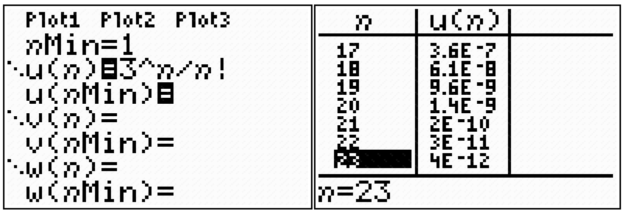
2022 C1 H2 Mathematics

Block Test Revision Package

1

Sequences and Series

Skill Set

s/n	Skills	Examples of questions involving the skills
1.	Evaluating limits (i) algebraically (ii) using GC	(i) **General formula** Perform long division first before evaluating limits <u>involving improper fractions</u>. $\lim_{n\to\infty}\frac{2n^2+3}{n^2+1}=\lim_{n\to\infty}\left(2+\frac{1}{n^2+1}\right)=2$ Alternatively, $\lim_{n\to\infty}\frac{2n^2+3}{n^2+1}=\lim_{n\to\infty}\left(\frac{2+\frac{3}{n^2}}{1+\frac{1}{n^2}}\right)=2$ (ii) **(Tutorial 1A Q1(c))** As $n\to\infty$, $\frac{n^2}{n^3+2014}\to 0$, therefore the sequence is convergent and the limit is 0. (iii) **(Tutorial 1A Q2(b))** $u_n=\frac{n+1}{n}=1+\frac{1}{n}$ when $n\to\infty$, $\frac{1}{n}\to 0$, therefore $u_n\to 1$. Thus u_n is convergent and limit is 1. (iv) **(Lecture Notes Example 4(c))** By keying in the general formula for the sequence, $u_n=\frac{3^n}{n!}$ into the GC, we can observe how the sequence behaves as n increases.  By scrolling down the table, we can see that $\lim_{n\to\infty}\frac{3^n}{n!}=0$.
2.	Replacement method to find (i) S_{n-1} from S_n, (ii) u_{n-1} from u_n.	(i) **(Lecture Notes Example 2)** We replace n by $(n-1)$ in $S_n=2n+n^2$ to obtain $S_{n-1}=2(n-1)+(n-1)^2$. (ii) Similarly, we replace n by $(n-1)$ in the expression of u_n to obtain u_{n-1}.

<table>
<tr><th>s/n</th><th>Skills</th><th>Examples of questions involving the skills</th></tr>
<tr><td>3.</td><td>Give answers in the context of a problem.</td><td>(i) (Tutorial 1A Q4(iii))
After solving for $r=-\frac{1}{2}$ or $r=1$, the value $r=1$ is rejected as the question states that sum to infinity S exists.
(ii) (Tutorial 1A Q5(iii))
After solving the inequality for n, we obtain $5.228 < n < 13.77$. However, the number of terms, n must be a positive integer. The final answer should be given as $\left\{n \in \mathbb{Z}^{+} : 6 \le n \le 13\right\}$.</td></tr>
<tr><td>4.</td><td>Recognize a new sequence is formed when certain terms are extracted from a given sequence.</td><td>(Lecture Notes Example 9)
A new sequence is formed using the even-numbered terms (i.e. $u_2, u_4, u_6, \ldots$) of an AP with first term a and common difference d.
• The new sequence is also an AP with first term $(a+d)$ and common difference $2d$.</td></tr>
<tr><td>5.</td><td>Formulate mathematical equations or formulae from a word problem.

Some methods include
- write out the first few terms and find the pattern for the general term

- Recognize AP or GP or their sums</td><td>(Tutorial 1A Question 11)
• Write down the first few amounts tracking every change to it.
<table>
<tr><th>n</th><th>Amount worth in account at <u>end</u> of nth month</th></tr>
<tr><td>1</td><td>$100(1+0.002)$</td></tr>
<tr><td>2</td><td>$100(1+0.002)^2$</td></tr>
<tr><td></td><td>…..</td></tr>
<tr><td>12</td><td>$100(1+0.002)^{12} = \$102.43$ (nearest cents)</td></tr>
</table>
○ For X% interest, use a factor of $1+\frac{X}{100}$
• Continue filling in the next few rows until you are confident of writing down in the nth row.
<table>
<tr><th>n</th><th>Amount worth in account at <u>end</u> of nth month</th></tr>
<tr><td>1</td><td>$100(1+0.002)$</td></tr>
<tr><td>2</td><td>$\big(100+100(1.002)\big)(1.002)$</td></tr>
<tr><td></td><td>…..</td></tr>
<tr><td>12</td><td>$100(1.002)+100(1.002)^2+100(1.002)^3+100(1.002)^{12}$
$=100(1.002)[1+1.002+\ldots(1.002)^{11}]$
$=\frac{100(1.002)[1-(1.002)^{12}]}{[1-1.002]}$
$=\$1215.71$</td></tr>
</table>

A number of other such word problems can be found in the lecture examples 18, 19, 20 and supplementary exercises question 3, 6, 9, 10, 12, 24, 25.</td></tr>
<tr><td>6.</td><td>Method of differences using partial fractions.</td><td>(Lecture Notes Example 26)
• Here is a quick check on partial fractions used for method of differences (only for MOD):</td></tr>
</table>

s/n	Skills	Examples of questions involving the skills
		For cancellations of terms to work in the method of differences, the coefficients of the partial fractions usually add up to 0. For the partial fraction $\frac{1}{8(2r-1)}-\frac{1}{4(2r+1)}+\frac{1}{8(2r+3)}$, we can see that $\frac{1}{8}+\left(-\frac{1}{4}\right)+\frac{1}{8}=0$. • Check that the terms are arranged in ascending order or descending order before writing out the rows of terms. This will enable you to cancel with ease. • Spot the terms for cancellations by looking at the denominators: $\sum_{r=1}^{n}\left[\frac{1}{8(2r-1)}-\frac{1}{4(2r+1)}+\frac{1}{8(2r+3)}\right]$ $=\left[\frac{1/8}{(1)}-\frac{1/4}{(3)}+\cancel{\frac{1/8}{(5)}}\right.$ $+\frac{1/8}{(3)}-\cancel{\frac{1/4}{(5)}}+\cancel{\frac{1/8}{(7)}}$ $+\cancel{\frac{1/8}{(5)}}-\cancel{\frac{1/4}{(7)}}+\cancel{\frac{1/8}{(9)}}$ $+\cdots$ $+\cancel{\frac{1/8}{(2n-3)}}-\cancel{\frac{1/4}{(2n-1)}}+\frac{1/8}{(2n+1)}$ $\left.+\cancel{\frac{1/8}{(2n-1)}}-\frac{1/4}{(2n+1)}+\frac{1/8}{(2n+3)}\right]$ It is easier to spot such trends using the denominators if one do not simplify the original form of the fractions, i.e. $\frac{1/8}{5},-\frac{1/4}{5},\frac{1/8}{5}$ to $\frac{1}{40},-\frac{1}{20},\frac{1}{40}$.
7.	Recognize method of differences when the question did not say so.	• **(Tutorial 1B Q6)** Verify that $\frac{1}{2x}-\frac{1}{(x-1)}+\frac{1}{2(x-2)}=\frac{1}{x(x-1)(x-2)}$. By using the above result, find $\sum_{n=3}^{N}\frac{1}{n(n-1)(n-2)}$.

s/n	Skills	Examples of questions involving the skills
		$\displaystyle\sum_{n=3}^{N}\frac{1}{n(n-1)(n-2)}=\frac{1}{2}\sum_{n=3}^{N}\left[\frac{1}{n}-\frac{2}{(n-1)}+\frac{1}{(n-2)}\right]$ $\displaystyle=\frac{1}{2}\begin{bmatrix}\frac{1}{3} & -\frac{2}{2} & +\frac{1}{1}\\ +\frac{1}{4} & -\frac{2}{3} & +\frac{1}{2}\\ +\frac{1}{5} & -\frac{2}{4} & +\frac{1}{3}\\ +\frac{1}{6} & -\frac{2}{5} & +\frac{1}{4}\\ \vdots & & \\ +\frac{1}{N-1} & -\frac{2}{N-2} & +\frac{1}{N-3}\\ +\frac{1}{N} & -\frac{2}{N-1} & +\frac{1}{N-2}\end{bmatrix}$ This question does not tell you to show by using method of difference. However, we can observe that a difference of similar looking terms $\frac{1}{2x}-\frac{1}{(x-1)}+\frac{1}{2(x-2)}$ is given which is an indication that method of difference may be the technique to employ. • **(Tutorial 1B Q9)** When the question asks you to show a difference $\mathrm{f}(r)-\mathrm{f}(r+1)$ of a given function, $\mathrm{f}(r)$, it is also an indication that method of difference may be the technique to employ.
8.	Add or subtract terms at the beginning of a summation	• **(Lecture Notes Example 24(i))** We want terms r = 10 to r = n, so from the known summation of r = 1 to r = n, we remove the terms we do <u>NOT</u> want, i.e. r = 1 to r = 9. Be careful to ensure the r = 10 is NOT removed. • **(Tutorial 1B Q6(last part))** Sometimes we need to add terms at the beginning instead, e.g. $\sum_{n=1}^{N}\frac{1}{n^3}=\frac{1}{1^3}+\frac{1}{2^3}+\sum_{n=3}^{N}\frac{1}{n^3}$. • For terms at the end of the summation, it is faster to do a direct replacement of the upper index value. e.g. Given $\sum_{n=1}^{N}\frac{1}{n^2}=\frac{1}{6}N(N+1)(2N+1)$, $\sum_{n=1}^{N+1}\frac{1}{n^2}=\frac{1}{6}(N+1)\left[(N+1)+1\right]\left[2(N+1)+1\right]$.
9.	Find a "new" sum using a given sum. Replace the variable	• **(Tutorial 1B Q7(iii))**

s/n	Skills	Examples of questions involving the skills
	in the term of the sum to get from the new sum to the old one.	$\sum_{r=1}^{n}\frac{r^2+5r+5}{(r+3)!}$ Replacing r by $r-1$ (the start and end values of the index must be changed correspondingly too) Replacing r by $r-1$ $\sum_{r=1}^{n}\frac{r^2+5r+5}{(r+3)!}$ $=\sum_{r-1=1}^{r-1=n}\frac{(r-1)^2+5(r-1)+5}{(r-1+3)!}$ $=\sum_{r=2}^{r=n+1}\frac{r^2-2r+1+5r-5+5}{(r+2)!}$ $=\sum_{r=2}^{n+1}\frac{r^2+3r+1}{(r+2)!}$ $=\sum_{r=1}^{n+1}\frac{r^2+3r+1}{(r+2)!}-\frac{1^2+3(1)+1}{3!}$ $=\frac{3}{2}-\frac{(n+1)+3}{((n+1)+2)!}-\frac{5}{6}$ $=\frac{2}{3}-\frac{n+4}{(n+3)!}$ • **(Tutorial 1B Q9(iii))** Replace r by $r-1$, (the start and end values of the index must be changed correspondingly too) Replacing r by $r-1$ $\sum_{r=1}^{n}\frac{r^2+5r+5}{(r+3)!}$ $=\sum_{r-1=1}^{r-1=n}\frac{(r-1)^2+5(r-1)+5}{(r-1+3)!}$ $=\sum_{r=2}^{r=n+1}\frac{r^2-2r+1+5r-5+5}{(r+2)!}$ $=\sum_{r=2}^{n+1}\frac{r^2+3r+1}{(r+2)!}$ $=\sum_{r=1}^{n+1}\frac{r^2+3r+1}{(r+2)!}-\frac{1^2+3(1)+1}{3!}$ $=\frac{3}{2}-\frac{(n+1)+3}{((n+1)+2)!}-\frac{5}{6}$ $=\frac{2}{3}-\frac{n+4}{(n+3)!}$

10.	Use found/proven general formula of a sum/sequence to show a given inequality.	**(Tutorial 1B Q6)** Given that $\sum_{n=3}^{N}\frac{1}{n(n-1)(n-2)}=\frac{1}{4}-\frac{1}{2N(N-1)}$, show that $\sum_{n=1}^{N}\frac{1}{n^3}<\frac{11}{8}$. • It is important to note that for inequality question, we need to draw the link between what is given. i.e., $\sum_{n=3}^{N}\frac{1}{n(n-1)(n-2)}=\frac{1}{4}-\frac{1}{2N(N-1)}$ and what is to be shown, $\sum_{n=1}^{N}\frac{1}{n^3}<\frac{11}{8}$. • We compare the original summand $\frac{1}{n(n-1)(n-2)}$ to the new summand $\frac{1}{n^3}$. It is easy to see that for $n\in\mathbb{Z}^+$, $n^3>n(n-1)(n-2)$, so $\frac{1}{n^3}<\frac{1}{n(n-1)(n-2)}$. If it is not obvious which summand is larger, you may need to expand the summand. For example, $(r+1)^3=r^3+3r^2+3r+1$ and $r(r+1)(r+2)=r^3+3r^2+2r$, we can tell $(r+1)^3>r(r+1)(r+2)$. • Then, we apply summation to both sides with the same index values of 3 to n. Note that 1 to n does not work here as $\frac{1}{n(n-1)(n-2)}$ cannot take n = 1 and n = 2. • Use skill 8 above to change the starting index value. The fact that $\frac{1}{4}$ in the original summation does not match with $\frac{11}{8}$ in the inequality to be proven is a hint that the starting index needs to be changed.

A (Method of Difference/Sigma Notation)

1. ACJC14/C1Mid-year/Q6

(i) Show that $\frac{9x^2+15x-2}{9x^2+15x+4}=A+\frac{B}{3x+1}+\frac{C}{3x+4}$, where A, B and C are constants to be found. [2]

(ii) Hence find $\sum_{r=1}^{n}\frac{9r^2+15r-2}{9r^2+15r+4}$ in terms of n. [3]

(iii) Explain if the series $\sum_{r=1}^{n}\frac{9r^2+15r-2}{9r^2+15r+4}$ is convergent. [1]

(iv) Find $\sum_{r=0}^{n-2}\frac{9(r+1)^2+15r+13}{9(r+1)^2+15r+19}$ in terms of n. [2]

2. **AJC14/C1Mid-year/Q7**

(i) Show that $\dfrac{2}{x(x+2)}=\dfrac{1}{x}-\dfrac{1}{x+2}$ where $x \neq -2, 0$. [1]

(ii) Let $\mathrm{f}(n)=(-1)^n\left(\dfrac{2}{n(n+2)}\right)$, where $n \in \mathbb{Z}^+$. By using the method of differences, show that $\displaystyle\sum_{n=1}^{N}\mathrm{f}(n)=\frac{1}{(N+1)(N+2)}-\frac{1}{2}$, where N is a positive even integer. [4]

(iii) For any positive integer M, determine $\displaystyle\sum_{n=1}^{M}\mathrm{f}(n)$ in terms of M.

Hence find $\displaystyle\sum_{n=1}^{\infty}\mathrm{f}(n)$. [4]

3. **JJC13/C2Mid-year/Q4(b)**

A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_n=\frac{1}{n!} \quad \text{for } n \geq 1.$$

(i) Show that $u_n-u_{n+1}=\dfrac{n}{(n+1)!}$. [2]

(ii) Find $\displaystyle\sum_{n=1}^{N}\frac{n}{(n+1)!}$ in terms of N. [3]

(iii) Hence show that for $N \geq 2$, the sum of the series

$$\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots+\frac{N}{(N+1)!}$$

is less than $\dfrac{1}{2}$. [3]

4. **CJC14/C1Mid-year/Q11**

(i) By considering u_r-u_{r+1}, where $u_r=\dfrac{1}{(r+1)!}$ for $r \geq 1$, show that

$$\sum_{r=1}^{N}\frac{r+1}{2(r+2)!}=\frac{1}{4}-\frac{1}{2(N+2)!}$$ [4]

(ii) Deduce that $\displaystyle\frac{1}{6} \leq \sum_{r=1}^{N}\frac{r+1}{2(r+2)!}<\frac{1}{4}$. [2]

(iii) Give a reason why the series $\displaystyle\sum_{r=0}^{\infty}\frac{r+1}{2(r+2)!}$ converges and write down its value. [3]

(iv) Find $\displaystyle\sum_{r=6}^{N}\frac{r}{2(r+1)!}$ in terms of N using the result in **(i)**. [3]

5. **HCI14/C1Mid-year/Q7**

(i) Express $\dfrac{r^2+5r+8}{r(r+1)(r+2)}$ in partial fractions. [3]

(ii) Hence find $\displaystyle\sum_{r=1}^{n}\frac{r^2+5r+8}{r(r+1)(r+2)}\left(\frac{1}{2^{r+2}}\right)$, give answer in the form $\dfrac{3}{8}-\mathrm{f}(n)$. [3]

(iii) Use your answer to part **(ii)** to show that $\displaystyle\sum_{r=1}^{n}\frac{1}{r2^{r+2}}<\frac{3}{8}$. [3]

6. **NJC14/C1Mid-year/Q6**

(i) Prove that $\cos[(n+1)\theta]-\cos[(n-1)\theta]=-2\sin(n\theta)\sin\theta$, where $n\in\mathbb{Z}$. [1]

(ii) Hence find $\displaystyle\sum_{n=1}^{N}\sin(n\theta)$. [3]

(iii) Deduce the exact value of $\sin\dfrac{\pi}{6}+\sin\dfrac{\pi}{3}+\sin\dfrac{\pi}{2}+\ldots+\sin\dfrac{29\pi}{6}$. [4]

7. **SAJC 2018/BT/7**

(i) Express $\dfrac{6r+18}{(r-1)r(r+2)}$ in the form $\dfrac{A}{r-1}+\dfrac{B}{r}+\dfrac{C}{r+2}$, where A, B and C are constants to be determined. [2]

(ii) Hence show that $\displaystyle\sum_{r=2}^{n}\frac{r+3}{(r-1)r(r+2)}=\frac{43}{36}-\frac{4}{3n}+\frac{1}{6(n+1)}+\frac{1}{6(n+2)}$. [4]

(iii) Use your answer to part **(ii)** to find $\displaystyle\sum_{r=2}^{n}\frac{r+4}{r(r+1)(r+3)}$. [3]

(iv) State the value of $\displaystyle\lim_{n\to\infty}\sum_{r=2}^{n}\frac{r+4}{r(r+1)(r+3)}$. [1]

(v) Hence, deduce $\displaystyle\sum_{r=2}^{\infty}\frac{r+4}{(r+3)^3}<\frac{41}{72}$. [3]

8. **RI19/C1Promo/Q3**

The sequence $u_1, u_2, u_3, \ldots$ is given by $u_n=\tan(n+2)\tan(n+3)$ for $n\geq 1$.

