HILLGROVE SECONDARY SCHOOL
PRELIMINARY EXAMINATION 2023
SECONDARY FOUR (EXPRESS)
[MARK SCHEME]

CANDIDATE NAME	( )		CLASS		-
CENTRE NUMBER	S		INDEX NUMBER		

Additional Mathematics 4049/02

Paper 2 **29 August 2023**

2 hours 15 minutes

Candidates answer on the Question Paper.

10.05 a.m. – 12.20 p.m.

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use	
TOTAL	**90**

Parent's/ Guardian's Signature: ____________________

Setters: Mdm Lee Li Lian

This document consists of **24** printed pages, including this page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$\left(a+b\right)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!\left(n-r\right)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}ab\sin C$$

1 A calculator must not be used in this question.

(a) Show that $\tan 15° = 2-\sqrt{3}$. [4]

$$\tan 15° = \tan(45° - 30°)$$
$$= \frac{\tan 45° - \tan 30°}{1+\tan 45° \tan 30°}$$
$$= \frac{1-\frac{\sqrt{3}}{3}}{1+1\times\frac{\sqrt{3}}{3}} \text{ [A1]}$$
$$= \frac{1-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}}$$
$$= \frac{3-\sqrt{3}}{3} \div \frac{3+\sqrt{3}}{3}$$
$$= \frac{3-\sqrt{3}}{3} \times \frac{3}{3+\sqrt{3}}$$
$$= \frac{3-\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}} \text{ [M1]}$$
$$= \frac{9-6\sqrt{3}+3}{9-3} \text{ [M1]}$$
$$= \frac{12-6\sqrt{3}}{6}$$
$$= 2-\sqrt{3} \text{ [A1]}$$

<u>Alternative Method</u>

$$\tan 15° = \tan(60° - 45°)$$
$$= \frac{\tan 60° - \tan 45°}{1+\tan 60° \tan 45°}$$
$$= \frac{\sqrt{3}-1}{1+\sqrt{3}\times 1} \text{ [A1]}$$
$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} \text{ [M1]}$$
$$= \frac{3-2\sqrt{3}+1}{2} \text{ [M1]}$$
$$= 2-\sqrt{3} \text{ [A1]}$$

(b) Use the result from **part (a)** to find an expression for $\sec^2 15°$, in the form $a+b\sqrt{3}$ where a and b are integers. [2]

$\sec^2 15° = 1+\tan^2 15°$

$= 1+\left(2-\sqrt{3}\right)^2$

$= 1+\underline{4-4\sqrt{3}+3}$ [A1]

$= 8-4\sqrt{3}$ [A1]

2 **(a)** Given that $\int_{-5}^{2} \mathrm{f}(x)dx = \int_{2}^{3} \mathrm{f}(x)dx = 5$, find $\int_{-5}^{3} 3\left[\mathrm{f}(x) - x\right]dx$. [4]

$$\int_{-5}^{3} 3\left[\mathrm{f}(x) - x\right]dx$$

$$= 3\int_{-5}^{3} \mathrm{f}(x)\ dx - 3\int_{-5}^{3} x\ dx$$

$$= 3\left[\underline{\int_{-5}^{2} \mathrm{f}(x)\ dx + \int_{2}^{3} \mathrm{f}(x)\ dx}\right] - 3\int_{-5}^{3} x\ dx \text{ [M1]}$$

$$= 3(5+5) - 3\left[\frac{x^2}{2}\right]_{-5}^{3} \text{ [M1]}$$

$$= 30 - \frac{3}{2}(9-25) \text{ [M1]}$$

$$= 54 \text{ [A1]}$$

(b) Differentiate $5x^2 \ln x$ with respect to x. Hence, find the value of $\int_1^3 5x \ln x \, dx$, giving your answer correct to 2 decimal places. [4]

