(TOPICAL REVISION)
RECURRENCE RELATIONS

1 A sequence is defined by the recurrence relation

$$x_{n+1} = px_n + q, \text{ where } -1 < p < 1, \ x_0 = 20, \ n \geq 0, \ p, q \in \mathbb{R}.$$

(i) If $x_1 = 18$ and $x_2 = 17$, find the values of p and q.

(ii) Explain, using limits or otherwise, why the condition $-1 < p < 1$ is required for the sequence to be convergent.

(iii) Find the limit of the sequence.

(iv) Find a formula for x_r in terms of r.

2 Trees are sprayed weekly with the pesticide, 'Killpest', whose manufacturers claim it will destroy 65% of all pests. Between the weekly sprayings, it is estimated that 500 new pests invade the trees.

A new pesticide, 'Pestkill', comes onto the market. The manufacturers claim it will destroy 85% of existing pests but it is estimated that 650 new pests per week will invade the trees.

Which pesticide will be more effective in the long term?

3 Using the substitution $x_n = \dfrac{y_{n+1}}{y_n} - 7$ where $y_n \neq 0$, show that the recurrence relation $x_{n+1}x_n + 7x_{n+1} + x_n + 15 = 0$ can be expressed as $y_{n+2} + ay_{n+1} + by_n = 0$, where a and b are constants to be found. Hence, find the general solution of x_n in the form of $\mathrm{f}(n) - 7$ where $\mathrm{f}(n)$ is an expression in terms of n. [5]

[DHS et.al./Prelim 9649/2020/01/Q1]

4 The sequence of positive real numbers $\{y_n\}$ is given by

$$y_1 = 3, \ y_2 = 2 \qquad \text{and} \qquad \frac{y_n}{y_{n+1}} = 3\left(\frac{y_{n-1}}{y_n}\right) - 2n + 8, \quad \text{for } n \geq 2.$$

(i) By using the substitution $u_n = \dfrac{y_n}{y_{n+1}} - n$, determine a first order recurrence relation in terms of y_{n+1}, y_n and n. [5]

[DHS/Promo 9649/2021/Q5(i)]

5 Terms in the sequences W_n and I_n are defined as follow.

$$W_n = \begin{cases} 10 & \text{for } n = 0, \\ 4W_{n-1} + 3I_{n-1} & \text{for } n \geq 1, \end{cases} \quad \text{and} \quad I_n = \begin{cases} 5 & \text{for } n = 0, \\ I_{n-1} - 2W_{n-1} & \text{for } n \geq 1. \end{cases}$$

Show that $W_n - 5W_{n-1} + 10W_{n-2} = 0$ for $n \geq 2$.

Hence find an expression for W_n in terms of *n*. [9]

[VJC Promo 9649/2021/Q5]

6 A bit is represented by a binary number i.e. 0 or 1. A string of *n* bits is constructed such that there are no two consecutive 0s. Let a_n be the number of ways to construct this string.

(i) Find a_1 and a_2.

(ii) By considering the cases for the string ending with either a 0 or 1, find a recurrence relation for a_n.

(iii) Find the number of ways to construct this type of string with 20 bits.

7 A walkway is laid with *n* slate tiles. The color of the slate tiles are either red, green or gray. The tiles are laid on the walkway such that no two red tiles are adjacent and tiles of the same color are indistinguishable. Let a_n be the number of ways to lay out *n* tiles in the walkway.

(i) Find a_1 and a_2.

(ii) Find a 2$^{\text{nd}}$ order recurrence relation for a_n. Hence express a_n in terms of *n*.

8 The terms in the sequence $F_0, F_1, F_2, F_3, \ldots$ satisfy the recurrence relation

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1.$$

(i) Find the general solution of this recurrence relation. [2]

(ii) Find an expression for F_n in terms of *n* given that $F_1 = 1$ and $F_2 = 1$. [2]

(iii) Hence find the exact value of $\sum_{n=0}^{\infty} \frac{F_n}{3^{n+1}}$, simplifying your answer as far as possible. [4]

[RI/FM/2019/MCT/Q1]

9 A marine biologist keeps a water tank of brine shrimp for her research. At the end of each week, she finds that the number of new-born shrimp is equal to half of the number of shrimp one week ago, while the number of dead shrimp in the tank is $\frac{9}{16}$ of the number of shrimp two weeks ago. The number of shrimp (in thousands) in the tank at the end of week *n* is given by x_n.

(i) Form a recurrence relation and solve the recurrence relation, given that $x_1 = 330$ and $x_2 = 360$. [6]

(ii) Find the number of shrimp at the end of week 11, giving your answer to the nearest thousand. [1]

(iii) Explain what happens to the shrimp population if the research continues indefinitely. [2]

[HCI et al/FM/2018/P1/Q2]

10 A website requires numerical passcodes only using the digits 0 to 9, with repetition allowed. In addition, a passcode is considered acceptable if the passcode contains an even number of the digit 5. For example, the 7-digit passcode 0505055 is acceptable but 0505005 is not acceptable. Let u_n be the number of n-digit acceptable passcodes.