(i) By considering $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$, show that

$$u_n=\frac{\tan(n+3)-\tan(n+2)}{\tan 1}-1.$$ [1]

(ii) Hence find $\displaystyle\sum_{r=2}^{n}u_r$ in terms of n. [3]

B (Arithmetic Progression and Geometric Progression)

9. RI19/C1Promo/Q2

An arithmetic series has first term a and common difference d, where a and d are non-zero. The 21st and 53rd terms of the arithmetic series are 91 and 155 respectively. Given also that the sum of its first n terms is 6600. Find the values of a, d and n. [4]

10. NYJC19/C1Promo/Q1

The sum, S_n, of the first n terms of a sequence $u_1,\ u_2,\ u_3, ...$ is given by

$$S_n = \ln\left(2^n 3^{n^2}\right).$$

Show that

(i) $u_n = \ln 2 + (2n-1)\ln 3$, [2]

(ii) the sequence is an arithmetic progression. [2]

11. AJC14/C1Mid-year/Q11

(a) An increasing arithmetic progression has first term a and common difference d. The nth term of the progression is denoted by T_n. $T_2,\ T_6$ and T_9 are consecutive terms of a geometric progression. The sum of all the odd-numbered terms of the arithmetic progression from the eleventh to the thirty-fifth term (i.e. $T_{11} + T_{13} + T_{15} + ... + T_{35}$) is 455.

(i) Find the common ratio of the geometric progression. [3]

(ii) Find the values of a and d. [3]

(b) John is considering two investment plans offered by a bank.
Plan A: John will invest \$27000 at the beginning of the first year. At the end of each year, he will receive \$1800 as interest in his account.
Plan B: John will invest \$27000 at the beginning of the first year and then a further \$200 at the beginning of each subsequent year. Due to interest, the amount in his account at the end of each year will increase by 4% of the amount he has at the beginning of that year.
John decides that, for either plan, he will not withdraw any money out of his account, but just leave it for any interest to build up.

(i) Show that at the end of n years, when the interest for the last year has been added, the amount that John will have in his account under plan B is

$$\$\left(32000(1.04)^n - 5200\right).$$ [4]

(ii) Find the total amount of interest John will receive under plan B at the end of n years. [2]

(iii) Find the least number of years John will need to invest under plan B for his total interest to exceed that of plan A. [2]

12. DHS14/C1Mid-year/Q13

(a) A geometric series has common ratio r and first term a. The sum to infinity of this series exceeds the sum of the first five terms by 2. Given that the sum to infinity is 64, find the values of r and a. [4]

(b) A motorcycle is travelling at a constant speed of 10 m/s. The rider spots an obstacle 34 m ahead and applies the brakes after 1 second. Thereafter, the distance travelled in each subsequent second is 0.7 times the distance travelled in the previous second. By considering the maximum distance travelled, show that the motorcycle will not hit the obstacle. [2]

(c) Initially, a car and a van were alongside each other at a red light traffic junction. When the traffic light turns green, they travel along the same straight road.

(i) The distance travelled by the car in each successive second after the traffic light turns green, follows an arithmetic progression with common difference 0.5 m. Given that the car travels 5 m in the 1st second, find the distance travelled in the 25th second. [2]

(ii) The total distance, in metres, travelled by the van in the first n seconds after the traffic light turns green, is given by $e^{0.2n}-1$. Find after how many complete seconds will the van overtake the car. [4]

13. RI14/C1Mid-year/Q8

(a) The seventeenth term of an arithmetic series is 73, and the thirty-third term of the series is 71. Find the first term and common difference of the series. [2]

Given that the sum of the first n terms is equal to the sum of the first $(n+1)$ terms of the series, find the value of n. [2]

(b) A geometric series has non-zero first term and sum to infinity S. Given that the first, fifth and sixth terms are consecutive terms of an arithmetic series, find the common ratio of the geometric series. [3]

If S = 10, find the least value of m for which the magnitude of the difference between S and the sum of the first m terms of the geometric series is less than 0.001. [3]

14. SAJC14/C2Mid-yearP2/Q1

(a) The sum of the first n terms of a sequence is given by the expression $9-\dfrac{5^n}{3^{n-2}}$. Show that this sequence is a geometric progression. [3]

(b) A sequence of numbers is grouped into sets $A_1=\{1,3\}$, $A_2=\{5,7,9\}$, $A_3=\{11,13,15,17\}$, … where the set A_n has $n+1$ elements.

(i) Show that the total number of elements in the first n sets can be expressed as $\dfrac{n}{2}(n+3)$. [2]

(ii) Hence find the first element of the set A_{n+1}, in terms of n. [2]

15. HCI14/C1Mid-year/Q8

John deposits \$1000 into Account A at the beginning of the first month. In the middle of this month, he receives an interest of \$10. In the middle of each subsequent month, the interest he receives is \$10 more than that in the previous month.

(i) Express the amount in the account at the end of the n^{th} month. [2]

(ii) At the end of which month will the amount in his account exceed \$2000 for the first time? [2]

John deposits another \$1000 into Account B on the same day as he deposits \$1000 into Account A. In the middle of each month, he receives a fixed interest of 6% of the amount of money in the account, and will withdraw \$10 on the next day after the interest is added.

(iii) Show that the amount in dollars in this account at the end of the n^{th} month is $\left(\frac{2500}{3}\right)1.06^n + \frac{500}{3}$. [4]

(iv) The amount in Account A exceeds the amount in Account B for the first time at the end of the k^{th} month. Find k. [2]

16. (IB May12/MathSLP2/TZ1/Q4 modified)

The Green Park Amphitheatre was built in the form of a horseshoe and has 20 rows. The number of seats in each row increase by a fixed amount, *d*, compared to the number of seats in the previous row. The number of seats in the sixth row, u_6, is 100, and the number of seats in the tenth row, u_{10}, is 124. u_1 represents the number of seats in the first row.

(a) Find the total number of seats in the amphitheatre. [6]

A few years later, a second level was added to increase the amphitheatre's capacity by another 1600 seats. Each row has four more seats that the previous row. The first row on this level has 70 seats.

(b) Find the number of rows on the second level of the amphitheatre. [4]

Frank is at the amphitheatre and receives a text message at 12:00. Five minutes later he forwards the text message to three people. Five minutes later, those three people forward the text message to three new people. Assume this pattern continues and each time the text message is sent to people who have not received it before.

The number of new people who receive the text message forms a geometric sequence 1, 3, …

(c) Calculate the total number of people who will have received the text message by 12:30. [2]

(d) Calculate the exact time at which a total of 29524 people will have received the text message. [3]

17. **EJC 2018/BT/2**

Mr. Daya decides to train for a marathon – of distance 42 195 metres – with the goal of completing it in under 4 hours. Each week, he runs 2 sessions of the same duration, and maintains an average speed of 3 metres per second. In the first week, Mr. Daya runs for 100 minutes in each session. For each subsequent week, the duration of each session is 10% longer compared to a session in the previous week.

(i) Show that the distance Mr. Daya runs in **each** session in the 3rd week of his training is 21 780 metres. [2]

In the final week of his training, the distance covered in each session will exceed 42 195 metres for the first time.

(ii) Find the number of weeks that Mr. Daya spends training. [2]

(iii) Hence, find the total distance, to the nearest metre, that Mr. Daya runs in **all** his training sessions. [2]

18. **EJC 2018/BT/3**

Let t_n be the n^{th} term of an arithmetic progression with first term a and common difference d. Define $u_1, u_2, u_3, \cdots, u_n, \cdots$ as follows:

$$u_1 = t_1 + t_2$$

$$u_2 = t_3 + t_4$$

$$u_3 = t_5 + t_6$$

$$\cdots$$

$$u_n = t_{2n-1} + t_{2n}$$

$$\cdots$$

Consider the sequence $u_1, u_2, u_3, \cdots, u_n, \cdots$ (i.e. the sequence with u_n as its n^{th} term).

(i) Express u_2 in terms of a and d. [1]

(ii) Express u_n in terms of a, d, and n. [2]

(iii) Hence or otherwise, show that the sequence $u_1, u_2, u_3, \cdots, u_n, \cdots$ is also an arithmetic progression. [2]

19. **SAJC 2018/BT/8**

Drug overdose is an increasingly serious issue globally. Therefore it is important that patients are aware if there is a possibility of drug overdose when taking medication.

(a) Doctor A advised his patient John to take 3 mg of the drug at the beginning of each day and increase the intake of the drug by 2 mg every subsequent day. To prevent drug overdose, Doctor A tells him that he can at most consume 50 mg of the drug over the entire course of medication. Find the last day in which John can still take his medication and not exceed the limit set by Doctor A. [3]

(b) Doctor B advised his patients David and Luke that they should each take 30 mg of the drug at the beginning of each day. It is known that the human body will burn off a fixed percentage of the drug by the end of the day due to human metabolism.

It is estimated that David's metabolism will burn off 60% of the drug in a day.

(i) Show that the amount of drug in David's body after 3 complete days is 18.72 mg. [2]

(ii) Find an expression in terms of n for the amount of drug in David's body after n complete days. [3]

(iii) It is known that if the level of drug left in the body reaches 54 mg, this could result in death. Explain with a reason if it is safe for David to continue taking the medication indefinitely. [2]

It is also known that metabolic rate of each person varies from person to person.

(iv) At the end of the 20th day since Luke had started taking the drug, he was admitted to hospital for suspected drug overdose. It is found that he had 53 mg of the drug in his body. Find the percentage of the drug left at the end of each day in Luke's body. [4]

20. VJC 2018/BT/7

The sequence $u_1,\ u_2,\ u_3,\ ...$ is a geometric progression with first term 3 and common ratio r.

(i) Write down u_k in terms of k and r. Hence, show that $\ln u_1,\ \ln u_2,\ \ln u_3,\ \ldots$ is an arithmetic progression. [3]

It is further given that $\sum_{k=1}^{30} \ln u_k = 45$.

(ii) Find the value of r. [2]

(iii) Show that $\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_3}, ...$ is a geometric progression. Explain why this progression is convergent and find the sum to infinity. [5]

21. VJC/2019C1BT/10

On 31 December 2009, Gordon put \$150000 into a retirement account which pays compound interest at a rate of 0.2% per month on the last day of each month.

(i) Find the amount in his retirement account at the end of June 2010 if no withdrawal from this account was made during this period. [2]

Starting from 1 January 2010, Gordon withdraws \$1000 on the first day of each month.

(ii) Find the amount in the account at the end of January 2010. [1]

(iii) Show that the amount in the retirement account at the end of the nth month is given by $\$\,k-351000(1.002)^n$ where k is a constant to be found. [5]

(iv) Find the year and the month for which the account is first fully depleted. [3]

(v) Find the amount he can withdraw on his last withdrawal. [1]

Answers

1 **(ii)** $n-\frac{1}{2}+\frac{2}{3n+4}$ **(iv)** $n-\frac{3}{2}+\frac{2}{3n+1}$

2 **(ii)** $\frac{1}{(N+1)(N+2)}-\frac{1}{2}$ **(iii)** $\sum_{n=1}^{M} \mathrm{f}(n)=\frac{(-1)^M}{(M+1)(M+2)}-\frac{1}{2}$, $-\frac{1}{2}$

3 **(ii)** $1-\frac{1}{(N+1)!}$ **4** **(iii)** $\frac{1}{2}$ **(iv)** $\frac{1}{1440}-\frac{1}{2(N+1)!}$

5 **(i)** $\frac{4}{r}-\frac{4}{r+1}+\frac{1}{r+2}$ **(ii)** $\frac{3}{8}-\frac{1}{2^{n+1}(n+1)}+\frac{1}{2^{n+2}(n+2)}$

6 **(ii)** $-\frac{1}{2\sin\theta}\left(\cos\left[(N+1)\theta\right]+\cos\left[N\theta\right]-\cos\theta-1\right)$ **(iii)** $\sqrt{3}+2$

(i) $\frac{6r+18}{(r-1)r(r+2)}=\frac{8}{r-1}-\frac{9}{r}+\frac{1}{r+2}$

(ii) $\frac{43}{36}-\frac{4}{3n}+\frac{1}{6(n+1)}+\frac{1}{6(n+2)}$

7 (iii) $\frac{41}{72}-\frac{4}{3(n+1)}+\frac{1}{6(n+2)}+\frac{1}{6(n+3)}$

(iv) $\frac{41}{72}$

(v) $\sum_{r=2}^{\infty}\frac{r+4}{(r+3)^3}<\frac{41}{72}$

8 (ii) $\frac{\tan(n+3)-\tan 4}{\tan 1}+1-n$

9 $a=51, d=2, n=60$

11 **(a) (i)** Common ratio = $\frac{3}{4}$ **(ii)** $a=-119$, $d=7$

(b) (ii) $32000(1.04)^n-200n-32000$ **(iii)** 23

12 **(a)** $r=\frac{1}{2}$, $a=32$ **(b)** Maximum distance $=33\frac{1}{3}$. Will not hit obstacle.

(c)(i) 17m **(ii)** After 30 seconds.

13 **(a)** $a = 75,\ d = -0.125$; $n = 600$ **(b)** -0.775 ; 37

14 **(b)(ii)** $n^2 + 3n + 1$

15 **(i)** $1000 + 5n^2 + 5n$ **(ii)** end of 14th month **(iv)** k = 15

16 **(a)** 2540 **(b)**16 **(c)** 1093 **(d)** 12:45

17 **(ii)** 10 weeks **(iii)** 573 747 m

18 **(i)** $u_2 = 2a + 5d$ **(ii)** $u_n = 2a + (4n-3)d$

19 **(a)** Day 6 **(b)(ii)** $20\left[1 - 0.4^n\right]$ **(b)(iv)** 63.9%

20 **(i)** $u_k = 3r^{k-1}$ **(ii)** $r = \sqrt[29]{\dfrac{e^3}{9}} = 1.03$ (3 s.f.) **(iii)** 12.2

21 $151 809.02 ; $149 298 ; $501000 - 351000(1.002)^n$; 179^{th} month; $ 87.87

2

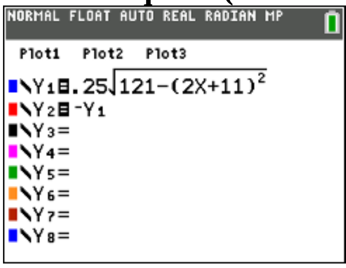
Graphs and Transformations

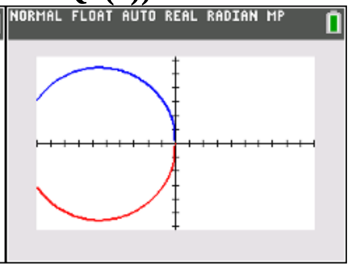
Skill Set

<table>
<tr><th>s/n</th><th>Skills</th><th>Examples of questions involving the skills</th></tr>
<tr><td>1.</td><td>Use of Zoom Functions and WINDOW (GC) to identify features of graphs:

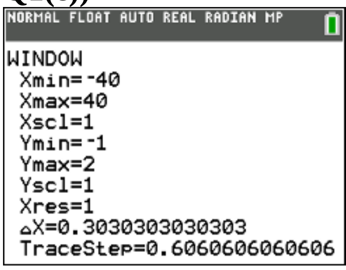
5:ZSquare
To enable viewing a graph without distortion (equal scales used for the x and y axes). Frequently used when sketching conics.