$$\frac{d}{dx}(5x^2 \ln x) = 5x^2 \times \frac{1}{x} + \ln x \times 10x$$

$$= 5x + 10x \ln x \text{ [A1]}$$

$$\int_1^3 (5x + 10x \ln x) \, dx = \left[5x^2 \ln x\right]_1^3 \text{ [M1]}$$

$$\int_1^3 5x \, dx + \int_1^3 10x \ln x \, dx = \left[5x^2 \ln x\right]_1^3$$

$$\int_1^3 10x \ln x \, dx = \left[5x^2 \ln x\right]_1^3 - \int_1^3 5x \, dx$$

$$\frac{1}{2}\int_1^3 10x \ln x \, dx = \frac{1}{2}\left[5x^2 \ln x\right]_1^3 - \frac{1}{2}\int_1^3 5x \, dx$$

$$\therefore \int_1^3 5x \ln x \, dx = \frac{1}{2}\left(45 \ln 3 - 5 \ln 1\right) - \underline{\frac{1}{2}\left[\frac{5x^2}{2}\right]_1^3} \text{ [A1]}$$

$$= \frac{1}{2}\left(45 \ln 3 - 5 \ln 1\right) - \frac{1}{2}\left(\frac{45}{2} - \frac{5}{2}\right)$$

$$= \frac{1}{2}\left(45 \ln 3 - 5 \ln 1\right) - \frac{1}{2}\left(\frac{45}{2} - \frac{5}{2}\right)$$

$$= 14.7187765$$

$$\approx 14.72 \text{ (to 2 d.p.) [A1]}$$

3 **(a)** Solve the equation $5^x - 25^{x-1} - 6 = 0$. [5]

$5^x - 25^{x-1} - 6 = 0$

$5^x - 5^{2x-2} - 6 = 0$

$5^x - \frac{(5^x)^2}{25} - 6 = 0$ [M1]

$25 \times 5^x - (5^x)^2 - 150 = 0$

Let $y = 5^x$,

$25y - y^2 - 150 = 0$

$y^2 - 25y + 150 = 0$

$(y-10)(y-15) = 0$ [M1]

$y = 10$ or $y = 15$

$5^x = 10$ $\quad$ $5^x = 15$ [A1]

$\lg 5^x = \lg 10$ $\quad$ $\lg 5^x = \lg 15$

$x\lg 5 = \lg 10$ $\quad$ $x\lg 5 = \lg 15$

$x = \frac{\lg 10}{\lg 5}$ $\quad$ $x = \frac{\lg 15}{\lg 5}$ [M1 for either shown]

$= 1.430676558$ $\quad$ $= 1.682606194$

≈ 1.43 (to 3 s.f.) $\quad$ ≈ 1.68 (to 3 s.f.) [A1 for both]

(b) **(i)** Given that $\log_{343} x^3 = \log_{49} y$, express y in terms of x. [3]

$\log_{343} x^3 = \log_{49} y$

$\frac{\log_7 x^3}{\log_7 343} = \frac{\log_7 y}{\log_7 49}$ [M1]

$\frac{3\log_7 x}{3\log_7 7} = \frac{\log_7 y}{2\log_7 7}$

$\log_7 x = \frac{1}{2}\log_7 y$ [A1]

$= \log_7 \sqrt{y}$

$x = \sqrt{y}$

$y = x^2$ [A1]

Alternative Method

$\log_7 x = \frac{1}{2}\log_7 y$ [A1]

$2\log_7 x = \log_7 y$

$y = x^2$ [A1]

(ii) Find the value of x for which $\log_{49}(x^2 + 11x) - \log_{343} x^3 = \frac{1}{\log_{49} 7}$. [3]

$\log_{49}(x^2 + 11x) - \log_{343} x^3 = \frac{1}{\log_{49} 7}$

$\log_{49}(x^2 + 11x) - \log_{49} x^2 = \frac{1}{\log_{49} 7}$ [M1]