(i) State the value of u_1. [1]

(ii) Formulate with clear reasoning, a first order recurrence relation of the form $u_n = au_{n-1} + b^{n-1}$, $n \geq 2$, where a, b are constants to be determined. [3]

(iii) Hence, by repeated substitution, find u_n in the form $u_n = \frac{1}{k}\left(a^n + b^n\right)$, $k \in \mathbb{Z}^+$. [4]

[TJC/FM/2019/P2//Q1]

11 **(a)** The sequence u_n, $n \geq 1$, satisfies the recurrence relation

$$2\,u_{n+2} - 3\,u_{n+1} + u_n = 1, \text{ where } u_1 = 1, u_2 = 2.$$

Without solving for u_n, prove that the sequence diverges. [2]

(b) The sequence v_n, $n \geq 0$, satisfies the recurrence relation

$$v_{n+2} - \sqrt{2}\,v_{n+1} + v_n = 0, \text{ where } v_0 = 1,\ v_1 = \sqrt{2}$$

(i) Express v_n as a single trigonometric function of n. [5]

(ii) Hence, write down all possible values of the sequence. [2]

[VJC/FM/2019/P2/Q3]

12 A researcher breeds fruitflies in a small jar for experimental purposes. Observations of the number of fruitflies at the end of each week suggest that the population satisfies the following recurrence relation,

$$u_{n+1} = \left(2 - q\right)u_n - 4,$$

where u_n denotes the number of fruitflies at the end of the n^{th} week and q denotes the proportion of fruitflies that die of natural causes over the week.

(i) Given that $q < 1$, by considering $u_{n+1} - u_n$, show that there must initially be more than $4\left(1-q\right)^{-1}$ fruitflies in the jar for the population of fruitflies to grow.

(ii) Suppose that there are 10 and 15 fruitflies in the jar at the end of the first and second weeks respectively. Find the value of q and calculate the number of fruitflies in the jar at the end of the 20^{th} week, giving your answer to 1 significant figure. Comment on the practicality of your answer.

(iii) Another researcher suggests using a differential equation to model the growth of the population of fruitflies instead of a recurrence relation. Should this suggestion be taken up? Explain your answer.

13 The life cycle of *kaka*, a newly-discovered micro-organism, consists of three stages, namely *nympha*, *iuvenis* and *adultus*. After one day, an existing *nympha kaka* will mature into an *iuvenis kaka*, an existing *iuvenis kaka* will mature into an *adultus kaka* and *adultus kaka* will remain as *adultus kaka*.

Kaka undergoes a special type of asexual reproduction. On the day when a *nympha kaka* matures into an *iuvenis kaka*, it produces one *nympha kaka*. On the day when an *iuvenis kaka* matures into an *adultus kaka*, it produces nine *nympha kaka*. On every subsequent day in its lifetime, an *adultus kaka* will continue to produce nine *nympha kaka*.

Stephen accidentally comes in contact with two *nympha kaka* on a particular day. Let u_n be the number of *kaka* in Stephen's body on the n^{th} day.

(i) Formulate a recurrence relation relating u_n, u_{n-1} and u_{n-2}, stating the values of u_1 and u_2. [3]

(ii) Find the number of *kaka* found in Stephen's body on the n^{th} day. [4]

(iii) State an assumption in your workings. [1]

[AJC/2016/promo/4]

14 In an interrogation procedure, a captured espionage will be given a 25 milligram dose of a truth serum every 4 hours. 15% of the truth serum present in his body is lost every hour. Let u_n be the amount of the serum in his body just after the n^{th} dose.

(i) Calculate, in milligrams, the amount of truth serum remaining in his body after 4 hours and just before the second dose is administered. [1]

(ii) Find a recurrence relation that u_n satisfies in the form $u_{n+1} = au_n + b$, where a and b are constants to be determined. [2]

(iii) Solve the recurrence relation stated in **(ii)**. [4]

(iv) It is known that the level of serum in the body has to be continuously above 20 milligrams before the victim starts to confess. Determine the number of doses that are needed before the interrogation can begin. [3]

(v) It is also known that 55 milligrams of this serum in the body is fatal to the human body. Is there any maximum length of time the serum can be administered so that the espionage can be kept alive? [2]

[NYJC/FM/2017/P1/Q10]

15 There is a high demand for Sunsung's smartphones in the world. In order to meet the high demand, the company Sunsung sets up a new manufacturing factory X at the beginning of September 2018. By the end of the first month, factory X manufactures 4500 smartphones and for every subsequent month, it targets to manufacture 6% more functional smartphones than in the previous month. Due to inevitable glitches in technology, 120 smartphones are found to be faulty per month. Let x_n be the number of functional smartphones manufactured by factory X at the end of the n^{th} month.