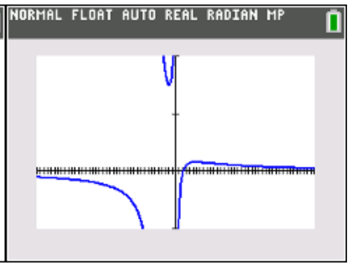
WINDOW
Mainly to adjust X_{min}, X_{max}, Y_{min}, Y_{max} to enable better display of important features of the graph in a particular region.</td><td>Use of ZSquare (Tutorial 2A Q2(a))





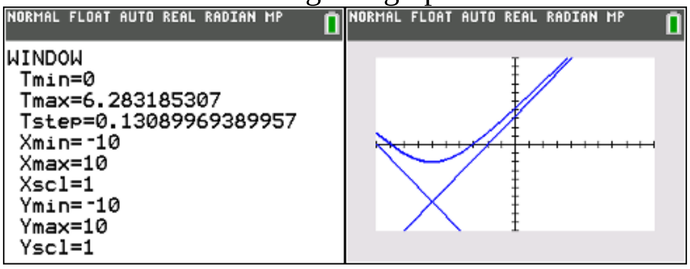
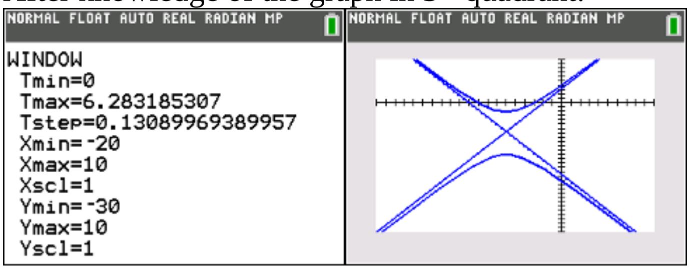
Use of WINDOW to view turning points (Tutorial 2A Q1(e))





</td></tr>
<tr><td>2.</td><td>Use of algebraic methods to prove that a curve lies/does not lie between two values of y.</td><td>(Tutorial 2A Q8(ii))
Express the equation of the graph in the form of a quadratic equation in x, and equate discriminant to less than 0.
$y=\dfrac{x^2+x-2}{x+3} \Rightarrow x^2+x(1-y)+(-2-3y)=0$
No points on $C \Rightarrow$ No y values $\Rightarrow$ No roots/solutions to the quadratic equation above:
$\therefore (1-y)^2-4(1)(-2-3y)<0 \Rightarrow -9<y<-1$</td></tr>
<tr><td>3.</td><td>Identify types of asymptotes from an equation of a graph.</td><td>• <u>Vertical asymptote</u>: Put denominator = 0
• <u>Horizontal asymptote</u>: Exists when degree of numerator equals to degree of denominator. Use long division method to find or take ratio of coefficients.
• <u>Oblique asymptote</u>: Exists when degree of numerator is one more than degree of denominator. Use long division to find.</td></tr>
</table>

s/n	Skills	Examples of questions involving the skills
4.	Identify lines of symmetry (especially *x*-axis or *y*-axis).	**Example** The curve C has equation $y=\dfrac{x^2}{(x+\lambda)(x-1)}$, where λ is a non-zero constant. Sketch C for the case where C is symmetrical about the *y*-axis: To be symmetrical about *y*-axis, $\mathrm{f}(x)=\mathrm{f}(-x)\Rightarrow$ $\dfrac{x^2}{(x+\lambda)(x-1)}=\dfrac{(-x)^2}{(-x+\lambda)(-x-1)}$ to deduce $\lambda=1$. The resultant expression is $\mathrm{f}(x)=\dfrac{x^2}{x^2-1}$. Or use the fact that to be symmetrical about the *y*-axis, all the terms in *x* must be of even powers. To be symmetrical about *x*-axis, all the terms in *y* must be of even powers.
5.	To sketch a graph with unknown(s) in equation.	**Example** The curve C has equation $y=x+\dfrac{\lambda}{x-1}$ where λ is a non-zero constant. Sketch C for the cases (i) $\lambda>0$ and (ii) $\lambda<0$. • Use GC to find out the shape of the graph by exploring different values of $\lambda>0$ such as taking $\lambda=2$ or 3 etc. • Note: Details in the final graph should still be in terms of λ.
6.	Identify a suitable graph to draw on the same sketch to help in finding the number of real roots of a given equation.	(**Tutorial 2A Q6(iii) and Q7(iv)**) • Manipulate until the original equation of the curve appears on one side of the equation. (**Tutorial 2A Q8(v)**) Substitute $y=\dfrac{x^2-2x+1}{x-2}=\dfrac{(x-1)^2}{x-2}$ into $9y^2=9a^2-a^2(x-3)^2\Rightarrow 9(x-1)^4=a^2(x-2)^2\left(9-(x-3)^2\right)$ • Note that the new equation is obtained by solving the equations of the curve and the ellipse simultaneously. • To obtain 3 real roots $\Rightarrow$ To obtain 3 points of intersection from the two graphs.
7.	Identify the types of conics based on their standard equations.	(**Tutorial 2A Q2(a)**) Circle $(2x+11)^2+4y^2=121$: Same coefficients of x^2 and y^2. (**Tutorial 2A Q2(b)**) Ellipse $y^2+4x^2-4x=7$: Coefficients of x^2 and y^2 have the same sign but different magnitude. (**Tutorial 2A Q2(c)**) Parabola $(y-3)^2=10-2x$: Entire equation, one variable has highest power 2, while the other has highest power 1.

s/n	Skills	Examples of questions involving the skills
		(Tutorial 2A Question 2(d)) Hyperbola $\frac{y^2}{7}-\frac{x^2}{5}=1$: Coefficients of x^2 and y^2 have opposite signs.
8.	Able to recognize/find the main features of each conic.	• Parabola – vertex • Ellipse – centre, horizontal and vertical widths • Circle – centre, radius • Hyperbola – centre, asymptotes, vertices
9.	Finding the equations of the asymptotes of a hyperbola.	**(Notes Example 16(b))** Hyperbola with equation $(y+1)^2-25(x-2)^2=25$: As $x\to\pm\infty$, $(y+1)^2\to 25(x-2)^2$ since a difference of 25 is negligible. $\therefore y\to -1\pm\sqrt{25(x-2)^2}$. Thus, equations of asymptotes are $y=-1+5(x-2)$ and $y=-1-5(x-2)$.
10.	Use of GC (including adjusting WINDOW) to sketch parametric curves.	**Example** $x=3\tan\theta-6,\ y=5\sec\theta-7$. Initial WINDOW setting and graph:  After knowledge of the graph in 3rd quadrant: 
11.	Use of substitution method or trigonometric identities to convert parametric equation to Cartesian equation.	**(Tutorial 2A Q4)** • (a) $x=t^2+1,\ y=2t$. Substitute $t=\frac{y}{2}$ into $x=t^2+1$. • (d) $x=2\cos t,\ y=\frac{1}{2}\sin t$. Use identity $\sin^2 A+\cos^2 A=1 \Rightarrow \left(\frac{x}{2}\right)^2+(2y)^2=1$.

s/n	Skills	Examples of questions involving the skills
12.	Able to obtain/sketch a transformed graph using at least two series of geometrical transformations (translation, scaling and reflection).	**(Tutorial 2B Q1(a))** • To obtain $y=-3\mathrm{f}\left(x+20\right)$ from $y=\mathrm{f}\left(x\right)$ using translation followed by scaling/reflection or scaling/reflection followed by translation. • Note that these two transformations can be done regardless of their sequence as one involves the *x*-axis while the other occurs along the *y*-axis.
13.	Able to do algebraic replacement to check the sequences of transformations.	**(Tutorial 2B Q1(c))** • To obtain $y=\mathrm{f}\left(3x+20\right)$ from $y=\mathrm{f}\left(x\right)$, use **either** of the following transformation sequences: 1. Translate by 20 units in the negative *x*-direction. 2. Scale parallel to the *x*-axis by a factor of $\frac{1}{3}$. $\mathrm{f}\left(x\right)\to\mathrm{f}\left(x+20\right)\to\mathrm{f}\left(3x+20\right)$ OR 1. Scale parallel to the *x*-axis by a factor of $\frac{1}{3}$. 2. Translate by $\frac{20}{3}$ units in the negative *x*-direction. $\mathrm{f}\left(x\right)\to\mathrm{f}\left(3x\right)\to\mathrm{f}\left(3\left(x+\frac{20}{3}\right)\right)\to\mathrm{f}\left(3x+20\right)$ • Note that the sequence of these two transformations matters as both involve the same *x*-axis.
14.	Able to use algebraic replacement to work backwards to obtain original equations from transformed graphs.	**(Tutorial 2B Q3 and Q6)** Always work 'forward' to check if the resulting curve/equation in the question can be obtained.
15.	Able to obtain the reciprocal graph $y=\frac{1}{\mathrm{f}(x)}$ from the graph of $y=\mathrm{f}\left(x\right)$, find the new region(s) where the curve will lie in and how the turning points and asymptotes are transformed.	**(Tutorial 2B Q8)** Take note of the transformation of the following features: vertical asymptotes → *x*-intercepts *x*-intercepts → vertical asymptotes maximum points → minimum points minimum points → maximum points oblique asymptote → horizontal asymptote $y = 0$ Take reciprocal of *y*-variable for *y*-intercepts, horizontal asymptotes and max/min points.
16	Able to obtain the graphs of $y=\left\|\mathrm{f}\left(x\right)\right\|$, $y=\mathrm{f}\left(\left\|x\right\|\right)$, from the graph of $y=\mathrm{f}\left(x\right)$.	**(Tutorial 2B Q2(a) and Q2(b))** • For $y=\left\|\mathrm{f}\left(x\right)\right\|$, reflect the parts of $y=\mathrm{f}\left(x\right)$ below the *x*-axis in the *x*-axis. • For $y=\mathrm{f}\left(\left\|x\right\|\right)$, remove the parts of $y=\mathrm{f}\left(x\right)$ on the left of the *y*-axis, then reflect and duplicate the remaining parts of $y=\mathrm{f}\left(x\right)$ in the *y*-axis.

A Curves Sketching and Conic Sections

1. CJC14/JC1Mid-year/Q1

The curve C has equation $\dfrac{(y-1)^2}{4}-(x-2)^2=1$.

(i) Sketch C, indicating clearly the equations of asymptotes, coordinates of axial intercepts and vertices, if any. [4]

(ii) Determine the value of r where $r>0$, such that the graph of $(x-2)^2+(y-1)^2=r^2$ will intersect C at exactly two points. [1]

2. CJC19/JC1Mid-year/Q7

The curve H has equation $9y^2+36y-4x^2+8x-4=0$.

Show that the equation $9y^2+36y-4x^2+8x-4=0$ can be written as $\dfrac{(y+2)^2}{4}-\dfrac{(x-1)^2}{9}=1$. [2]

State the lines of symmetry of H. [1]

Sketch the curve H, stating the coordinates of any points of intersection with the axes, the coordinates of any turning points and the equations of any asymptotes. [4]

The curve J has equation $2n(x-1)^2+(y+2)^2=2n$, where n is a positive constant.

Deduce the range of values of n such that H and J intersect at least twice. [3]

3. ASRJC19/JC1Mid-year/Q3

The curve C_1 has equation $y=\left|\dfrac{x}{x-4}\right|$.

The curve C_2 has equation $x^2+y^2-4x-6y+9=0$.

(i) Sketch C_1 and C_2 on the same diagram, stating the equation of the asymptotes. [4]

(ii) Find the points of intersection of C_1 and C_2. [2]

4. RVHS14 /JC1Mid-year/Q12

The curve C has equation

$$y=\frac{ax^2+2x-4}{x+b},$$

where a and b are constants. It is given that one of the asymptotes of C has equation $y=2x$.

(i) Find the values of a and b. [3]

(ii) By differentiation, what can be said about the gradient of C? [3]

(iii) Sketch C, showing its asymptotes and stating the coordinates of the points of intersection with the axes. [3]

(iv) By drawing a sketch of another suitable curve in the same diagram, deduce the number of distinct real roots of the equation

$$(x+1)^4+(2x^2+2x-4)^2=4(x+1)^2$$ [3]

5. **EJC18/JC1Mid-year/Q7**

The curve C has equation $y=\dfrac{x^2+\lambda x+\lambda}{x+1}$, where $\lambda \in \mathbb{R}$.

(i) Show that the values of y cannot lie between the values $\lambda-4$ and λ. [3]

(ii) Sketch the curve C for the case where $\lambda=5$, stating the coordinates of any points of intersection with the axes, any turning points, and the equations of any asymptotes.

[4]

(iii) By drawing another line on the same diagram, find the range of values of α such that

$$\frac{x^2+5x+5}{x+1}=\alpha(x+1)+3$$

has two distinct roots. [2]

6. **JJC13/JC1CT/Q4**

(a) The curve C_1 has equation $y=\dfrac{x-1}{x-2}$. The curve C_2 has equation $\left(x-2\right)^2+\left(y-1\right)^2=2$.

(i) Sketch C_1 and C_2 on the same diagram, stating the coordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

(ii) Hence, deduce the number of roots of the equation $\left(x-2\right)^2+\left(\dfrac{1}{x-2}\right)^2=2$.

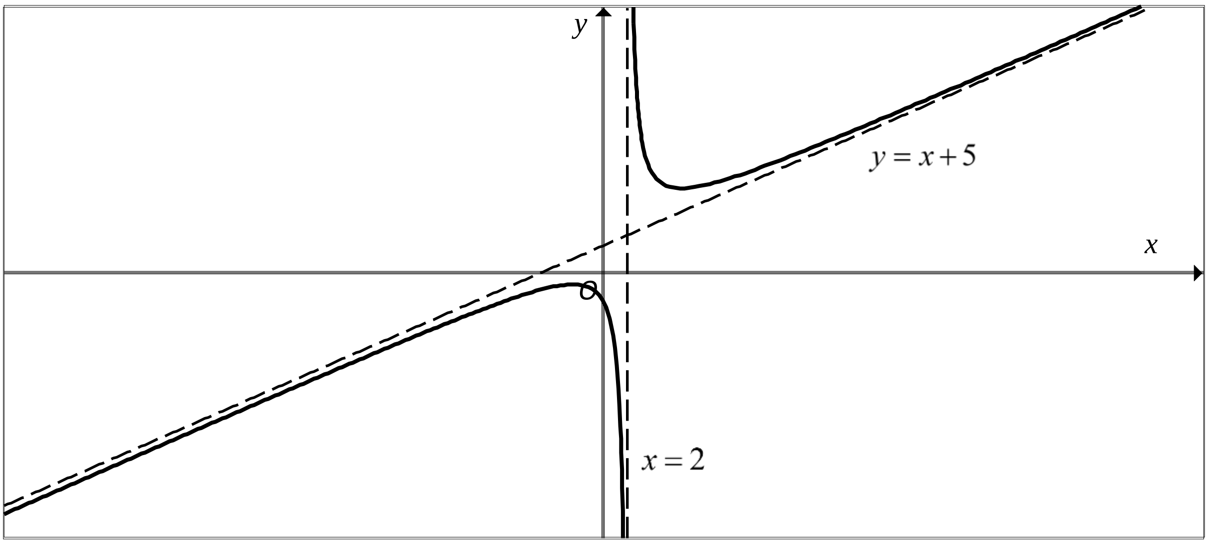
[2]

(iii) Another curve C_3 has the equation $\left(x-2\right)^2+\left(y-1\right)^2=h$.

State the range of values of $h \in \mathbb{R}$ such that C_1 and C_3 intersect at 4 points.

[1]

(b) A sketch of the curve $y=\dfrac{Ax^2+Bx+11}{x+C}$, where A, B and C are constants, is shown below. The lines $x=2$ and $y=x+5$ are asymptotes to the curve. Find the values of A, B and C. [3]



7. **MJC18/JC1Mid-year/Q3**

(i) A curve C_1 has equation $x^2 - 4x - 4y^2 - 12 = 0$. By completing the square, show that the equation of C_1 can be expressed in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where a, b, h and k are real constants. Sketch C_1, indicating clearly, the axial intercepts and equations of asymptotes. [4]

(ii) Another curve C_2 is defined by the parametric equations

$$x = 5 + 3\sin\theta, \quad y = 2\cos\theta, \qquad \text{where } 0 < \theta \le 2\pi.$$

Find the cartesian equation of C_2. [2]

(iii) Sketch C_2 on the same diagram as C_1, indicating clearly the axial intercepts and any other relevant features. (You need not find the points of intersection between C_1 and C_2.) [3]

(iv) Show algebraically that the *x*-coordinate of the points of intersection of C_1 and C_2 satisfies the equation $\frac{(x-2)^2}{4^2} + \frac{(x-5)^2}{3^2} = 2$. Use your calculator to find this *x*-coordinate. [3]

B Parametric Equations

8. NJC18 /JC1Mid-year/Q6b

A curve C_2 is given by the parametric equations $x = t^3 + t$, $y = -e^t + 2e^{-t}$, $-1 \le t \le 1$.

(i) Sketch C_2, indicating clearly where the curve crosses the *x*- and *y*-axes. [4]

(ii) Find the coordinates of the point of intersection of C_2 and the line $y = x - 1$. [3]

9. SAJC/2019C1BT/5

A curve *C* is defined by the equation $y = \dfrac{-4x^2 + 8kx - 5k^2 + 4}{x - k}$, $x \ne k$.

(i) Find the range of values of *k* such that *C* has two stationary points. [5]

(ii) It is given that *C* has an oblique asymptote which cuts the *y*-axis at the point (0 , 4). Find the value of *k*. [2]

(iii) Using the value of *k* in **(ii)**, sketch the curve *C*, stating the equations of asymptote(s), exact coordinates of turning point(s) and axial intercept(s), if any. [3]

(iv) A curve C_1 is defined by the following parametric equations

$$x = 1 + \tan t \ , \ y = b\sec t, \ 0 \le t \le 2\pi, \ b > 0 \ .$$

Find the cartesian equation of the curve C_1. [2]

(v) Find the range of values of *b* such that there are at most two intersection points between the curves *C* and C_1. [2]

10. VJC14/JC1Mid-year/Q12

A curve *C* has parametric equations

$$x = 16 - \sqrt{t^2 + 9}, \quad y = \frac{t-3}{t}, \text{ for } t \in \mathbb{R}.$$

(i) Find the coordinates of the points where *C* crosses the *x*- and *y*-axes. [3]

(ii) Show that *C* has no stationary point. [3]

(iii) Find the equation of the vertical asymptote. [1]

(iv) Given that $y \to a$ as $x \to -\infty$, find *a*. [2]

(v) Sketch *C*. [3]

11. MJC11/JC1BT/Q3(modified)

The curve C is given by the parametric equations

$$x = a\sin\theta + 2 \quad , \quad y = 3\cos\theta \text{, where } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \text{ and } a > 0.$$

(i) State the range of values for x and y. [2]

(ii) Sketch C for which $0 < a < 2$. [4]

12. EJC 2020 C1 BT Q6

The curve C_1 has parametric equations

$$x = 3\sin\theta + 4, \qquad y = 3\cos\theta - 3, \quad \text{where } 0 \le \theta \le 2\pi,$$

and the curve C_2 has equation $x^2 - y^2 = 4$.