$\log_{49}(x^2 + 11x) - \log_{49} x^2 = \frac{1}{\frac{\log_7 7}{\log_7 49}}$

$\log_{49} \frac{x^2 + 11x}{x^2} = 2$ [M1]

$\frac{x^2 + 11x}{x^2} = 49^2$

$\frac{x^2 + 11x}{x^2} = 2401$

$2401x^2 = x^2 + 11x$

$2400x^2 - 11x = 0$

$x(2400x - 11) = 0$

$x = 0$ (N.A.) or $x = \frac{11}{2400}$ [A1]

4 A particle travels in a straight line so that, t seconds after leaving fixed point, O, its velocity is, v ms^{-1}, is given by $v = t^2 - 8kt + 6k$, where k is a constant. The minimum velocity of the particle occurs when $t = 12$.

(a) Show that $k = 3$. [2]

$v = t^2 - 8kt + 6k$

Acceeration, $a = \dfrac{dv}{dt}$

$= 2t - 8k$ [A1]

When $t = 12$, $\dfrac{dv}{dt} = 0$

$24 - 8k = 0$

$k = 3$ (shown) [A1]

(b) Determine whether the particle will return to O during its journey. [4]

$v = t^2 - 24t + 18$

Displacement, $s = \dfrac{t^3}{3} - \dfrac{24t^2}{2} + 18t + c$, where c is a constant

When $t = 0, s = 0$,

$c = 0$

$\therefore s = \dfrac{t^3}{3} - 12t^2 + 18t$ [A1]

When $s = 0$,

$\dfrac{t^3}{3} - 12t^2 + 18t = 0$ [M1]

$t^3 - 36t^2 + 54t = 0$

$t(t^2 - 36t + 54) = 0$

$t = 0$ or $t^2 - 36t + 54 = 0$

$$t = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(1)(54)}}{2(1)}$$

$$= \frac{36 \pm \sqrt{1080}}{2}$$

$= 34.43167673$ or 1.568323275 [A1]

$\therefore$ Yes, the particle will return to O at $t \approx 1.57$s and 34.4s. [A1]

(c) Find the total distance travelled by the particle in the first 2 seconds. [3]

When $v = 0$,

$t^2 - 24t + 18 = 0$

$t = \dfrac{24 \pm \sqrt{24^2 - 4\times1\times18}}{2\times1}$

$= \dfrac{24 \pm \sqrt{504}}{2}$

$= 23.22497216$ or 0.7750278397

≈ 23.2 or 0.775 [M1]

$s = \dfrac{t^3}{3} - 12t^2 + 18t$

When $t = 0, s = 0$ m

When $t = 0.7750278397, s = 6.897661467$ m [Either this or below M1]

When $t = 2, s = -9\frac{1}{3}$ m

Total distance travelled by the particle in the first 2 seconds

$= 2\times 6.897661467 + 9\frac{1}{3}$

$= 23.12865627$

≈ 23.1 m [A1]

5 It is given that $f(x) = 11 - ax - x^2 = 36 - (b + x)^2$, where a and b are both positive, for all real values of x.

(a) Find the value of a and of b. [2]

$f(x) = 11 - ax - x^2 = 36 - (b + x)^2$

$f(0) = 11 = 36 - b^2$

$b^2 = 25$

$b = 5$ [A1] or $b = -5$ (N.A., $\because b$ is positive)

$f(1) = 11 - a - 1 = 36 - (5 + 1)^2$

$10 - a = 0$

$a = 10$ [A1]

(b) Determine if $f(x)$ has a maximum or minimum value, state this value. [2]

$f(x) = 11 - 10x - x^2 = 36 - (5 + x)^2$

The maximum [A1] value of $f(x) = 36$ [A1]

(c) Find the range of values of x for which $f(x)$ is positive. [3]

