n} e^{i\frac{2r\pi}{n}} = \mathrm{Re}(0) = 0$

To find $\sum_{r=1}^{n} \cos\frac{4r\pi}{n}$

Consider $z^{2n} - 1 = 0 \Rightarrow (z^2)^n - 1 = 0$

Let $z^2 = w$ then the roots of $w^n - 1 = 0$ are $\left(\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}\right)^2 = \cos\frac{4r\pi}{n} + i\sin\frac{4r\pi}{n}$

Sum of roots = 0 $\Rightarrow \sum_{r=1}^{n}(\cos\frac{4r\pi}{n}+i\sin\frac{4r\pi}{n})=0$

Considering the real part: $\sum_{r=1}^{n}\cos\frac{4r\pi}{n}=0$

Method 2:

$$\sum_{r=1}^{n}\cos\frac{4r\pi}{n}=\mathrm{Re}\sum_{r=1}^{n}\mathrm{e}^{\mathrm{i}\frac{4r\pi}{n}}=\mathrm{Re}\left[\frac{\mathrm{e}^{\mathrm{i}\frac{4\pi}{n}}\left(1-\mathrm{e}^{\left(\mathrm{i}\frac{4\pi}{n}\right)n}\right)}{1-\mathrm{e}^{\left(\mathrm{i}\frac{4\pi}{n}\right)}}\right]=\mathrm{Re}\left[\frac{\mathrm{e}^{\mathrm{i}\frac{4\pi}{n}}\left(1-\mathrm{e}^{\mathrm{i}4\pi}\right)}{1-\mathrm{e}^{\left(\mathrm{i}\frac{4\pi}{n}\right)}}\right]=\mathrm{Re}\left[\frac{\mathrm{e}^{\mathrm{i}\frac{4\pi}{n}}(1-1)}{1-\mathrm{e}^{\left(\mathrm{i}\frac{4\pi}{n}\right)}}\right]=\mathrm{Re}(0)=0$$

(iii) The angle OP_r makes with the positive x-axis is $\frac{2r\pi}{n}$

Using cosine rule in ΔOAP_r and given $AP_r = d_r$

$$d_r^2=OA^2+1-2(OA)\cos\frac{2r\pi}{n}=\frac{1}{4}+1-\cos\frac{2r\pi}{n}\quad\text{for } r=1,2,3,\ldots,\ n$$

$$d_r^4=\left(\frac{5}{4}-\cos\frac{2r\pi}{n}\right)^2=\frac{25}{16}-\frac{5}{2}\cos\frac{2r\pi}{n}+\cos^2\frac{2r\pi}{n}$$

$$=\frac{25}{16}-\frac{5}{2}\cos\frac{2r\pi}{n}+\frac{1+\cos\frac{4r\pi}{n}}{2}=\frac{33}{16}-\frac{5}{2}\cos\frac{2r\pi}{n}+\frac{1}{2}\cos\frac{4r\pi}{n}$$

$$\sum_{r=1}^{n}(d_r)^4=\sum_{r=1}^{n}\left(\frac{33}{16}-\frac{5}{2}\cos\frac{2r\pi}{n}+\frac{1}{2}\cos\frac{4r\pi}{n}\right)$$

$$=\frac{33}{16}n\quad\text{as}\quad\sum_{r=1}^{n}\cos\frac{2r\pi}{n}=0\ \text{and}\ \sum_{r=1}^{n}\cos\frac{4r\pi}{n}=0\quad\text{from (ii)}$$

Method 2:

$$d_r^2=\left(\frac{1}{2}-\cos\frac{2r\pi}{n}\right)^2+\left(0-\sin\frac{2r\pi}{n}\right)^2$$

Question 9

Fig.1 below shows a section of a cylindrical pipe. Two regions, P and Q, within the pipe are separated by a barrier. The length of the barrier is L m.

The temperature in region P is greater than the temperature in region Q, so that there is a heat transfer from P to Q by conduction through the barrier, as indicated by the arrow.

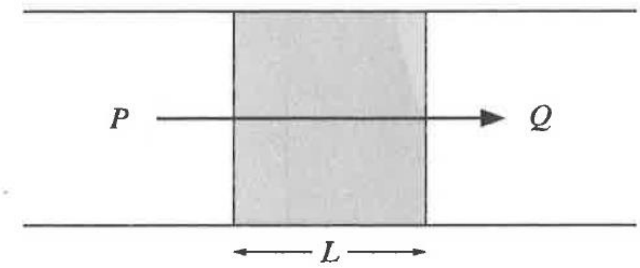


Fig. 1

The temperature T within the barrier varies according to the distance x m (measured from the left-hand boundary). The rate of heat transfer at this point is denoted by H, which is taken to be proportional to the product of the area of cross-section, A, of the barrier at that point and the rate of change, with respect to x, of T.