(i) Sketch C_1 and C_2 on the same diagram, stating clearly the co-ordinates of any points intersection with the axes and the equations of any asymptotes. [5]

(ii) Show algebraically that the points of intersection of C_1 and C_2 satisfy the equation $3(\sin^2\theta - \cos^2\theta) + 8\sin\theta + 6\cos\theta + 1 = 0$. [2]

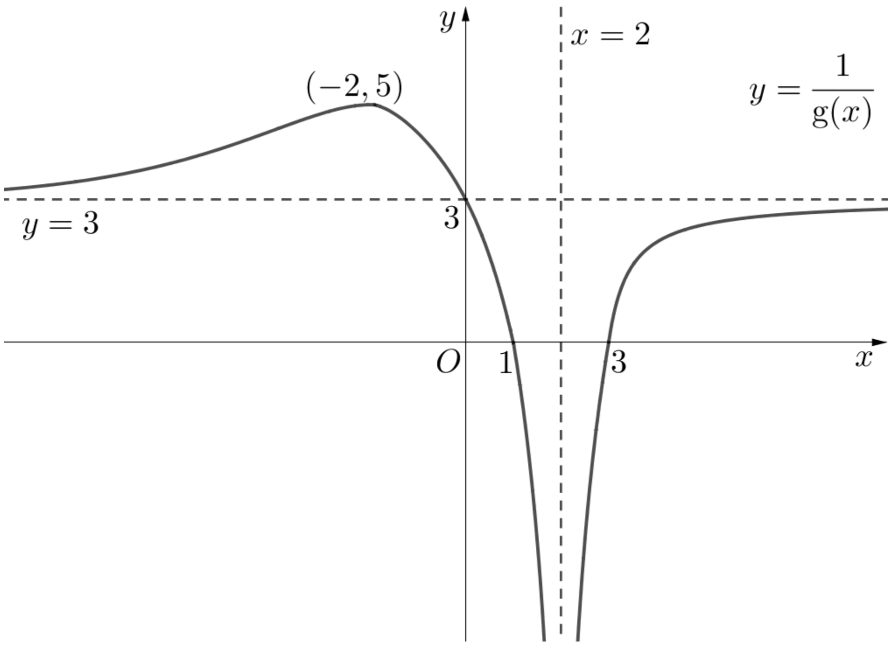
(iii) Hence, find the co-ordinates of the points of intersection of C_1 and C_2. [3]

C Graph Transformations

13. NJC/2019C1BT1/5

(a) Describe a pair of transformations which transforms the curve with equation $\frac{x^2}{6^2} + \frac{(y+3)^2}{2^2} = 1$ on to the circle with centre at the origin and radius 6 units. [3]

(b) The diagram shows the curve with equation $y = \frac{1}{\mathrm{g}(x)}$. The curve passes through the points with coordinates $(1, 0)$, $(3, 0)$ and $(0, 3)$, and has a maximum point at $(-2, 5)$. The lines $x = 2$ and $y = 3$ are asymptotes to the curve.



Sketch, on separate diagrams, the curves of

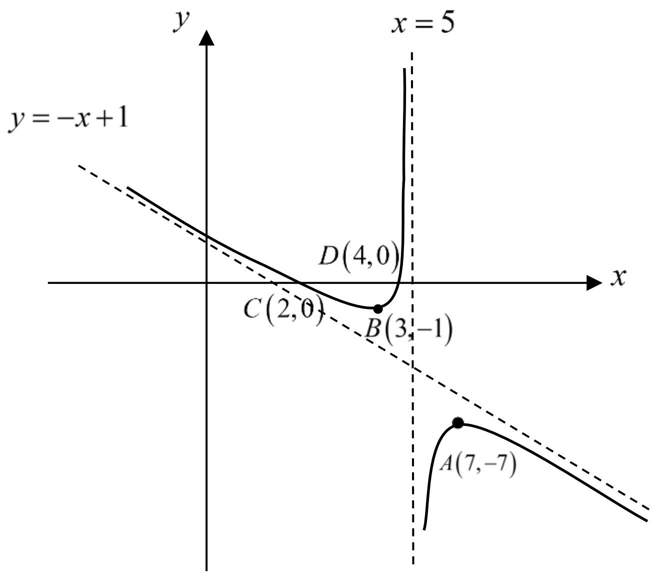
(i) $y = \dfrac{1}{g(2-x)}$, [3]

(ii) $y = g(x)$, [3]

stating the equations of any asymptotes, and the coordinates of turning points and of points where the curves cross the axes.

14. AJC14/JC1Mid-year/Q9

The diagram below shows the graph of $y = g(x)$. The curve has a maximum point at $A(7,-7)$ and a minimum point at $B(3, -1)$, and crosses the x-axis at the points $C(2, 0)$ and $D(4, 0)$. The lines $y = -x+1$ and $x = 5$ are asymptotes to the curve.



On separate diagrams, sketch the graphs of

(i) $y = \mathrm{g}(1-x)$, [3]

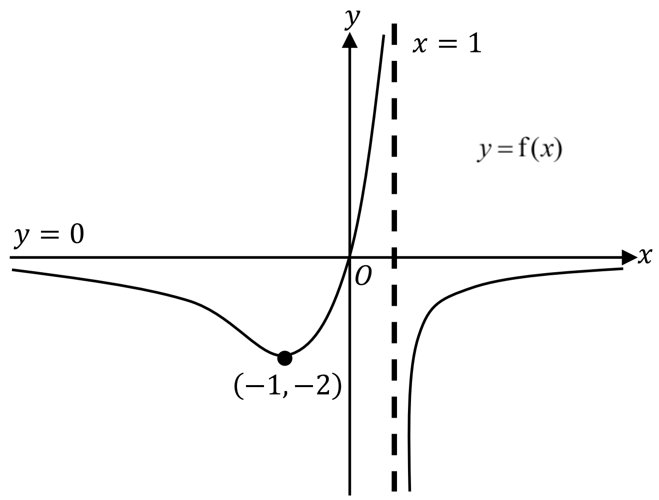
(ii) $y = \dfrac{1}{\mathrm{g}(x)}$, [3]

showing clearly in each case the coordinates of the axial intercepts, turning point(s), and the equations of asymptotes, if any.

The inequality $\mathrm{g}(1-x) > a$, where a is a constant, has the solution set $\{x \in \mathbb{R} : x > -4\}$. Find the set of values of a. [2]

15. CJC18/JC1Mid-year/Q10

(a) The diagram shows the graph of $y = \mathrm{f}(x)$. The curve crosses the x-axis at the origin and has a minimum point at $(-1,-2)$. The lines $y = 0$ and $x = 1$ are asymptotes to the curve.



On separate diagrams, sketch the graphs of

(i) $y = \frac{1}{2}\mathrm{f}(x) + 2$, [3]

(ii) $y = \mathrm{f}(|x-1|)$, [3]

(iii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

indicating clearly the equations of any asymptote(s), the coordinates of turning point(s) and any points where the curve crosses the x- and y- axes, whenever possible.

(b) A curve C_1 undergoes, in succession, the following transformations:

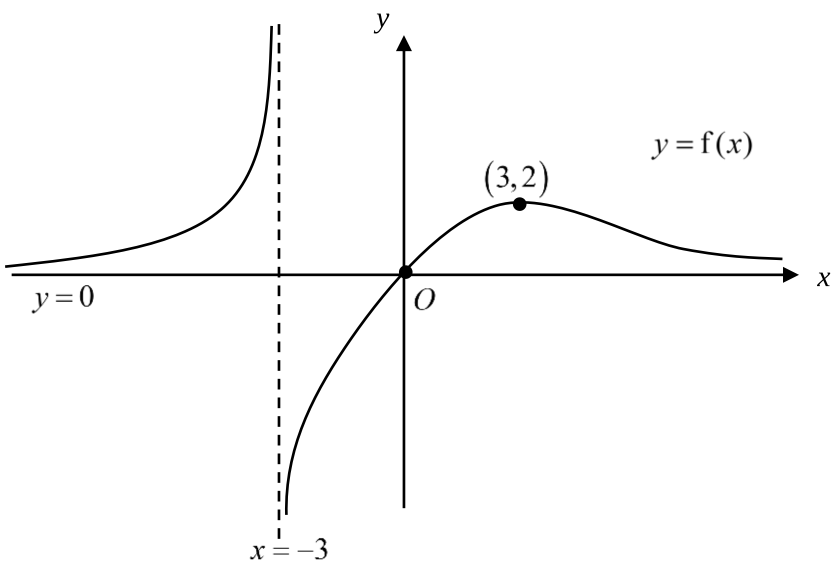
A: Translation of 4 units in the negative y-direction
B: Scaling parallel to the x-axis by a scale factor of 3
C: Translation of 2 units in the positive x-direction.

The equation of the resulting curve, C_2, is $y=\dfrac{x^2-2}{x+1}$.

Find the equation of the original curve C_1. [3]

16. CJC19/JC1Mid-year/Q11

The diagram shows the graph of $y=\mathrm{f}(x)$.



On separate diagrams, indicating clearly the equations of any asymptotes, the coordinates of turning points, and the coordinates of any points of intersection with the x- and y-axes, sketch the graphs of

(i) $y=\mathrm{f}\left(\dfrac{1}{2}x\right)-1$, [4]

(ii) $y=\mathrm{f}\left(|x|\right)$, [3]

(iii) $y=\dfrac{1}{\mathrm{f}\left(x\right)}$. [4]

17. **DHS19/JC1BT/Q4**

A curve C_1 has equation $4y^2 = 9 + 16x^2$. It undergoes two transformations in this order:

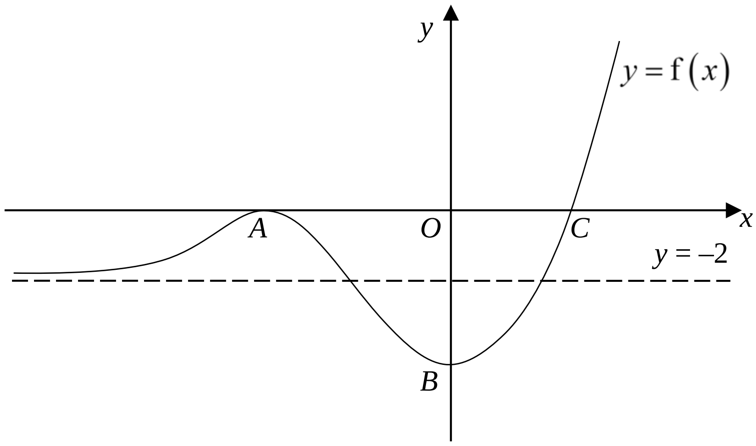
- translate 1 unit in the positive x-direction,
- scale parallel to the y-axis by factor k, where k is a positive constant.

(i) Find the equation of the resulting curve C_2, leaving your answer in terms of k. [2]

(ii) Sketch C_2 indicating clearly the centre, equations of the asymptotes and the coordinates of the vertices. [3]

18. **YIJC/2019C1BT/4**

The diagram shows the curve $y = \mathrm{f}(x)$ with an asymptote $y = –2$. The curve has turning points at A and B and crosses the x-axis at the point C. The coordinates of A, B and C are (–3, 0), (0, –4) and (2, 0) respectively.



Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}(-x) + 3$, [2]

(ii) $y = \mathrm{f}(|x| - 1)$, [2]

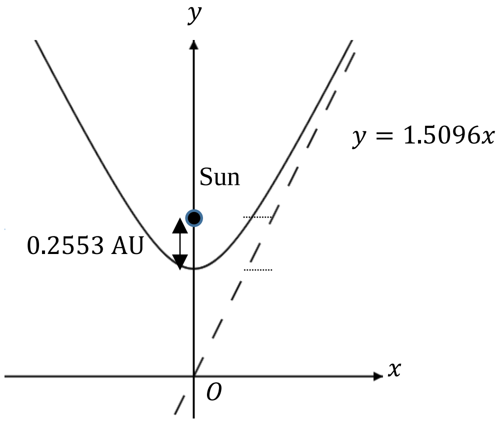
(iii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

stating the equations of any asymptotes and the coordinates of the points corresponding to A, B and C where appropriate.

D Real Life Applications

19. CJC18/JC1Mid-year/Q1

Oumuamua is the first known interstellar object to pass through the Solar System. Oumuamua has a hyperbolic trajectory of the inner Solar System, with the Sun as the focus. The diagram below shows the trajectory of Oumuamua relative to the Sun and the axes.



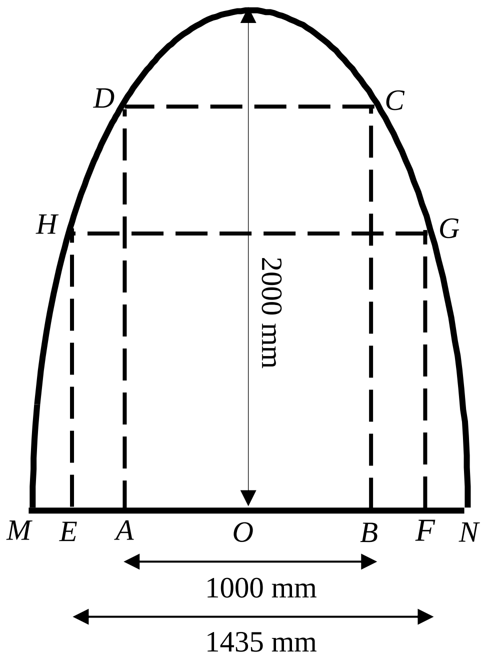
The perihelion (the shortest distance of an object from the Sun) is 0.2553 Astronomical Unit (AU) while the Sun is 1.5351 AU from the origin, O. The trajectory approaches the asymptote $y = 1.5096x$.

A possible model of the trajectory is $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$, where $a > 0$ and $b > 0$.

Find the values of a and b. [3]

20. A tunnel is built to facilitate the transportation of goods by trains between Country X and Country Y. Due to differences in the rail systems between the two countries, two types of tracks are used – the international track with track gauge of 1435 mm and the narrow track with track gauge of 1000 mm (in rail transport, track gauge is the spacing on a railway track). It is known that the cross-section of the tunnel is a half ellipse with centre O and width MN (see diagram). The maximum height of the tunnel is 2000 mm. To standardize the volumes of the goods to be transported, the areas $ABCD$ and $EFGH$ are made equal. Find the width of the tunnel MN, giving your answer to the nearest mm.

[4]

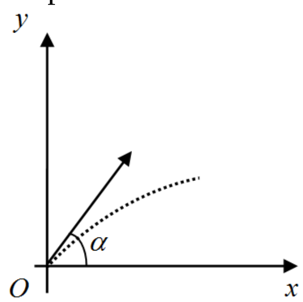


21. NJC16/C1BT/Q9

Toy Rocket A is launched at an angle α to the horizontal ground, where $0<\alpha<\frac{\pi}{2}$. As shown in the diagram below, its path in the air after t seconds is approximated by

$$x=(10\cos\alpha)t\,,\quad y=(10\sin\alpha)t-5t^2,$$

where the origin O is the launch point.



The horizontal distance between the launch point and the landing point is called the *range*.

(a) Find, in terms of α,

(i) the time taken for Toy Rocket A to hit the ground, [2]

(ii) the *range* of Toy Rocket A. [1]

State the angle to the horizontal at which Toy Rocket A should be launched to obtain the maximum *range*. Justify your answer. [2]

(b) It is known that Toy Rocket A is fired at an angle of $\frac{\pi}{3}$ to the horizontal ground.

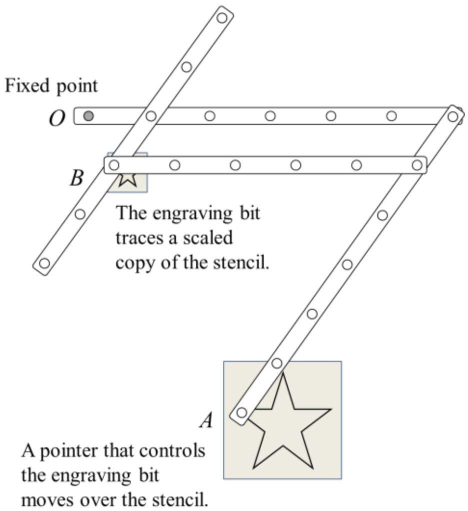
(i) Show that its path in the air before hitting the ground can be described by the Cartesian equation $y=\sqrt{3}x-\frac{1}{5}x^{2}$. [2]

(ii) Sketch the path of Toy Rocket A on an x – y plane, indicating the exact coordinates of the turning point and the point where it hits the ground. [3]

(c) Toy Rocket B is fired from the origin O in the positive x-direction. Its path in the air follows the parabola $y=10\sqrt{3}x-4x^{2}$.
Describe a sequence of **two** transformations which maps the path of Toy Rocket A onto the path of Toy Rocket B. [2]

22. EJC 2020 C1 BT Q9a

(a) Engraving is the process of incising a design or text onto a surface. For instance, serial numbers are commonly engraved onto items such as machine parts, in order to identify them individually. Today, computers are used for engraving, but in the past the job required a mechanical linkage known as a pantograph.