When $f(x) > 0$,

$36 - (5 + x)^2 > 0$

$36 - (25 + 10x + x^2) > 0$

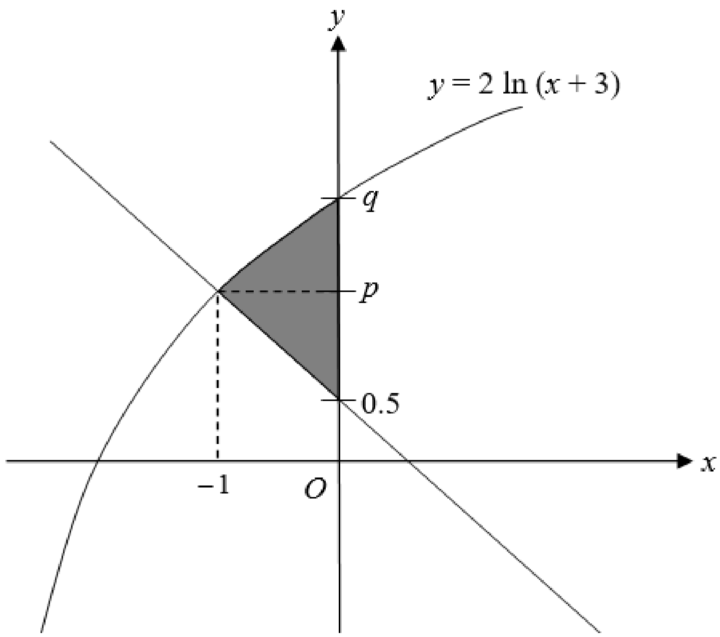
$36 - 25 - 10x - x^2 > 0$

$x^2 + 10x - 11 < 0$ [M1]

$(x - 1)(x + 11) < 0$ [M1]

$-11 < x < 1$ [A1]

6 In the diagram, the curve $y = 2\ln(x+3)$ cuts the y-axis at $(0, q)$. A line, which meets the curve at $(-1, p)$ cuts the y-axis at $(0, 0.5)$.



(a) State the exact value of p and of q. [2]

$p = 2\ln 2$ [Accept: $\ln 4$] [B1]

$q = 2\ln 3$ [Accept: $\ln 9$] [B1]

(b) Calculate the area of the shaded region. [4]

$$\text{Area of } \Delta = \frac{1}{2}(1)(\ln 4 - 0.5)$$
$$= 0.4431471806 \text{ [A1]}$$

$$y = 2\ln(x+3)$$
$$e^{\frac{y}{2}} = x+3$$
$$x = e^{\frac{y}{2}} - 3$$

Area of shaded region

$$= 0.4431471806 + \left| \int_{\ln 4}^{\ln 9} e^{\frac{y}{2}} - 3 \, dy \right| \text{ [M1]}$$
$$= 0.4431471806 + \left| \left[2e^{\frac{y}{2}} - 3y \right]_{\ln 4}^{\ln 9} \right| \text{ [M1]}$$
$$= 0.4431471806 + \left| \left(2e^{\frac{\ln 9}{2}} - 3\ln 9 \right) - \left(2e^{\frac{\ln 4}{2}} - 3\ln 4 \right) \right|$$
$$= 0.4431471806 + 0.4327906486$$
$$= 0.8759378292$$
$$\approx 0.876 \text{ units}^2 \text{(to 3 s.f.) [A1]}$$

7 **(a)** A formula for working out the braking distance, *d* for a vehicle travelling at a speed *v*, is $d = av^3 + bv^2$, where a and b are constants. Values of *d* for different values of *v* have been collected. [4]

Explain how a straight line can be drawn to represent the formula, and state how the values of *a* and *b* could be obtained from the line.