(i) Write down the expression for H. [1]

In an application of Fourier's law, it is assumed that $\frac{\mathrm{d}H}{\mathrm{d}x} = 0$.

(ii) In this case, show that $\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + \left(\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}\right)\frac{\mathrm{d}T}{\mathrm{d}x} = 0$. [3]

(iii) In the case when the barrier is a cylinder, find an expression for T in terms of x. [2]

(iv)

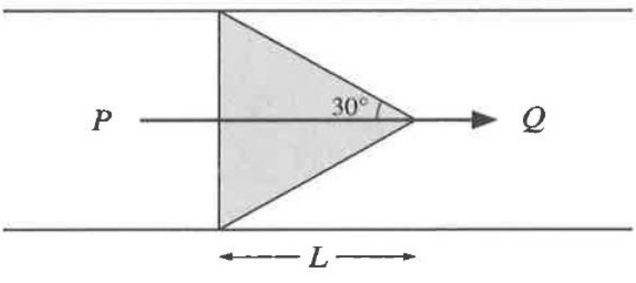
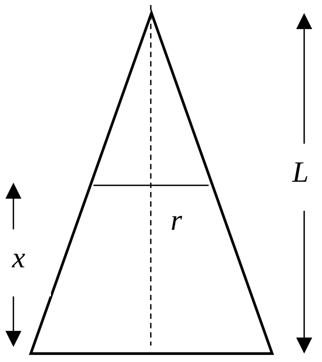


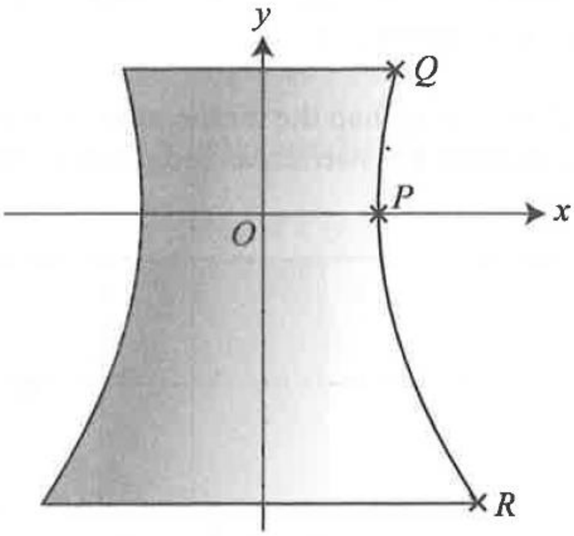
Fig. 2

Fig. 2 shows the case when the barrier between P and Q is a right circular cone, of semi-vertical angle 30°, with the vertex in Q. Suppose that the equation in part (ii) is also valid in this case. Find an expression for T in terms of x. [6]

(i)	$H \alpha A\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)$ $H = -kA\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)$, where $k > 0$. is the constant of proportionality
(ii)	$H = -kA\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)$ Differentiate w.r.t x: $\frac{\mathrm{d}H}{\mathrm{d}x} = -kA\left(\frac{\mathrm{d}^2T}{\mathrm{d}x^2}\right) - k\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)$ Since $\frac{\mathrm{d}H}{\mathrm{d}x} = 0$, $k\left[A\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)\right] = 0$ $\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + \frac{1}{A}\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right) = 0$ (shown)
(iii)	Since barrier is a cylinder, cross section A (which is a circle) is a constant, thus $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ $\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = 0$ Integrating w.r.t. x: $\frac{\mathrm{d}T}{\mathrm{d}x} = c_1$ Integrating w.r.t x: $T = c_1x + c_2$, where c_1 and c_2 are arbitrary constants
(iv)	 Let r be the radius of the cross section of the cone at distance x m. $\tan 30^\circ = \frac{r}{L-x} \Rightarrow r = \frac{1}{\sqrt{3}}(L-x)$ $A = \frac{\pi}{3}(L-x)^2$ Sub $A = \frac{\pi}{3}(L-x)^2$ into differential equation in part (ii), $\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + \frac{3}{\pi(L-x)^2}\left(-\frac{2\pi}{3}(L-x)\right)\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right) = 0$ $\frac{\mathrm{d}^2T}{\mathrm{d}x^2} - \frac{2}{(L-x)}\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right) = 0$ ------ (*) Integrating factor: $\mathrm{e}^{\int -\frac{2}{L-x}\mathrm{d}x} = \mathrm{e}^{2\ln(L-x)} = (L-x)^2$