The diagram shows a simple pantograph, in the x-y plane, consisting of two pairs of parallel rigid rods, joined together at four joints. The rods are free to rotate at these joints. One end of the pantograph is fixed at the origin O represented by the point (0, 0). Points (x, y) are defined relative to the fixed point O. A pointer is attached to A, while the engraving bit is attached to B. The engraver controls the pantograph by moving the pointer over a stencil, which then causes the engraving bit to trace out a copy of the shape.

(i) A pantograph is set up such that the engraving bit, point B, always lies on the same line as OA as the pointer moves. The ratio of the distances $OA:OB$ is set at $5:1$. Describe a sequence of geometric transformations which maps the shape of the stencil (at A) to the shape of the engraving (at B). [2]

(ii) Suppose that the shape on the stencil is a circle with centre at (h,k) and radius r. State the equation of the circle. Find the equation of the engraved shape, showing your working clearly. [3]

(iii) State the area enclosed by the engraved shape. [1]

Answers

A Curves Sketching and Conic Sections

1. $r = 2$

2. $x=1$ and $y=-2$; $n \geq 2$

3.(ii) Intersection points: (2, 1), (3.28, 4.54).

4.(i) $a = 2$, $b = 1$

4.(ii) The gradient of C is positive for all $x \in \mathbb{R},\ x \neq -1$.

4.(iv) 4 distinct real roots

5.(iii) $\alpha > 1$

6.(aii) 2 **(aiii)** $h > 2$ **(b)** $A=1,\ B=3,\ C=-2$

7(ii) $\dfrac{(x-5)^2}{3^2}+\dfrac{y^2}{2^2}=1$

7(iv) $x = 6.99$ (3 s.f)

B Parametric Equations

8.(ii) (0.602, –0.398)

9.(i) $k < -2$ or $k > 2$ **(ii)** $k = 1$

9.(iv) $\left(\dfrac{y}{b}\right)^2-(x-1)^2=1$

9.(v) $b \geq 4$

10.(i) (11.8, 0), (0, 0.809), (0, 1.19)

10.(iii) $x = 13$ **(iv)** $a = 1$

11.(i) $2-a < x < 2+a$, $0 < y \leq 3$

12.(iii) $\theta = 2.5083 \Rightarrow (5.78, -5.42)$

$\theta = 5.6014 \Rightarrow (2.11, -0.671)$

C Graph Transformations

13. Scale parallel to the y-axis by a factor of 3. Translate in the positive y-direction by 9 units. OR
Translate in the positive y-direction by 3 units. Scale parallel to the y-axis by a factor of 3.

14. $\{a \in \mathbb{R} : -7 \leq a < -1\}$

15.(b) $y=\dfrac{9x^2+24x+14}{3x+3}$

17.(i) $$4\left(\frac{y}{k}\right)^2=9+16(x-1)^2$$

D Real Life Applications

19 $a=0.848\ \left(3\text{ s.f}\right), b=1.2798$

20 1749 mm

21(a)(i) $2\sin\alpha$ seconds

21(a)(ii) Range is $20\cos\alpha\sin\alpha$ or $10\sin 2\alpha$.

Rocket A should be launched at $\frac{\pi}{4}$

21 (c) The graph traced by Toy Rocket A is scaled by a scale factor of 5 parallel to the *y*-axis and then scaled by a scale factor of $\frac{1}{2}$ parallel to the *x*-axis. (Or vice versa)

22 (a)(i) Scaling parallel to *y*-axis with scale factor 1/5;
Scaling parallel to *x*-axis with scale factor 1/5

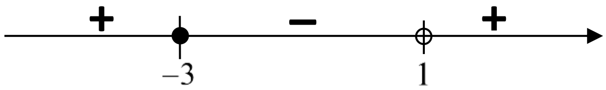
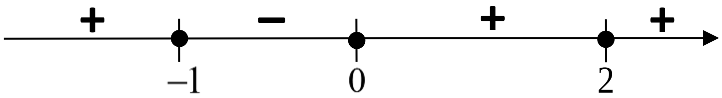
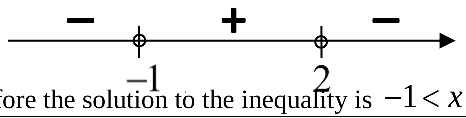
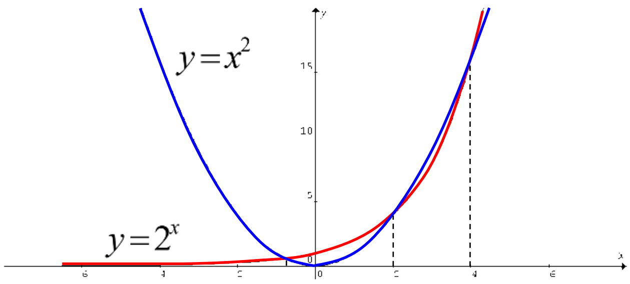
22 (a)(ii) $$\left(x-\frac{h}{5}\right)^2+\left(y-\frac{k}{5}\right)^2=\left(\frac{r}{5}\right)^2$$

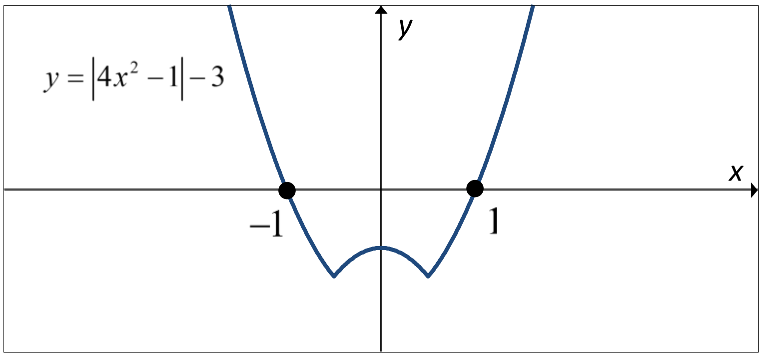
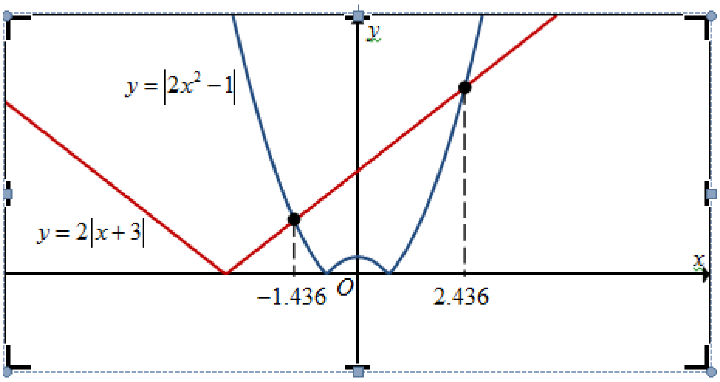
22 (a)(iii) $A=\frac{\pi r^2}{25}$

3

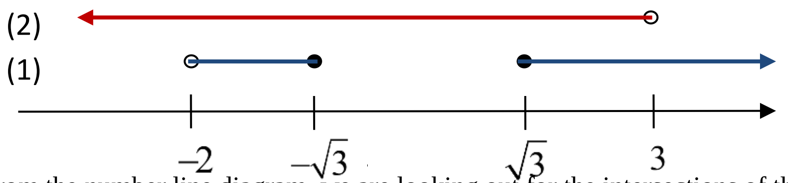
Inequalities and Systems of Equations

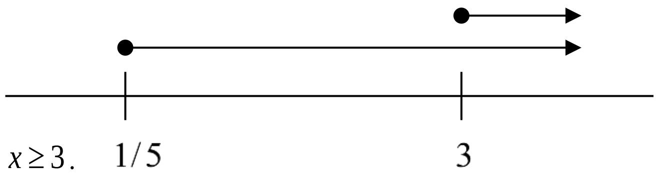
Skill Set

s/n	Skills	Examples of questions involving the skills
1.	Manipulate inequality into the form $\frac{\mathrm{f}(x)}{\mathrm{g}(x)} > 0$ and solve the inequality using algebraic method, where a number line is drawn using the critical values found. Note that the signs will not be alternating when there is a repeated root of even power.	**(Lecture Notes Example 4)** Find the solution of $\frac{3x+1}{x-1} \le 2$. $\frac{3x+1}{x-1} - 2 \le 0 \Rightarrow \frac{3x+1-2(x-1)}{x-1} \le 0 \Rightarrow \frac{x+3}{x-1} \le 0$.  Hence solution is $-3 \le x < 1$. **(Lecture Notes Example 3)** $x(x+1)(x-2)^2 \le 0$  $-1 \le x \le 0$ or $x = 2$
2.	Complete the square to deduce if a quadratic expression is always positive or always negative.	**(Lecture Notes Example 2)** Solve $(x^2+2x+5)(x+1)(2-x) > 0$. Note that $x^2+2x+5 = (x+1)^2+4$ is always positive. [Or, discriminant = $2^2 - 4(1)(5) < 0$ and coefficient of x^2 is positive, implies x^2+2x+5 is always positive.] Thus, $(x^2+2x+5)(x+1)(2-x) > 0 \Rightarrow (x+1)(2-x) > 0$. We draw the number line using the critical values -1 and 2.  Therefore the solution to the inequality is $-1 < x < 2$.
3.	Use the graphical method to solve inequalities if exact answers are not demanded, or if the expressions involve logarithmic, exponential, trigonometric functions etc. Arrange the inequality to the form $\mathrm{f}(x) > 0$ or $\mathrm{f}(x) > \mathrm{g}(x)$.	**(Inequalities Tutorial Q2(d))** $x^2 < 2^x$  The two graphs intersect at points with *x*-coordinates –0.767, 2, 4. Therefore, $-0.767 < x < 2$ or $x > 4$.

s/n	Skills	Examples of questions involving the skills
	If exact solutions are required, graphical method can be used to identify the points of intersection. We then proceed to solve the equations to obtain the exact values of the points of intersection.	**(Lecture Notes Example 7)** From $\left\|4x^2-1\right\|>3 \Rightarrow \left\|4x^2-1\right\|-3>0$. Thus we sketch the graph $y=\left\|4x^2-1\right\|-3$.  From the diagram, instead of finding points of intersection as in previous examples, here, we find the axis intercepts which are given as $(-1, 0)$ and $(1, 0)$. To satisfy the inequality $\left\|4x^2-1\right\|>3$, we identify the range of values of x such that the curve lies above the x-axis. Thus, the answer is $x<-1$ or $x>1$. **(Lecture Notes Example 8)** Find the range of values of x for which $\left\|\frac{2x^2-1}{x+3}\right\|>2$, giving your answer in exact form.  we solve for x in $\left\|\frac{2x^2-1}{x+3}\right\|=2$ Consider: $\frac{2x^2-1}{x+3}=-2$ or $\frac{2x^2-1}{x+3}=2$ Upon simplifying, we obtain $2x^2+2x+5=0$ or $2x^2-2x-7=0$ $2x^2+2x+5=0$ has no real solution while $2x^2-2x-7=0$ gives $x=\frac{1-\sqrt{15}}{2}$ or $x=\frac{1+\sqrt{15}}{2}$

s/n	Skills	Examples of questions involving the skills
		Thus, the x-coordinates of the intersecting points are $x=\frac{1-\sqrt{15}}{2}$ and $x=\frac{1+\sqrt{15}}{2}$. $\therefore \left\|\frac{2x^2-1}{x+3}\right\|>2 \Rightarrow x<\frac{1-\sqrt{15}}{2}$ or $x>\frac{1+\sqrt{15}}{2}$, $x\neq -3$.

4.	Solve inequality of the form $f(x)\le g(x)\le h(x)$ by splitting it into two sub-inequalities $f(x)\le g(x)$ and $g(x)\le h(x)$, and then find the intersection of the solutions for each inequality.	**(Lecture Notes Example 5)**
		From $\frac{2x+3}{x+2}\le x<\frac{x^2-3x-4}{x-3}$,
		consider (1) $\frac{2x+3}{x+2}\le x$ **AND** (2) $x<\frac{x^2-3x-4}{x-3}$.
		For (1): Rewrite $\frac{2x+3}{x+2}\le x \Rightarrow \frac{x^2-3}{x+2}\ge 0 \Rightarrow \frac{(x-\sqrt{3})(x+\sqrt{3})}{x+2}\ge 0$.
		This gives $-2<x\le-\sqrt{3}$ or $x\ge\sqrt{3}$ as the solution for (1).
		For (2): Similarly, rewrite $x<\frac{x^2-3x-4}{x-3}$ as $0<\frac{-4}{x-3}$.
		This gives $x<3$ as the solution for (2).
		Combining (1) and (2):
		
		From the number line diagram, we are looking out for the intersections of the partial solutions' regions. In this case, the combined answer is
		$-2<x\le-\sqrt{3}$ or $\sqrt{3}\le x<3$.

5.	Solving inequalities involving modulus.	**(Lecture Notes Example 6)** $\|2x+1\|\le 3x-2 \Rightarrow -(3x-2)\le 2x+1\le 3x-2$ $\Rightarrow 2-3x\le 2x+1\le 3x-2$ Thus $2-3x\le 2x+1$ **AND** $2x+1\le 3x-2$ $x\ge\frac{1}{5}$ **AND** $x\ge 3$  $\therefore$ answer is $x\ge 3$. **(Inequalities Tutorial Q1(g))** $\|5x-8\|>2x+1$ $5x-8<-2x-1$ OR $5x-8>2x+1$ $7x<7$ OR $3x>9$ $x<1$ OR $x>3$

<table>
<tr><th>s/n</th><th>Skills</th><th>Examples of questions involving the skills</th></tr>
<tr><td>6.</td><td>Identify suitable replacement for x to find the solution to new inequalities.</td><td>(Lecture Notes Example 9)
Solving $\frac{x}{x+8} \le \frac{1}{x-1}$, we obtain
$-8 < x \le -2$ or $1 < x \le 4$
To obtain $\frac{x+1}{x+9} \le \frac{1}{x}$ from $\frac{x}{x+8} \le \frac{1}{x-1}$, replace x with $x+1$.
$-8 < x+1 \le -2$ or $1 < x+1 \le 4$
So, $-9 < x \le -3$ or $0 < x \le 3$</td></tr>
<tr><td>7.</td><td>Formulate an equation or a system of linear equations from a problem situation

Find the numerical solution of equation (including system of linear equations) using a graphic calculator</td><td>(Lecture Notes Example 11)
Three students (A, B, C) decided to raise money by selling shirts for three weeks. The following table shows the number of shirts sold during each week and the target amount to be raised by each student. To help meet the target, they would set the unit price at the start of each week. Find the unit prices they charged in each of the three weeks.
<table>
<tr><th></th><th>A</th><th>B</th><th>C</th></tr>
<tr><td>Week 1</td><td>8</td><td>7</td><td>2</td></tr>
<tr><td>Week 2</td><td>2</td><td>3</td><td>3</td></tr>
<tr><td>Week 3</td><td>6</td><td>1</td><td>4</td></tr>
<tr><td>Total Amount ($)</td><td>105</td><td>76</td><td>55</td></tr>
</table>
Let the unit prices in each week be x, y and z respectively.
Note: Define your variables clearly and write out the 3 equations.
We have 3 equations:
$8x+2y+6z=105$
$7x+3y+z=76$
$2x+3y+4z=55$
From GC, $x=7.5$, $y=6$, $z=5.5$
Therefore, the unit prices are $7.5 , $6 and $5.5 respectively.</td></tr>
</table>

Inequalities

1. RVHS11/C1BT/Q1

Given that x is real, prove that x^2-2x+3 is always positive. [1]

Without using a calculator, solve the inequality $\frac{x^2-2x+3}{x^3-4x^2-x+4} \ge 0$. [3]

2. RI10/C1BT/Q2

Sketch, on a single diagram, the graphs of $y=\mathrm{e}^{2x-1}$ and $y=x+1$.

Hence solve the inequality $\mathrm{e}^{2x-1} > x+1$.

Deduce the solution of the inequality $\mathrm{e}^{-(2x+1)} > 1-x$. [4]

3. RI11/C1BT/Q6

(a) Solve graphically the inequality $|x-3| > \ln x$ where $x > 0$. [3]

Hence, find the solution of $\left|\frac{1-3x}{x}\right| + \ln x > 0$ where $x > 0$. [3]

(b) Without using a calculator, solve the inequality $\frac{2x^2-7x+6}{x^2-x-2} < 1$. [4]

4. ACJC10/C1BT/Q3

Solve the inequality $\frac{1}{x-a} < 4|x-a|$, where a is a positive constant, leaving your answer in terms of a. [3]

5. HCI14/C1BT/Q4

(i) Without using a calculator, solve the inequality $\frac{2x^2+4}{(x-1)(1-2x)} \leq -1$. [4]

(ii) Hence, find the exact set of values of x for which $\frac{2\sin^2 x+4}{(\sin x-1)(1-2\sin x)} \leq -1$, where $0 \leq x \leq 2\pi$. [4]

6. AJC16/C1BT/Q8

Sketch the curve with equation $y = |x^2-5x|+2$, labeling your graph clearly.