$d = av^3 + bv^2$

$\frac{d}{v^2} = av + b$ [A1]

Plot $\frac{d}{v^2}$ against *v*. [B1]

Gradient $= a$ [B1]

Vertical-intercept $= b$ [B1]

<u>Alternative Method</u>

$d = av^3 + bv^2$

$\frac{d}{v^3} = a + b\left(\frac{1}{v}\right)$ [A1]

Plot $\frac{d}{v^3}$ against $\frac{1}{v}$. [B1]

Gradient $= b$ [B1]

Vertical-intercept $= a$ [B1]

(b) The value, $\$V$, of an art piece has been increasing each year from 2008 to 2020. An auctioneer claims that the increase is exponential and so can be modelled by an equation in the form

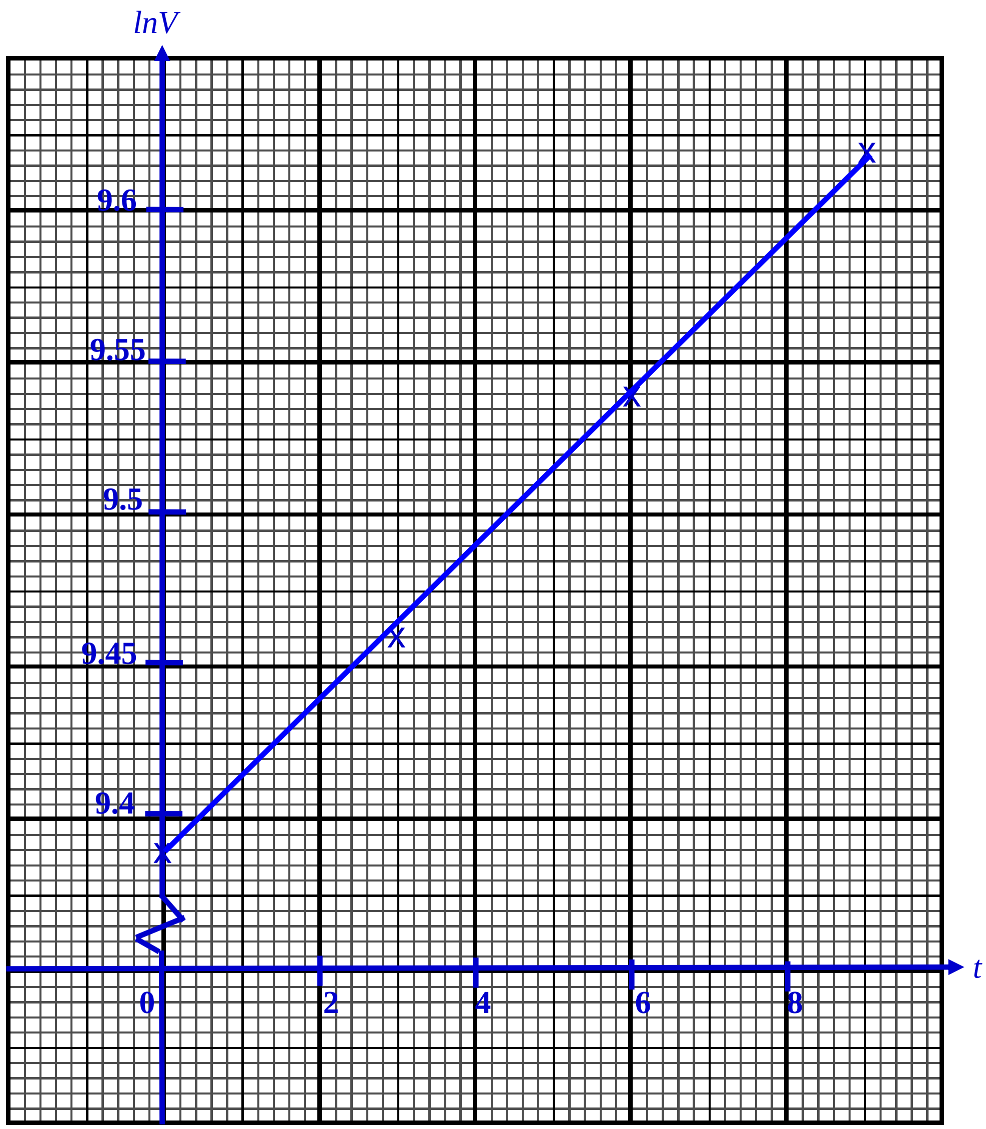
$$V = V_o e^{kt},$$

where V_o and k are constants and t is the time in years since 1st January 2008. The table below gives values of V and t for some of the years from 2008 to 2017.