	Multiplying (*) throughout by integrating factor: $\left(L-x\right)^2 \frac{\mathrm{d}^2T}{\mathrm{d}x^2} - 2\left(L-x\right)\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right) = 0$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(L-x\right)^2 \frac{\mathrm{d}T}{\mathrm{d}x}\right] = 0$ Integrating w.r.t x: $\left(L-x\right)^2 \frac{\mathrm{d}T}{\mathrm{d}x} = C_1$ $\frac{\mathrm{d}T}{\mathrm{d}x} = C_1\left(L-x\right)^{-2}$ Integrating w.r.t x: $T = \int C_1\left(L-x\right)^{-2}\,\mathrm{d}x$ $= \frac{C_1}{L-x} + C_2$, where C_1 and C_2 are arbitrary constants

Question 10



The picture shows the outline of a cooling tower. Coordinates axes have been superimposed onto this outline for ease of reference. The shape of the cooling tower is a section of a hyperboloid, which is created when a hyperbola is rotated about one of its axes of symmetry, in this case the y-axis.

The defining curve for the outline of the cooling tower is the hyperbola H which contains the points $P(42, 0)$, $Q(45.5, 55)$ and $R(70,-176)$, shown on the picture. P lies on the x-axis, the second axis of symmetry of H. All distances are in metres.

(i) Determine a Cartesian equation for H. [3]

(ii) Show that the volume contained by the tower is $\dfrac{2\,388\,309}{4}\pi$ cubic metres. [You do not need to consider the thickness of the tower's walls. You are required to show all necessary calculus working in order to justify the answer.] [5]

(iii) Find the integral which, when evaluated, will give exactly the surface area of the tower. Use your calculator to evaluate this integral numerically, giving your answer in square metres to the nearest integer. [7]

(i)	Since H is symmetrical about the x-axis and y-axis, the centre of H is (0, 0) Let equation of H be $\dfrac{x^2}{42^2}-\dfrac{y^2}{b^2}=1$ Sub $Q(45.5, 55)$ into equation of H: $\dfrac{55^2}{b^2}=\dfrac{25}{144} \Rightarrow b^2=132^2$ $\therefore$ Cartesian equation of H is $\dfrac{x^2}{42^2}-\dfrac{y^2}{132^2}=1$.

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<tr><td>(ii)</td><td>

$$\frac{x^2}{42^2}-\frac{y^2}{132^2}=1\Rightarrow x^2=42^2\left(1+\frac{y^2}{132^2}\right)$$

Using cylindrical disc method,

Required volume = $42^2\pi\int_{-176}^{55}\left(1+\frac{y^2}{132^2}\right)\mathrm{d}y$

$$=42^2\pi\left[y+\frac{y^3}{52272}\right]_{-176}^{55}$$

$$=42^2\pi\left[55+\frac{55^3}{52272}-\left(-176-\frac{176^3}{52272}\right)\right]$$

$$=\frac{2388309}{4}\pi\ \mathrm{m}^3\ \text{(shown)}$$

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<tr><td>(iii)</td><td>

$$x^2=42^2\left(1+\frac{y^2}{132^2}\right)$$

Differentiate w.r.t y:

$$2x\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{49y}{242}\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{49y}{484x}$$

Surface area of the tower = $2\pi\int_{-176}^{55}x\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y$

$$=2\pi\int_{-176}^{55}x\sqrt{1+\frac{49^2y^2}{484^2x^2}}\ \mathrm{d}y$$

$$=2\pi\int_{-176}^{55}x\frac{\sqrt{484^2x^2+49^2y^2}}{484x}\ \mathrm{d}y$$

$$=\frac{2}{484}\pi\int_{-176}^{55}\sqrt{20328^2\left(1+\frac{y^2}{132^2}\right)+49y^2}\ \mathrm{d}y$$

$$=\frac{1}{242}\pi\int_{-176}^{55}\sqrt{20328^2+26117y^2}\ \mathrm{d}y$$

$$\approx 73782\ \mathrm{m}^3\ \text{(to the nearest integer)}$$

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