On the same diagram, sketch the line with equation $y = 10-5x$. [4]

Hence, solve the inequality $|x^2-5x| > 8-5x$, giving your answer in exact form. [4]

7. CJC16/C1BT/Q2

(i) Sketch the graph of $\frac{(x+1)^2}{4}-(y-3)^2 = 1$, showing clearly the coordinates of the vertices and equations of the asymptotes. [3]

(ii) Hence or otherwise, solve the inequality $3+\sqrt{\frac{(x+1)^2}{4}-1} > 2x$. [2]

8. IJC16/C1BT/Q5

(i) Prove that $3x^2-3x+1$ is always positive. [2]

(ii) Using an algebraic method, solve the inequality $\frac{x}{2x-1} \leq \frac{1}{3x-1}$. [3]

(iii) Hence solve $\frac{x^2}{2x^2-1} \leq \frac{1}{3x^2-1}$, leaving your answer in exact form. [3]

9. NJC16/C1BT/Q4

Without using a calculator, find the set of values of x that satisfies the inequality

$$\frac{29+3x}{9-x^2} \geq 4. \qquad [4]$$

Hence solve the inequality $\dfrac{3|x|-29}{x^2-9} \geq 4$ exactly. [4]

10. SRJC16/C1BT/Q6

Without the use of a graphing calculator, solve $\dfrac{3x-13}{x^2+x-12} \leq 1$. [4]

Hence, solve exactly the inequality $\dfrac{\ln x^3 - 13}{(\ln x)^2 + \ln x - 12} \leq 1$. [3]

11. RI 2020 C1 BT Q8

Do not use a calculator in answering this question.

(i) Solve the inequality $\dfrac{x-2}{x^2-x} \geq 1$. [3]

(ii) Hence, solve the inequalities

(a) $\dfrac{2-e^x}{e^x - e^{2x}} \geq 1$, [2]

(b) $\dfrac{x-3}{x^2-3x+2} \leq 1$. [2]

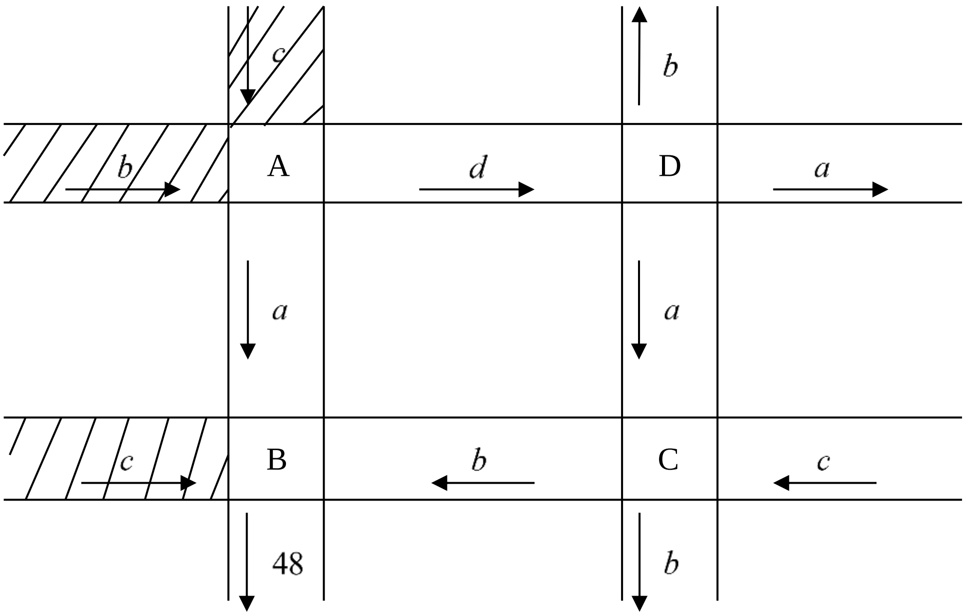
(iii) Deduce the set of values of x satisfying

$$\left(\frac{2-e^x}{e^x-e^{2x}}-1\right)\left(\frac{x-3}{x^2-3x+2}-1\right)\left(\frac{x-2}{x^2-x}-1\right) > 0 \qquad [2]$$

System of Equations

12. MJC13/Promo/Q3

The diagram below shows the traffic flow of vehicles in four traffic junctions A, B, C and D. Each arrow indicates the direction of the vehicles entering or leaving the junction. The unknown constants *a, b, c* and *d* indicate the number of vehicles entering or leaving a particular junction. It is given that the total number of vehicles entering a traffic junction must be equal to the total number of vehicles leaving that same junction. There are 48 vehicles leaving junction B.



(i) Determine the values of $a,\ b,$ c and d. [3]

(ii) The shaded region indicates the presence of an Electronic Road Pricing (ERP) gantry located at that road. It is known that each gantry charges a fixed price of \$0.50 per vehicle. How much revenue will be collected in total by the gantries in these regions? [1]

13. HCI11/C1BT/Q4

The amount of annual profit P of a company may be predicted by the model

$$P = at + b + \frac{c}{t+4}$$

where *a, b* and *c* are constants and *t* is the number of years after 2000.

In 2000, the annual profit of the company was \$160 000.

In 2001, the annual profit was \$38 000 more than the annual profit in 2000.

In 2002, the annual profit was $1\frac{1}{2}$ times the annual profit in 2000.

(i) Determine the values of *a*, *b* and *c*. [3]

(ii) If instead the model is $P=-at+b+\dfrac{c}{t+4}$, using the values of a, b and c found in part (i), find the year after 2000 when the amount of annual profit P first becomes zero. [3]

14. RI11/C1BT/Q1

On the 10th of March 2011, Dada bought 10 PineApple, 50 Googol and 300 Macrohard shares, spending a total of $40040 with the price of 1 Googol share equivalent to the sum of the prices of 1 PineApple and 10 Macrohard shares. The following day, the prices of PineApple, Googol and Macrohard shares plunged by 10%, 15% and 20% respectively. This resulted in Dada making a loss of $6227. Find the respective prices of 1 PineApple, 1 Googol and 1 Macrohard shares on the 10th of March 2011. [3]

15. HCI14/C1BT/Q3

The equation of a curve is given by

$$Ax^2+By^2+Cx+Dy+13=0$$

where A, B, C and D are constants.

(i) Find $\dfrac{dy}{dx}$ in terms of x and y. [3]

The curve has a stationary point at $(1,-1)$ and the tangent to the curve at $(3,-2)$ is parallel to the y-axis.

(ii) Find the values of A, B, C and D. [4]

16. VJC11/C1BT/Q1

LTA is trying to adjust the ERP rates for cars, lorries and motorcycles at the Orchard Road gantry. The number of the different types of vehicles passing through the gantry on 3 randomly chosen time periods for 3 different days is summarized in the table below. The table also shows the projected amount of revenue LTA plans to collect for the 3 different days. Determine the ERP rates for the different types of vehicles. [3]

	Cars	Lorries	Motorcycles	Revenue ($)
Day 1	123	91	210	788.5
Day 2	175	98	210	910
Day 3	154	103	190	850.5

In order to curb the number of lorries passing through the Orchard Road gantry, LTA decides to raise the ERP rates for lorries by 20%. Determine the new amount of revenue collected on day 3 if the rates for other vehicles remain unchanged. [1]

17. ASRJC/2019C1BT1/2

In a particular shop, the total price of a box of chocolates, a box of biscuits and a packet of nuts is $73.40 during normal season. The shop owner decided to offer a discount during the Mothers' Day season, where every three boxes of chocolates would be given a 15% discount, and for every two boxes of biscuits purchased, a discount of $5 would be given.

Mary and John went to the shop during the sale and made the following purchase.

	Boxes of Chocolates	Boxes of Biscuits	Packets of nuts	Total Cost
Mary	4	6	3	$297.97
John	6	5	2	$322.39

Find the usual unit price of each item and hence find the total savings made by Mary. [4]

18. RI/2019C1BT/2

At Gardens by the Bay, the Singapore Resident admission rates for the Audio Tour and the Cooled Conservatories are as follows:

Audio Tour
Adult: $8 Senior Citizen ($\geq$ 60 years old): $5 Child (3-12 years old): $3

Cooled Conservatories
Adult: $20 Senior Citizen ($\geq$ 60 years old): $15 Child (3-12 years old): $12

A tour group of Singapore Residents purchased 45 tickets to join the Audio Tour. The total amount for the tickets was $290.

(i) Find the different possible number of tickets that were purchased for the children in the tour group. [4]

The tour group then paid $785 for 45 tickets to visit the Cooled Conservatories.

(ii) Find the total amount that was spent on the senior citizens' tickets for the Audio Tour and Cooled Conservatories. [2]

Answers

1. $-1 < x < 1$ or $x > 4$ **2.** $x < -0.944$ or $x > 0.792$; $x > 0.944$ or $x < -0.792$

3. **(a)** $0 < x < 2.21$ or $x > 4.51$; $x > 0.453$ or $0 < x < 0.222$;

(b) $-1 < x < 4,\ x \neq 2$

4. $x < a$ or $x > a + \dfrac{1}{2}$

5. **(i)** $-1 \leq x < \dfrac{1}{2}$ or $x > 1$ **(ii)** $\left\{ x \in \mathbb{R} :\ 0 \leq x < \dfrac{\pi}{6} \text{ or } \dfrac{5\pi}{6} < x \leq 2\pi \right\}$

6. $x < -2\sqrt{2}$ or $x > 5 - \sqrt{17}$ **7.** $x \leq -3$ or $1 \leq x < 2.09$

8. **(ii)** $\dfrac{1}{3} < x < \dfrac{1}{2}$ **(iii)** $\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{2}}$ or $-\dfrac{1}{\sqrt{2}} < x < -\dfrac{1}{\sqrt{3}}$

9. $-3 < x \leq -\dfrac{7}{4}$ or $1 \leq x < 3$; $-3 < x \leq -\dfrac{7}{4}$ or $\dfrac{7}{4} \leq x < 3$

10. $a = 8, b = 16, c = 24$ and $d = 32$ $0 < x < e^{-4}$ or $x > e^{3}$ or $x = e$

11. (i) $0 < x < 1$ (ii)(a) $x < 0$ (ii)(b) $x < 1$ or $x > 2$ (iii) $(-\infty, 2) \setminus \{0, 1\}$

12. **(i)** $a = 8, b = 16, c = 24$ and $d = 32$ **(ii)** \$32

13. **(i)** a = 50 000, b = 100 000, c = 240 000; **(ii)** 2003

14. $P = \$326$, G = \$582, M = \$25.60

15. **(i)** $\dfrac{dy}{dx} = -\dfrac{2Ax + C}{2By + D}$ **(ii)** $A = 1, B = 4, C = -2, D = 16$

16. 902

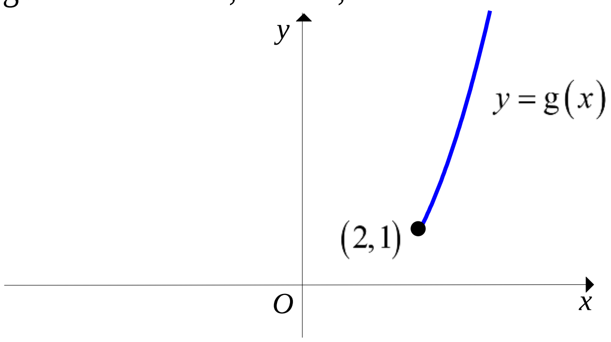
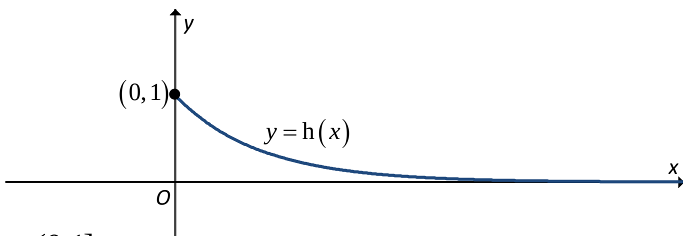
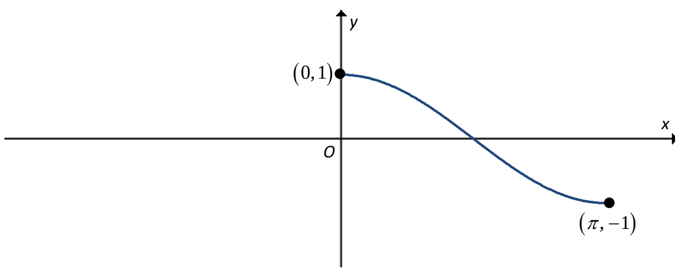
17. chocolate \$36.40; biscuits \$24.25; nuts \$12.75; \$31.38

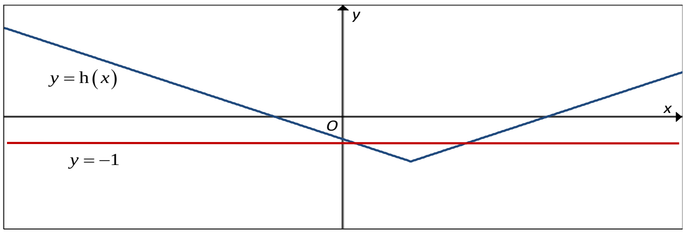
18. 2, 5, 8, 11 *or* 14; \$300

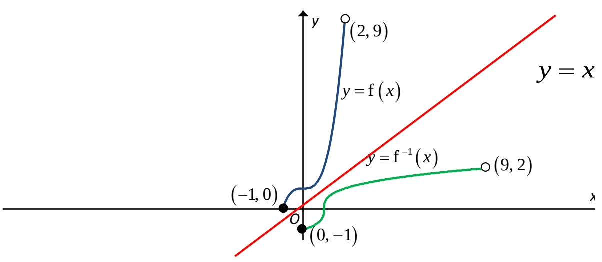
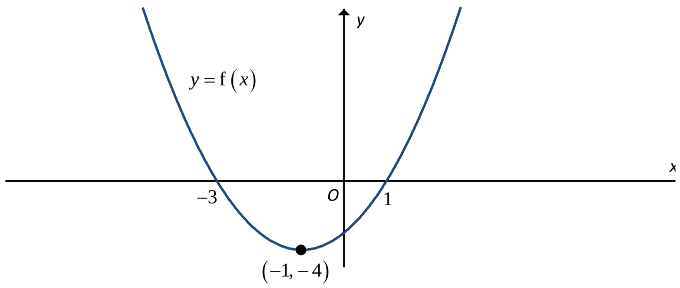
4

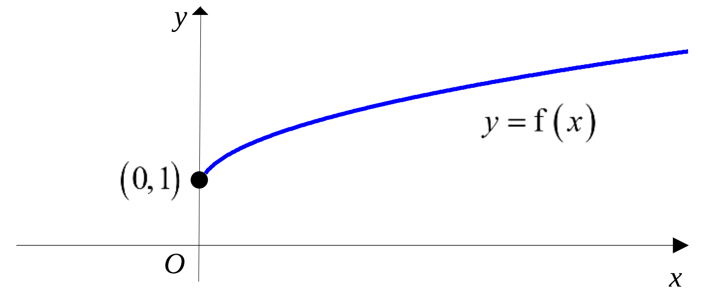
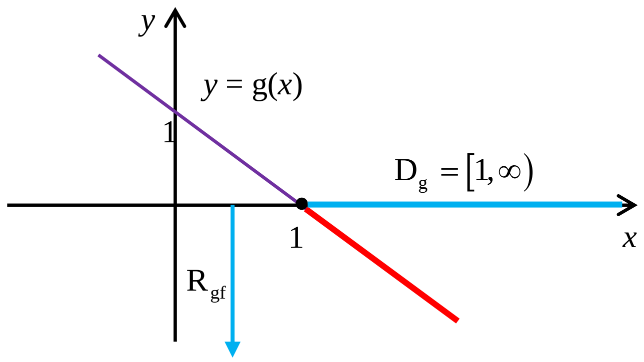
Functions

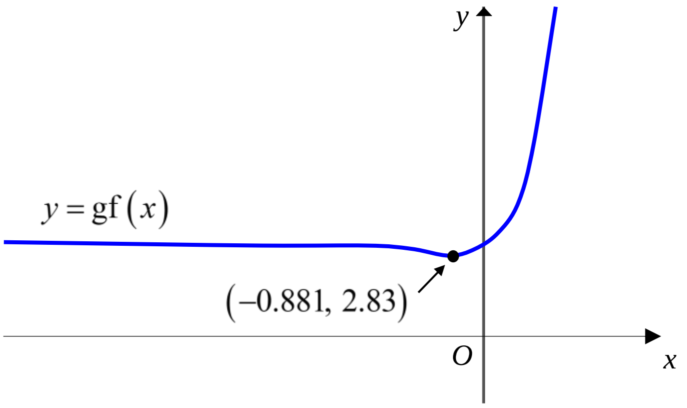
Skill Set

s/n	Skills	Examples of questions involving the skills
1.	Sketch the graph of the function according to the domain. - Students tend to sketch the graph without referring to the domain given by the function.	**(Lecture Notes Example 1(b))** $\mathrm{g}: x \to x^2 - 2x + 1, \quad x \in \mathbb{R}, x \geq 2$ 
2.	Find the range of the function by drawing accurate graphs. - Students must realize that the accuracy of the range found depends on accurate graphs. Do pay attention to any asymptotes, turning points and other features of the graph.	**(Lecture Notes Example 2(c))** $\mathrm{h}: x \to \mathrm{e}^{-2x}, \quad x \in \mathbb{R}, x \geq 0$  $R_{\mathrm{h}} = (0,1]$ Here the <u>horizontal asymptote is an important feature for figuring out the range</u>.
3.	Determine if the function is one to one. <u>Method 1</u>: By applying the 'Horizontal Line Test' and stating the correct reason to justify whether the function is one to one.	<u>When the function is one to one:</u> **(Lecture Notes Example 3(b))** $\mathrm{g}: x \to \cos x, \quad x \in \mathbb{R}, 0 \leq x \leq \pi$,  Every horizontal line intersects the graph of $y = \mathrm{g}(x)$ at no more than one point, thus the function g is a one to one function. Therefore its inverse exists.