Year	2008	2011	2014	2017
t years	0	3	6	9
$\$V$	12000	12900	13900	15000

(i) Plot $\ln V$ against t and draw a straight line graph to show that the model is valid for the years 2008 to 2020. [2]

t years	0	3	6	9
$\ln V$ [B1]	9.39	9.46	9.54	9.62



Best-fit line with vertical-axis intercept [B1]

(ii) Estimate the value of V_o and k. [3]

$$V = V_o e^{kt}$$
$$\ln V = \ln V_o e^{kt}$$
$$= \ln V_o + \ln e^{kt}$$
$$= \ln V_o + kt \ln e$$
$$= kt + \ln V_o$$

Vertical-axis intercept = $InV_o = 9.39$

$$V_o = e^{9.39}$$
$$= 11968.09933$$
$$\approx 12000 \text{ (to 3 s.f.) [A1]}$$

Gradient, $k = \dfrac{9.59 - 9.41}{8 - 0.8}$ [M1]

$= 0.025$ [A1] [Accept $\pm 0.0025 =$ 0.0225 to 0.0275]

(iii) Explain the significance of the value of V_o. [1]

It refers to the value of the art piece on 1st January 2008. (Accept initial value of the art piece. [B1]

(iv) Assuming that the model is still appropriate, estimate the value of the art piece on 1st January 2020. [2]

$$V = 11968.09933 e^{0.02465277778t}$$

When $t = 12$, $V = 11968.09933 e^{0.025(12)}$ [M1]

$= 16155.24429$

$\therefore$ The value of the art piece was about \$16155.24 [A1]

Accept \$15677.78 to \$16647.24

8 (a) Show that $\sin\left(\frac{\pi}{2}-x\right)=\cos x$ where x is measured in radians. [2]

$$\sin\left(\frac{\pi}{2}-x\right)=\sin\frac{\pi}{2}\cos x-\cos\frac{\pi}{2}\sin x \text{ [M1]}$$
$$=1\times\cos x-0\times\sin x$$
$$=\cos x \text{ (shown) [A1]}$$

(b) A musician wants to superimpose two sound waves to form an overall sound. Two such sound waves are $\mathrm{f}(t)$ and $\mathrm{g}(t)$ where, for $t\geq 0$ (in seconds),

$\mathrm{f}(t)=12\sin\left(\frac{t}{4}\right)+3\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)$ and $\mathrm{g}(t)=3\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)-4\sin\left(\frac{t}{4}\right)$.

The overall sound $\mathrm{C}(t)$ is found by adding the two sound waves $\mathrm{f}(t)$ and $\mathrm{g}(t)$.

Using the result from (a),

(i) show that the overall sound wave $\mathrm{C}(t)$ may be written in the form [2]

$$\mathrm{C}(t)=a\sin\left(\frac{t}{4}\right)+b\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)$$

where a and b are integers to be determined.

$$\mathrm{C}(t)=12\sin\left(\frac{t}{4}\right)+3\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)+3\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)-4\sin\left(\frac{t}{4}\right)$$
$$=8\sin\left(\frac{t}{4}\right)+6\sin\left(\frac{\pi}{2}-\frac{t}{4}\right) \text{ [A1]}$$

$\therefore a=8$ and $b=6$ [A1]

$C(t)$ may also be written in the form $C(t) = R\sin\left(\frac{t}{4}+\alpha\right)$, where R is a positive constant and α is an acute angle measured in radians.