(i) Write down the recurrence relation for x_n, stating clearly the initial condition. [2]

Factory X is instructed to deliver a monthly order of at least 6500 functional smartphones.

(ii) Solve the recurrence relation and hence determine the earliest time in which factory X is able to meet the required monthly delivery order. [5]

Another factory Y owned by Sunsung targets to manufacture k% more functional smartphones each month than in the previous month. Due to the aged machinery, the number of smartphones found to be faulty per month is 3.5% of the number of functional smartphones manufactured in the previous month. An initial number of 4000 functional smartphones are manufactured by the factory by the end of September 2018.

(iii) Write down the recurrence relation, in terms of k, for the number of functional smartphones manufactured by factory Y at the end of the nth month, stating clearly the initial condition. [2]

(iv) Find the least value of k if factory Y is tasked to deliver a monthly order of at least 6500 functional smartphones within the same timeline taken by factory X in part **(ii)**. [2]

Explain whether the recurrence relation model for factory Y is sustainable in the long run. [1]

[NJC/FM/2018/P1/Q9]

16 A new mobile operator, POMO, is offering an attractive mobile plan for all its newly signed up customers. During its launch on 1st January 2019, POMO sees a take up rate of 100 customers per month for the first 2 months. On the first day of each subsequent month, POMO is able to attract 20 customers more than the previous month total while at the same time, loses 1% of the total number of customers in the previous month.

Let c_n denotes the total number of customers under POMO at the end of n^{th} month after its launch.

(i) Show that $c_3 = 418$. [2]

(ii) Write down a recurrence relation for c_n where $n \geq 2$. Hence solve this recurrence relation. [3]

To compete with POMO, another mobile operator, Moonhub, decides to offer a new mobile plan to increase its customer base starting 1st May 2019. As a result, 200 customers sign up for the new plan in each the first 2 months of its launch. It is observed that at the start of each subsequent month, the new plan manages to attract twice the total number of customers under the same plan in the previous month but loses a number of customers equal to the difference in the total number of customers under the same plan in the previous 2 months.

Let d_n denotes the total number of customers under the new plan offered by Moonhub at the end of n^{th} month after its launch.

(iii) Show that $d_n - 2d_{n-1} - d_{n-2} = 0,$ for $n \geq 3.$ Hence find an expression for d_n in terms of n for $n \geq 1.$ [5]

(iv) At the end of which month will the total number of customers under the new mobile plan offered by Moonhub first exceed the total number of customers under POMO? Justify your answer clearly. [2]

[NJC/FM/2019/MYE/P1/Q9]

17 Ben, with k dollars initially, repeatedly wagers on a game of chance until he has no more money left, or achieves a total amount of N dollars. In each game he wins a dollar with probability p or loses a dollar with probability $1-p$. It can be assumed that each game is played independently of another. The probability that Ben achieves N dollars given that he has k dollars initially is given by R_k. It is known that R_k satisfies the recurrence relation

$$R_k = \left(\frac{1}{p}\right)R_{k-1} - \left(\frac{1-p}{p}\right)R_{k-2},$$

with boundary conditions $R_0 = 0$ and $R_N = 1$.

(i) Given that $p \neq \frac{1}{2}$, solve the above recurrence relation, giving your answer in terms of p. [6]

(ii) If $p = \frac{9}{19}$, explain whether it is more likely for Ben to achieve 20 dollars with an initial 10 dollars, or to achieve 120 dollars with an initial 100 dollars. [3]

(iii) If $p = \frac{1}{2}$, find the probability to achieve N dollars given that he starts with k dollars. [3]

[RI/FM/2017/P1/Q8]

18 **Do not use a calculator in answering this question.**

The sequence of Fibonacci numbers f_n, where $f_0 = 1$ and $f_1 = 1$, is defined by the recurrence relation

$$f_n = f_{n-1} + f_{n-2}\ ,\ n \in \mathbb{Z}^+,\ n \geq 2$$

The characteristic equation for solving the recurrence relation $m^2 - m - 1 = 0$ has roots ψ and ϕ where $\psi < \phi$. By first finding the exact value of ϕ, show that

$$f_n = k\left(\phi^{n+1} - \psi^{n+1}\right),$$

where k is a real number to be determined in exact form. [4]

Another sequence of positive real numbers g_n is defined by

$$g_n = \frac{f_n}{f_{n-1}},\ n \in \mathbb{Z}^+$$

(i) Show that $g_{n+1} = 1 + \frac{1}{g_n}$. By using the fact $\phi = 1 + \frac{1}{\phi}$, deduce that $g_{n+1} - \phi = \frac{\phi - g_n}{\phi g_n}$. [2]