s/n	Skills	Examples of questions involving the skills
		When the function is not one to one: **(Lecture Notes Example 3(c))** $\mathrm{h}: x \mapsto \lvert x-1\rvert-2, \quad x\in\mathbb{R}$.  Draw a horizontal line on your graph to show that it intersects the curve at two points. Thus h is not a one to one function. OR Give a counterexample like $x=-1, x=3$ gives the same *y* value = 0.
4.	Find the rule of the inverse function by letting *y* = f(*x*) and making *x* as the subject eventually. (a) When f(*x*) is a quadratic function. <u>Method 1:</u> Complete square <u>Method 2:</u> Express the quadratic expression in the form of $ax^2+bx+c=0$ and find *x* in terms of *y* using $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. (b) When f(*x*) is a logarithmic function - Realize that inverse of a ln function is the exponential function.	**(Lecture Notes Example 5)** $\mathrm{f}: x\to x^2+2x-3, \quad \text{for } x\in\mathbb{R}$. From $y=\mathrm{f}(x)$, we have $y=x^2+2x-3$ $\Rightarrow y=(x+1)^2-4$ $\Rightarrow x=-1\pm\sqrt{y+4}$ Since $x\geq-1$, therefore we take $x=-1+\sqrt{y+4}$. Thus $\mathrm{f}^{-1}(y)=-1+\sqrt{y+4}\Rightarrow \mathrm{f}^{-1}(x)=-1+\sqrt{x+4}$ From $y=\mathrm{f}(x)$, we have $y=x^2+2x-3$ $\Rightarrow x^2+2x-3-y=0$ $\Rightarrow x=\dfrac{-2\pm\sqrt{4-4(1)(-3-y)}}{2}$ $\Rightarrow x=\dfrac{-2\pm\sqrt{4(4+y)}}{2}$ $\Rightarrow x=-1\pm\sqrt{4+y}$ Since $x\geq-1$, therefore we take $x=-1+\sqrt{y+4}$. Thus $\mathrm{f}^{-1}(y)=-1+\sqrt{y+4}\Rightarrow \mathrm{f}^{-1}(x)=-1+\sqrt{x+4}$ **(Tutorial Q1(d))** $\mathrm{f}: x\mapsto(\ln x)^2, \quad x\in\mathbb{R},\ x>1$ Let $y=(\ln x)^2$ $\sqrt{y}=\ln x$ since $x>1,\ \ln x>0$

s/n	Skills	Examples of questions involving the skills
	(c) When f(x) is a modulus function - Remove modulus sign by resolving modulus function into 1 of the 2 expressions. i.e. $\lvert x-1\rvert=(x-1)$ or $-(x-1)$	$\Rightarrow \mathrm{e}^{\sqrt{y}}=x$ [Note: $\mathrm{e}^{\ln x}=x$] $\Rightarrow x=\mathrm{e}^{\sqrt{y}}$ $\therefore \mathrm{f}^{-1}(x)=\mathrm{e}^{\sqrt{x}}$ **(Tutorial Q1(e))** $\mathrm{f}: x\mapsto\left\lvert\frac{2}{x-1}\right\rvert,\quad x\in\mathbb{R},\ x<1$ Let $y=\left\lvert\frac{2}{x-1}\right\rvert$ Then, $y=-\frac{2}{x-1}$ since $x<1$ $-y=\frac{2}{x-1}\Rightarrow -\frac{1}{y}=\frac{x-1}{2}\Rightarrow x=1-\frac{2}{y}$ $\therefore\ \mathrm{f}^{-1}(x)=1-\frac{2}{x}$
5.	Find the domain and range of the inverse function using the relation:	Domain of f^{-1} = Range of f Range of f^{-1} = Domain of f
6.	Sketch the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ on the same diagram Important to note: (a) change of coordinates from (x, y) to (y, x) after reflecting the graph of $y=\mathrm{f}(x)$ in the line $y=x$ (b) same scale on both axes (using ZSQAURE function on TI84PlusC).	**(Lecture Notes Example 4(iv))** 
7.	Restrict the domain of f so that f^{-1} exists.	**(Lecture Notes Example 5(i))** $\mathrm{f}: x\to x^2+2x-3.\quad$ for $x\in\mathbb{R}$  From the graph, we observe that for the inverse to exist, i.e. for f is one to one, the largest domain we can go for is $x\geq -1$. Thus the smallest value of p is -1.

s/n	Skills	Examples of questions involving the skills
8.	State the range of f and domain of g to check the existence of composite function gf.	Students need to check $R_{\mathrm{f}} \subseteq \mathrm{D_g}$ for gf exists. Students must <u>state the range of f and domain of g explicitly</u> before concluding if the former is a subset of the latter. **(Lecture Notes Example 6)** $\mathrm{f}: x \mapsto 1+\sqrt{x},\ x \in \mathbb{R}, x \geq 0$ and $\mathrm{g}: x \mapsto 1-x,\ x \in \mathbb{R}$.  From the graph, $R_{\mathrm{f}}=[1,\infty)$. Since $R_{\mathrm{f}}=[1,\infty) \subset \mathbb{R}=D_{\mathrm{g}}$, gf exists.
9.	Find the rule of composite function gf and its domain	**(Extension of lecture notes example 6)** $\mathrm{f}: x \to 1+\sqrt{x},\ x \in \mathbb{R}, x \geq 0$ and $\mathrm{g}: x \to 1-x,\ x \in \mathbb{R}$ $\mathrm{gf}(x)=\mathrm{g}(\mathrm{f}(x))=\mathrm{g}(1+\sqrt{x})=1-(1+\sqrt{x})=-\sqrt{x}$ $D_{\mathrm{gf}}=D_{\mathrm{f}}=[0,\infty)$
10.	Find the range of gf using <u>Method 1:</u> Do in stages. $D_{\mathrm{gf}}=D_{\mathrm{f}} \xrightarrow{\mathrm{f}} R_{\mathrm{f}} \xrightarrow{\mathrm{g}} R_{\mathrm{gf}}$	**(Lecture Notes Example 6)** $\mathrm{f}: x \to 1+\sqrt{x},\ x \in \mathbb{R}, x \geq 0$ and $\mathrm{g}: x \to 1-x,\ x \in \mathbb{R}$ $D_{\mathrm{gf}}=D_{\mathrm{f}}=[0,\infty) \xrightarrow{\mathrm{f}} R_{\mathrm{f}}=[1,\infty) \xrightarrow{\mathrm{g}} ?$ Use graph of $y = \mathrm{g}(x)$, substitute $[1,\infty)$ into function g.  $R_{\mathrm{gf}}=(-\infty,0]$
	<u>Method 2:</u> Graphical method Sketch the graph of $y=\mathrm{fg}(x)$ <u>Limitations:</u> - The graph of fg may not be easy to draw accurately. - This method is recommended if fg is a simple function.	**(Lecture Notes Example 9(ii) Method 2)** Sketch the graph of $y = \mathrm{gf}(x)$ according to the domain of f. Find the range of gf. $y=\mathrm{gf}(x)=\mathrm{e}^{x}+\dfrac{2}{\mathrm{e}^{x}+1}+1, \quad x \in \mathbb{R}$. Sketch the graph of the composite function gf subjected to its domain.

s/n	Skills	Examples of questions involving the skills
		 From the graph, $R_{\mathrm{gf}} = [2.83, \infty)$.
11.	Solving the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ is the same as solving $\mathrm{f}(x) = x$, provided that the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ only meet at the line $y = x$.	**(Tutorial Q6(iv))** Since the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = x$ will meet at the same point, to find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, we could solve $\mathrm{f}(x) = x$, ie. $x^2 - 2x - 3 = x$.
12.	Know the difference between the graphs of $\mathrm{f}^{-1}\mathrm{f}$ and $\mathrm{f}\mathrm{f}^{-1}$	**(Tutorial Q7(iii))** Given $\mathrm{f}: x \mapsto (x-1)^2 + 2, \ x \in \mathbb{R}, \ x < 1$ What is the difference between the graph of $\mathrm{f}^{-1}\mathrm{f}$ and $\mathrm{f}\mathrm{f}^{-1}$? $\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{f}\mathrm{f}^{-1}(x) = x$ [This result is true all the time, regardless of f]. Therefore $\mathrm{f}^{-1}\mathrm{f}$ and $\mathrm{f}\mathrm{f}^{-1}$ share the same rule. However, the domain of $\mathrm{f}^{-1}\mathrm{f}$ = domain of f = $(-\infty, 1)$ while domain of $\mathrm{f}\mathrm{f}^{-1}$ = domain of $\mathrm{f}^{-1} = (2, \infty)$.

1. AJC12/C1BT/Q9(a)

Function h is defined by

$$\mathrm{h}: x \mapsto \mathrm{e}^{(3-x)^2}, \quad 0 \le x \le 3.$$

(i) Find the function h^{-1}. [5]

(ii) Find the range of values of x such that $\mathrm{h}^{-1}\mathrm{h}(x) = \mathrm{h}\mathrm{h}^{-1}(x)$. [3]

2. DHS10/C1BT/Q6

The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{1-x}{2-x}, \qquad x > 2,$$
$$\mathrm{g}: x \mapsto x \ln x, \qquad x \ge 1.$$

(i) Show that f^{-1} exists and express f^{-1} in a similar form. [4]

(ii) Find the exact value of x for which $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

(iii) Show that gf exists and solve exactly the equation $(\mathrm{gf})^{-1}(x) = 3$. [3]

3. **NJC11/C1BT/Q11(b)&(c)**

The functions g and h are defined as follows:

$$\text{g} : x \mapsto \ln(x+2), \quad x > -2,$$

$$\text{h} : x \mapsto x(x^2 - x - 1), \quad -\frac{1}{3} < x < 1.$$

(a) Find the value of $\text{h}^{-1}\left(-\frac{1}{2}\right)$. [2]

(b) Show that the composite function gh exists. Find the composite function gh and its exact range. [5]

4. **VJC11/C1BT/Q7**

The functions g and h are defined by

$$\text{g} : x \mapsto 3^x, \ x \in \mathbb{R},$$

$\text{h} : x \mapsto ax + b, \ x \in \mathbb{R},$ where a and b are non-zero constants.

(i) Find $\text{h}^{-1}(x)$. [2]

(ii) Given that $\text{h}^{-1}(2) = \text{g}(2)$ and the graphs of h^{-1} and g meet on the y-axis, show that $a = \frac{1}{4}$ and find the value of b. [3]

(iii) With the values of a and b found in **(ii)**, find $(\text{gh})^{-1}(3)$. [3]

5. **DHS11/C1BT/Q8**

The functions f and g are defined as follows:

$$\text{f} : x \mapsto -x^3 + 1, \ x \in \mathbb{R},$$

$$\text{g} : x \mapsto \text{e}^{2x} - 2, \ x > -b, b > 0.$$

(i) Define f^{-1} in a similar form. [3]

(ii) Hence, without finding g^{-1}, find x such that $\text{fg}^{-1}(x) + 7 = 0$. [3]

6. **TJC10/C1BT/Q9**

The functions f and g are defined by

$$\text{f} : x \mapsto 2\sqrt{1-4x^2} \ \text{ for } x \in \mathbb{R}, \ c \le x \le \frac{1}{2}$$

$$\text{g} : x \mapsto \ln(\text{e} - x) \ \text{ for } x \in \mathbb{R}, x < \text{e}$$

(i) In the case where $c = -\frac{1}{2}$,

(a) sketch the graph of $y = \text{f}(x)$, showing clearly the intercepts on both axes. [1]

(b) show that the function gf exists and state the range of gf, giving your answer in exact values. [3]

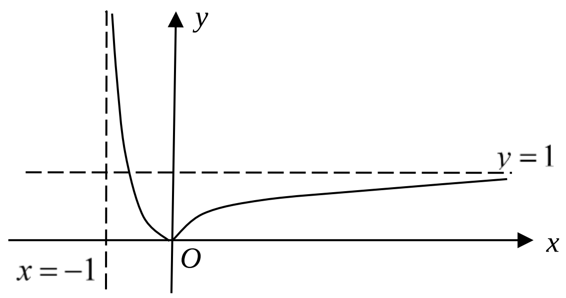
(ii) Find the minimum value of c such that f^{-1} exists. [1]

(iii) Using the value of c found in part **(ii)**, sketch on the same diagram, the graphs of $y = \text{f}(x)$ and $y = \text{f}^{-1}(x)$, illustrating clearly the relationship between the two graphs. [3]

7. AJC11/C1BT/Q11

(a) The function h is given by $\mathrm{h}: x \mapsto a + \dfrac{1}{x-a}, x \neq a$. Find $\mathrm{h}^2(x)$ and hence determine $\mathrm{h}^7(x)$. [3]

(b) The diagram below shows the graph of function f defined on $(-1, \infty)$.



(i) State the restricted domain of f such that f^{-1} exists and the range of f remains unchanged. [1]

(ii) With the restricted domain of f found in part **(i)**, sketch the graphs of f^{-1} and $\mathrm{f}^{-1}\mathrm{f}$ in a single diagram, showing clearly any asymptotes. [2]

8. AJC10/C1BT/Q12

(a) The functions f and g are defined by:

$$\mathrm{f}(x) = \frac{1}{x^2}, x \in \mathbb{R}, x \neq 0 \quad \text{and} \quad \mathrm{g}(x) = \mathrm{e}^{|x|}, x \in \mathbb{R}$$

Show that the composite function gf exists. Define gf and find its range. [5]

(b) The function h is defined by:

$$\mathrm{h}(x) = \begin{cases} \sqrt{a^2 - x^2}, 0 \leq x \leq a \\ -\sqrt{a^2 - x^2}, -a < x < 0 \end{cases},$$

where a is a positive constant.

(i) By sketching the graph of $y = \mathrm{h}(x)$, show that h^{-1} exists and $\mathrm{h}^{-1} = \mathrm{h}$. [3]

(ii) Evaluate $\mathrm{h}^5\left(-\dfrac{a}{2}\right)$ exactly, giving your answer in terms of a. [2]

(iii) For $0 \leq x \leq a$, find in terms of a, the x-coordinate of the point of intersection of the graphs $y = \mathrm{h}(x)$ and $y = \mathrm{hh}^{-1}(x)$, giving your answer in exact form. [2]

9. VJC10/C1BT/Q11

The functions f and g are defined, for $x \in \mathbb{R}$, by

$$\mathrm{f}: x \mapsto x^2 + 1 \quad \text{and} \quad \mathrm{g}: x \mapsto x - 3.$$

(i) Find f(3x). [1]

(ii) Use an algebraic method to solve $|\mathrm{fg}(x) - \mathrm{gf}(x)| \leqslant \sqrt{3}\, x + 4$, giving your answer in exact form. [4]

The function h is defined by $\mathrm{h}: x \mapsto \dfrac{5x-8}{x-1}, \ x \in \mathbb{R}, x > 3.$

(iii) Solve the equation h(x) = x. [2]

(iv) Find the expression for $\mathrm{h}^{-1}(x)$. [2]

10. RI11/C1BTP1/Q9

The functions f and g are defined as follows:

$$\mathrm{f}(x)=\begin{cases}(x-2)^2+1 & \text{for } x\in\mathbb{R},\ x\le 2,\\ x^2-3 & \text{for } x\in\mathbb{R},\ x>2,\end{cases}$$

$$\mathrm{g}(x)=\frac{3x+4}{x+1} \text{ for } x\in\mathbb{R}, x\ge 0.$$

(i) Sketch the graph of $y=\mathrm{f}(x)$. [2]

(ii) Find the range of g. Hence, find $\mathrm{fg}(x)$. [3]

(iii) State the equation of the asymptote of $y=\mathrm{fg}(x)$. [1]

The function h is defined as $\mathrm{h}: x\mapsto \mathrm{f}(x),\ \text{for } x\in\mathbb{R},\ x\le k$.

(iv) State the largest value of k for which h has an inverse. [1]

(v) With this value of k found in (iv), find the inverse of h in similar form. [3]

11. TPJC15/C1BT/Q1

It is given that

$$\mathrm{f}(x)=\begin{cases}5-x^2 & \text{for } 0<x\le 2\ ,\\ 2x-3 & \text{for } 2<x\le 4\ .\end{cases}$$

and that $\mathrm{f}(x)=\mathrm{f}(x+4)$ for all real values of x.

(i) Evaluate $\mathrm{f}(5)+\mathrm{f}(2015)$. [2]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $-3\le x\le 9$. [3]

12. VJC15/C1BT/Q2

It is given that

$$\mathrm{f}(x)=\begin{cases}\cos^{-1}x & \text{for } 0\le x<1,\\ \dfrac{\pi}{2}x-\dfrac{\pi}{2} & \text{for } 1\le x<2,\end{cases}$$

and that $\mathrm{f}(x)=\mathrm{f}(x+2)$ for all real values of x.