(ii) Find the value of $\tan\alpha$ and R. [4]

$$C(t) = 8\sin\left(\frac{t}{4}\right) + 6\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)$$

$$= 8\sin\left(\frac{t}{4}\right) + 6\cos\left(\frac{t}{4}\right) \text{ [A1]}$$

$$8\sin\left(\frac{t}{4}\right) + 6\cos\left(\frac{t}{4}\right) = R\sin\left(\frac{t}{4}+\alpha\right)$$

$$= R\sin\frac{t}{4}\cos\alpha + R\cos\frac{t}{4}\sin\alpha \text{ [M1]}$$

$R\cos\alpha = 8$....................................(1)

$R\sin\alpha = 6$....................................(2)

$$\frac{(2)}{(1)}: \tan\alpha = \frac{6}{8}$$

$$= \frac{3}{4} \text{ [A1]}$$

$$R = \sqrt{8^2+6^2} = 10 \text{ [A1]}$$

(iii) Find the time, in seconds, at which the overall sound wave is first at its minimum. [3]

$\tan\alpha = \dfrac{3}{4}$

$\alpha = 0.6435011088$ radians [A1]

$\mathrm{C}(t) = 10\sin\left(\dfrac{t}{4} + 0.6435011088\right)$

When $\mathrm{C}(t) = -10$

$10\sin\left(\dfrac{t}{4} + 0.6435011088\right) = -10$ [M1]

$\sin\left(\dfrac{t}{4} + 0.6435011088\right) = -1$

$\dfrac{t}{4} + 0.6435011088 = \dfrac{3\pi}{2}$

$\dfrac{t}{4} = 4.068887872$

$t = 16.27555149$

≈ 16.3 (to 3 s.f.) [A1]

9 Three points are given by $P(3,-3)$, $Q(11,1)$ and $R(9,5)$.

(a) Show that angle *PQR* is 90°. [3]

$$\text{Gradient of } PQ, m_{PQ} = \frac{-3-1}{3-11}$$
$$= \frac{-4}{-8}$$
$$= \frac{1}{2} \text{ [A1]}$$
$$\text{Gradient of } QR, m_{QR} = \frac{5-1}{9-11}$$
$$= \frac{4}{-2}$$
$$= -2 \text{ [A1]}$$

Since $m_{PQ} \times m_{QR} = -1,$ then PQ is perpendicular to QR and

$P\hat{Q}R = 90°$ (shown) [A1]

<u>Alternative Method</u>

$$\text{Length of } PQ = \sqrt{(11-3)^2 + (1-(-3))^2} \text{ [M1]}$$
$$= \sqrt{80} \text{ units}$$
$$\text{Length of } QR = \sqrt{(11-9)^2 + (1-5)^2}$$
$$= \sqrt{20} \text{ units}$$
$$\text{Length of } PR = \sqrt{(3-9)^2 + (-3-5)^2}$$
$$= 10 \text{ units}$$

Since <u>$PQ^2 + QR^2 = PR^2$</u> [A1] ,∴ by converse of <u>Pythagoras' Theorem</u>, [A1]

PQ is perpendicular to QR and $P\hat{Q}R = 90°$ (shown)

(b) Explain why *P*, *Q* and *R* lie on a circle with diameter *PR*. [1]

By converse of <u>right angle in a semicircle</u>, since $P\hat{Q}R = 90°$, then *P*, *Q* and *R* lie on a circle with diameter *PR*.

(c) Find the equation of the circle in general form. [3]

$PR = \sqrt{(3-9)^2 + (-3-5)^2}$

$= 10$ units

Radius $= \frac{1}{2} \times 10$

$= 5$ units [A1]

Midpoint of $PR = \left(\frac{3+9}{2}, \frac{-3+5}{2} \right)$

= (6, 1)

Centre of the circle = (6, 1) [A1]

Equation of the circle:

$(x-6)^2 + (y-1)^2 = 5^2$

$x^2 - 12x + 36 + y^2 - 2y + 1 = 25$

$x^2 - 12x + y^2 - 2y + 12 = 0$ [A1]

(d) Explain why the tangent to the circle at Q is parallel to the y-axis. [2]

Let M be the centre of the circle, $M = (6,1)$

Equation of the radius, $MQ: y = 1$ is a horizontal line [B1]

Equation of tangent at $Q: x = 11$ is a vertical line [B1]

$\therefore$ The tangent to the circle at Q is parallel to the y-axis.