Hence show that

(a) $g_{n+1} > \phi$ if $0 < g_n < \phi$

$g_{n+1} < \phi$ if $g_n > \phi$, [2]

(b) $\left|g_n - \phi\right| \le (\phi - 1)\phi^{1-n}$. [3]

For any sequences u_n, v_n and w_n such that $u_n \le v_n \le w_n$, *Squeeze Theorem* states that if both $\lim_{n\to\infty} u_n$ and $\lim_{n\to\infty} w_n$ exists and that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} w_n = L$, then $\lim_{n\to\infty} v_n = L$. Use *Squeeze Theorem* to find the exact value of $\lim_{n\to\infty} g_n$. [2]

(ii) Show that $g_{n+2} = \dfrac{2g_n + 1}{g_n + 1}$ and hence by using the fact $\phi = \dfrac{2\phi + 1}{\phi + 1}$, show that

$g_n < g_{n+2} < \phi$ if $0 < g_n < \phi$

$\phi < g_{n+2} < g_n$ if $g_n > \phi$. [6]

Use part **(i)** and **(ii)** of the question to describe the behavior of the sequence g_n. [2]

[ASRJC/FM/2019/P1//Q10]

<u>Answers</u>

1(i) $p = \frac{1}{2},\ q = 8$ **(iii)** $\lim_{n\to\infty} x_n = 16$ **(iv)** $x_r = 16 + 4\left(\frac{1}{2}\right)^r,\ r \ge 0$

2 Pestkill

3 $x_n = \dfrac{A(2^{n+1}) + B(4^{n+1})}{A(2^n) + B(4^n)} - 7$

4 $\dfrac{y_n}{y_{n+1}} = 3^n + n - \dfrac{5}{2}$

5 $W_n = 10^{\frac{n}{2}}\left(10\cos(0.659n) + 15.5\sin(0.659n)\right)$

6(i) 2, 3 **(ii)** $\therefore a_n = a_{n-1} + a_{n-2},\ n \ge 3$ **(iii)** 17711 ways

7(i) 3, 8 **(ii)** $a_n = 2\left(a_{n-1} + a_{n-2}\right)$, $a_n = \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)(1+\sqrt{3})^n + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)(1-\sqrt{3})^n,\ n \ge 1$

8(i) $F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$ **(ii)** $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ **(iii)** $\frac{1}{5}$

9(i) $x_n = 240\left(\frac{3}{4}\right)^n + 200n\left(\frac{3}{4}\right)^n$ **(ii)** 103 000 (to nearest thousand)

10(i) 9 **(ii)** $u_n = 8u_{n-1} + 10^{n-1}$ **(iii)** $u_n = \frac{1}{2}\left(8^n + 10^n\right)$

11(b)(i) $v_n = \sqrt{2}\cos\left(\frac{n\pi}{4} - \frac{\pi}{4}\right)$ $\left[\text{or } \sqrt{2}\sin\left(\frac{n\pi}{4} + \frac{\pi}{4}\right)\right]$ **(ii)** $0,\ \pm 1, \pm\sqrt{2}$.

12(ii) $q = 0.1$, $u_{20} = 1000000$

13(i) $u_n = 2u_{n-1} + 8u_{n-2},\ n \geq 3$, $u_1 = 2$, $u_2 = 4$ **(ii)** $u_n = \frac{1}{3}(4)^n - \frac{1}{3}(-2)^n$

14(i) 13.1 **(ii)** $u_{n+1} = 0.522u_n + 25$ **(iii)** $u_n = 52.3(1 - 0.522^n)$ **(iv)** 3 **(v)** No

15(i) $x_n = 1.06x_{n-1} - 120,\ n > 1,\ x_1 = 4380$ **(ii)** $x_n = 2380\left(1.06^{n-1}\right) + 2000,\ n \geq 1$**,** End Aug 2019

(iii) $y_n = \left(\frac{96.5 + k}{100}\right)y_{n-1},\ n > 1,\ y_1 = 4000$ **(iv)** 8.01

16(ii) $c_n = 55.6(1.99)^n - \frac{2000}{99}$ **(ii)** $d_n = 50\sqrt{2}\left(1 + \sqrt{2}\ \right)^n - 50\sqrt{2}\left(1 - \sqrt{2}\right)^n$

(iii) End June 2020

17(i) $R_k = \left[1 - \left(\frac{1-p}{p}\right)^k\right] \div \left[1 - \left(\frac{1-p}{p}\right)^N\right]$ **(iii)** $R_k = \frac{k}{N}$

18(i)(last) $\lim_{n\to\infty} g_n = \phi$