(e) Find the equation of the tangent to the circle at R. [2]

Gradient of $PR, m_{PR} = \frac{5-(-3)}{9-3}$

$= \frac{8}{6}$

$= \frac{4}{3}$

Gradient of the tangent at $R = -\frac{3}{4}$ [A1]

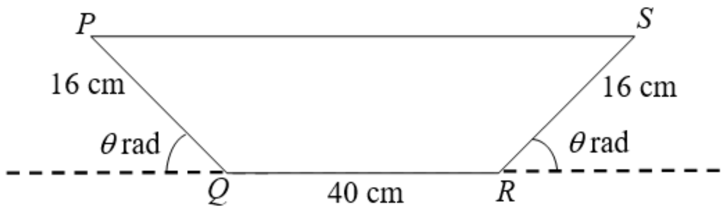
Equation of the tangent at R:

$\frac{y-5}{x-9} = -\frac{3}{4}$

$4y - 20 = -3x + 27$

$4y = -3x + 47$ [A1]

10 The diagram shows the vertical cross-section $PQRS$ of an open trough made from plastic sheeting. The lengths of PQ, QR and RS are 16 cm, 40 cm and 16 cm respectively. The trough rests with QR on horizontal ground and both PQ and RS are inclined at θ radians to the ground.



(a) Show that the area, A cm^2, of the cross-section $PQRS$ is given by [4]

$$A = 640\sin\theta + 128\sin 2\theta .$$

$PS = 40 + 2(16\cos\theta)$

$= 40 + 32\cos\theta$ [A1]

Height of the trough $= 16\sin\theta$ [A1]

$A = \frac{1}{2}\left(40 + 40 + 32\cos\theta\right)(16\sin\theta)$

$= 8\sin\theta(80 + 32\cos\theta)$

$= 640\sin\theta + 256\sin\theta\cos\theta$ [A1]

$= 640\sin\theta + 128(2\sin\theta\cos\theta)$

$= 640\sin\theta + 128\sin 2\theta$ (shown) [A1]

(b) Given that θ can vary, find the value of θ for which the trough can hold a maximum amount of water. [5]

$A = 640\sin\theta + 128\sin 2\theta$

$\dfrac{dA}{d\theta} = 640\cos\theta + 256\cos 2\theta$

When $\dfrac{dA}{d\theta} = 0$,

$640\cos\theta + 256\cos 2\theta = 0$ [M1]

$5\cos\theta + 2\cos 2\theta = 0$

$5\cos\theta + 2(2\cos^2\theta - 1) = 0$

$4\cos^2\theta + 5\cos\theta - 2 = 0$

$\cos\theta = \dfrac{-5 \pm \sqrt{5^2 - 4(4)(-2)}}{2(4)}$ [M1]

$= \dfrac{-5 \pm \sqrt{57}}{8}$

$= \dfrac{-5 + \sqrt{57}}{8}$ or $\dfrac{-5 - \sqrt{57}}{8}$ (N.A. $\because -1 \le \cos\theta \le 1$)

$\theta = 1.246407756$

≈ 1.25 radians (to 3 s.f.) [A1]

$\dfrac{d^2A}{d\theta^2} = -640\sin\theta - 512\sin 2\theta$ [A1]

When $\theta = 1.246407756$,

$\dfrac{d^2A}{d\theta^2} = -915.9780788 \underline{\le 0}$,

$\therefore$ A is a maximum. [A1]