(i) Sketch the graph given by $y=\mathrm{f}(x)$ for $-3\le x\le 4$. [3]

(ii) For $0\le x\le 4$, solve the inequality $\mathrm{f}(x)>\dfrac{\pi}{3}$, giving your answer in the exact form. [3]

13. CJC16/C1BT/Q11

Functions f and g are defined by

$$\mathrm{f}: x\mapsto \ln\left(x^2+2x+5\right) \quad \text{for} \quad x\in\mathbb{R},\ x>-5,$$

$$\mathrm{g}: x\mapsto x^2-4 \quad \text{for} \quad x\in\mathbb{R}.$$

(i) Show that fg exists. Find $\mathrm{fg}(x)$, stating the domain and the range of fg. [5]

(ii) Give a reason why f does not have an inverse. [2]

(iii) The function f has an inverse if its domain is restricted to $x>k$. State the least value of k for which the function f^{-1} exists. [1]

Using the value of k found in **(iii)**,

(iv) find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

(v) sketch on the same diagram, the graphs of f and f^{-1}, showing clearly the relationship between the graphs and the coordinates of the endpoints. [3]

14. **TJC16/C1BT/Q11**

The function f is defined by $\mathrm{f}: x \mapsto 1+\sqrt{2-x}, \quad x \in \mathbb{R},\ 0 \le x \le 2$.

(i) Find f^{-1}, stating its domain. [3]

(ii) By sketching the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ on the same diagram, show that the equation $\mathrm{f}(x)-\mathrm{f}^{-1}(x)=0$ has 3 real roots. [3]

Find the roots of the equation $\mathrm{f}(x)-\mathrm{f}^{-1}(x)=0$, leaving your answers in exact form. [3]

The function g is defined by $\mathrm{g}: x \mapsto a^{-x}, \quad x \in \mathbb{R},\ x \ge 0$ where $a>1$.

(iii) Show that the composite function gf exists and define it in a similar form. [3]

(iv) By sketching the graph of $y=\mathrm{g}(x)$, find the range of gf in terms of *a*. [2]

15. **NJC18/C1BT/Q5**

The function f is defined by

$$\mathrm{f}: x \mapsto \begin{cases} x+2, & x<-1, \\ \mathrm{e}^{1-x^2}, & x \ge -1. \end{cases}$$

(i) Sketch the graph of f and state its exact range. [3]

(ii) Explain why f^{-1} does not exist. [1]

(iii) State the largest integer value of *k* such that the restriction function

$$\mathrm{g}: x \mapsto \begin{cases} x+2, & x<-1, \\ \mathrm{e}^{1-x^2}, & -1 \le x \le k, \end{cases}$$

has an inverse. [1]

(iv) With the value of *k* found in part **(iii)**, find g^{-1} in a similar form. [4]

(v) On the same diagram, sketch the graph of g and g^{-1}, indicating clearly the line of symmetry and its equation. [2]

16. **VJC 2020 C1 BT Q5**

The domain of a function f is $(0,4]$ and

$$\mathrm{f}(x)=\begin{cases}2x-x^2 & \text{for } 0<x\leqslant 2,\\ \mathrm{f}(x-2)+1 & \text{for } 2<x\leqslant 4.\end{cases}$$

(i) Find $\mathrm{f}(2)$ and $\mathrm{f}(4)$. [2]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $0<x\leqslant 4$. [3]

(iii) The function g is defined by $\mathrm{g}:x\mapsto|x-2|$ for $x\in\mathbb{R},\ 0\leq x<5$. Does the composite function fg exist? Justify your answer. [2]

Answers

1. **(i)** $\mathrm{h}^{-1}(x)=3-\sqrt{\ln x},\quad 1\leq x\leq \mathrm{e}^9$ **(ii)** $[1,3]$

2. **(i)** $\mathrm{f}^{-1}:x\mapsto\dfrac{2x-1}{x-1},\ x>1$; **(ii)** $\dfrac{3+\sqrt{5}}{2}$; **(iii)** 2ln2

3. **(a)** 0.403;

(b) $\mathrm{gh}(x)=\ln\left(x^3-x^2-x+2\right)$; $\mathrm{D_{gh}}=\mathrm{D_h}=\left(-\dfrac{1}{3},1\right)$; Range of gh $=\left(0,\ln\dfrac{59}{27}\right)$

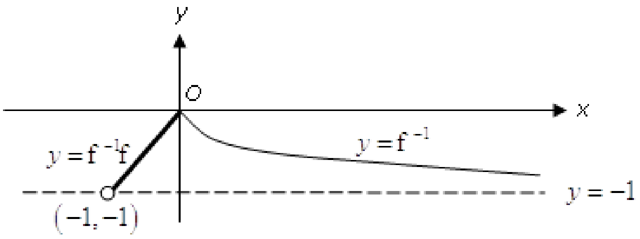
4. **(i)** $\mathrm{h}^{-1}(x)=\dfrac{x-b}{a}$; **(ii)** $a=\dfrac{1}{4},\ b=-\dfrac{1}{4}$; **(iii)** 5

5. **(i)** $\mathrm{f}^{-1}:x\mapsto(1-x)^{\frac{1}{3}},\ x\in\mathbb{R}$; **(ii)** e^4-2

6. **(i)(b)** $\mathrm{R_f}=[0,2]\subseteq(-\infty,\mathrm{e})=\mathrm{D_g}$; $\mathrm{R_{gf}}=[\ln(\mathrm{e}-2),1]$; **(ii)** min $c=0$

7. **(a)** $\mathrm{h}^7(x)=a+\dfrac{1}{x-a}$; **(b)(i)** $(-1,0]$;

(b)(ii)



8. **(a)** $\mathrm{gf}(x)=\mathrm{e}^{\frac{1}{x^2}},x\in\mathbb{R},x\neq 0$, $\mathrm{R_{gf}}=(1,\infty)$; **(b)(ii)** $-\dfrac{\sqrt{3}}{2}a$; **(iii)** $\dfrac{a}{\sqrt{2}}$

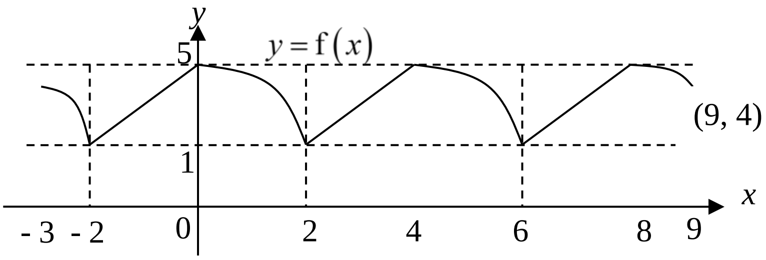
9. **(i)** $9x^2+1$; **(ii)** $\dfrac{8}{6+\sqrt{3}}\leq x\leq\dfrac{16}{6-\sqrt{3}}$; **(iii)** $x=4$;

(iv) $\mathrm{h}^{-1}(x)=\dfrac{x-8}{x-5},\ \dfrac{7}{2}<x<5$

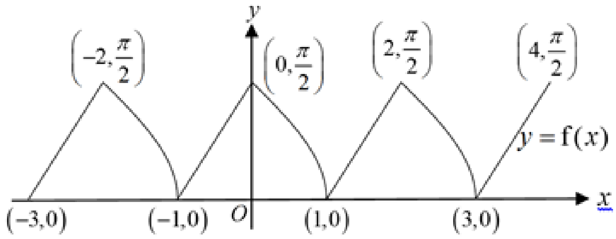
10. **(ii)** $\mathrm{fg}(x)=\dfrac{6x^2+18x+13}{x^2+2x+1}$; **(iii)** $y=6$; **(iv)** 2;

(v) $\mathrm{h}^{-1}:x\to 2-\sqrt{x-1},\quad x\in\mathbb{R},x\geq 1$

11. **(i)** 7; **(ii)**



12. **(i)**

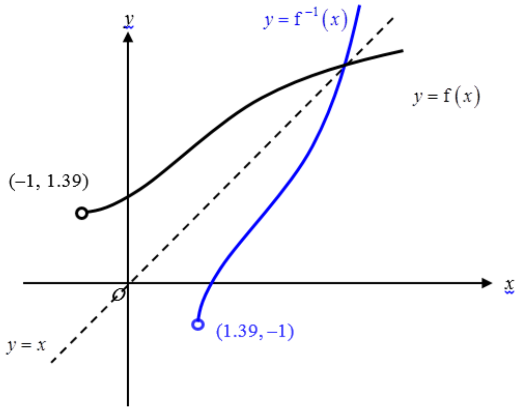


(ii) $0 \le x < \dfrac{1}{2}$ or $\dfrac{5}{3} < x < \dfrac{5}{2}$ or $\dfrac{11}{3} < x \le 4.$

13. **(i)** $\mathrm{D_{fg}} = (-\infty, \infty)$, $\mathrm{R_{fg}} = [1.39, \infty)$ **(iii)** $k = -1$

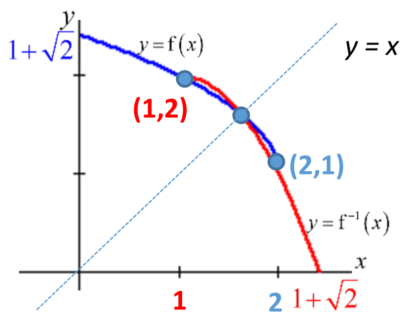
(iv) $\mathrm{f}^{-1}(x) = -1 + \sqrt{\mathrm{e}^x - 4}$, $\mathrm{D_{f^{-1}}} = \mathrm{R_f} = (1.39, \infty)$

(v)



14. **(i)** $\mathrm{f}^{-1}(x) = 2 - (x-1)^2$, $\mathrm{D_{f^{-1}}} = \left[1, 1+\sqrt{2}\right]$

(ii)

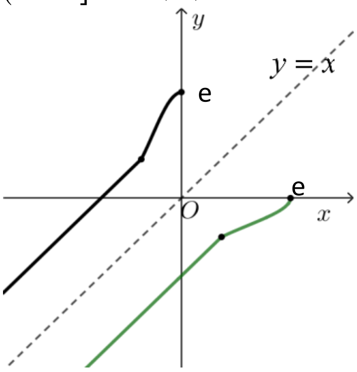


$x = 1,\ x = \dfrac{1+\sqrt{5}}{2}$ or $x = 2$

(iii) $\mathrm{gf} : x \to a^{-1-\sqrt{2-x}}, x \in \mathbb{R},\ 0 \le x \le 2$ **(iv)** $\mathrm{R_{gf}} = \left[a^{-1-\sqrt{2}}, a^{-1}\right]$

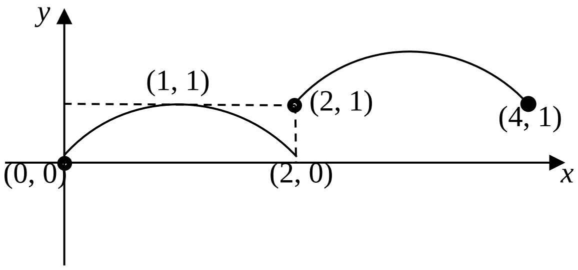
15. **(i)** $R_f = (-\infty, e]$ **(iii)** $k = 0$ **(iv)** $g : x \mapsto \begin{cases} x-2 & x<1, \\ -\sqrt{1-\ln x} & 1 \le x \le e. \end{cases}$

(v)



16. **(i)** $f(2)=0,\ f(4)=1$ **(iii)** fg does not exist

(ii)



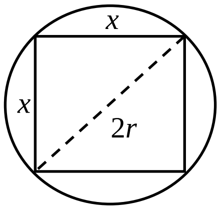
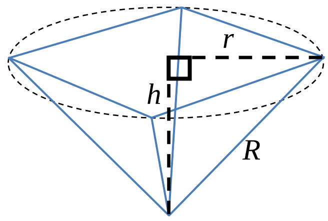
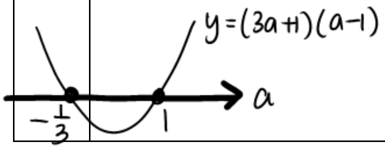
5

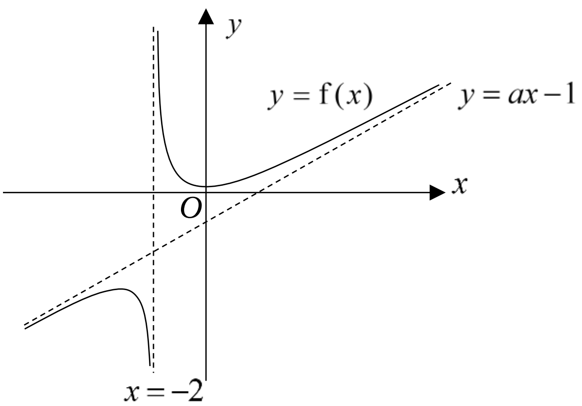
Differentiation and Its Applications

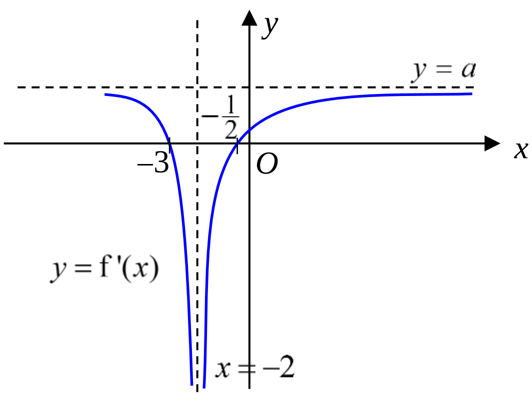
Skill Set

No.	Skills	Examples of questions involving the skills
1.	Techniques of differentiation using • product rule • quotient rule • chain rule	Refer to Summary in Lecture Notes (Pg. 18)
2.	Differentiation of • logarithm & exponential functions • trigonometric functions • inverse trigonometric functions	Refer to Summary in Lecture Notes (Pg. 18-19)
3.	Implicit differentiation	Tutorial 5A Q3a $x+\sqrt{xy}=3$ $1+\frac{1}{2\sqrt{xy}}\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0$ $x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-2\sqrt{xy}-y$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2\sqrt{xy}-y}{x}$
4.	Higher order differentiation	Tutorial 5A Q6b $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+xy=3y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ Differentiating w.r.t. x, $2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+x\frac{\mathrm{d}y}{\mathrm{d}x}+y=3\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+3y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ $x\frac{\mathrm{d}y}{\mathrm{d}x}+y=\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+3y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$
5.	Differentiation involving parametric equations	Tutorial 5A Q7 $x=t^2-3t$; $y=\ln t$ $\frac{\mathrm{d}x}{\mathrm{d}t}=2t-3$; $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{t}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{\frac{1}{t}}{2t-3}=\frac{1}{2t^2-3t}$
6.	Simplify inverse trigonometric functions before differentiation	Tutorial 5A Q3d $\sin^{-1}(x+y)=xy$ $x+y=\sin(xy)$ Differentiate w.r.t x: $1+\frac{\mathrm{d}y}{\mathrm{d}x}=\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cos(xy)$

No.	Skills	Examples of questions involving the skills
		$1+\frac{\mathrm{d}y}{\mathrm{d}x}=\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cos(xy)$ $\frac{\mathrm{d}y}{\mathrm{d}x}\left(1-x\cos(xy)\right)=y\cos(xy)-1$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y\cos(xy)-1}{1-x\cos(xy)}$
7.	Simplify logarithm functions using properties of logarithm before differentiation	Tutorial 5A Q1d $\frac{\mathrm{d}}{\mathrm{d}x}\ln\sqrt{\frac{x-a}{x+a}}=\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}[\ln(x-a)-\ln(x+a)]$ $=\frac{1}{2}[\frac{1}{x-a}-\frac{1}{x+a}]$ $=\frac{1}{2}\frac{x+a-(x-a)}{(x-a)(x+a)}$ $=\frac{1}{2}\frac{2a}{x^2-a^2}$ $=\frac{a}{x^2-a^2}$
8.	Always work in radians for angles	Tutorial 5A Q1c $\frac{\mathrm{d}}{\mathrm{d}x^\circ}\left[\cos^2(x^\circ)+\tan(x^\circ)\right]$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos^2\left(\frac{\pi x}{180}\right)+\tan\left(\frac{\pi x}{180}\right)\right]$ $=\left(2\cos\frac{\pi x}{180}\right)\left(-\sin\frac{\pi x}{180}\right)\left(\frac{\pi}{180}\right)+\left(\sec^2\frac{\pi x}{180}\right)\left(\frac{\pi}{180}\right)$ $=\frac{\pi}{180}\left(\sec^2\frac{\pi x}{180}-\sin\frac{\pi x}{90}\right)$ $=\frac{\pi}{180}\left(\sec^2 x^\circ-\sin 2x^\circ\right)$
9.	Finding equations of tangents and normals where curve is defined (i) implicitly (ii) parametrically	(i) Lecture Notes Example 15 $4x^2-y^2=11$ $8x-2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\Rightarrow \therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{8x}{2y}=\frac{4x}{y}$ At (3,5), gradient of tangent = $\frac{12}{5}$ Hence equation of tangent to curve at (3,5) is $y-5=\frac{12}{5}(x-3)$, i.e. $y=\frac{12}{5}x-\frac{11}{5}$ (ii) Tutorial 5B Q1 $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{a}{t^2}$, $\frac{\mathrm{d}y}{\mathrm{d}t}=a\left(1+\frac{2}{t^3}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{a\left(1+\frac{2}{t^3}\right)}{-\frac{a}{t^2}}=-\left(t^2+\frac{2}{t}\right)$ When $t=2$, $x=\frac{3a}{2}$ $y=\frac{7a}{4}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-5$ $\therefore$ equation of normal is $y-\frac{7a}{4}=\frac{1}{5}\left(x-\frac{3a}{2}\right)$ $20y-35a=4x-6a$ $20y=4x+29a$
10.	Properties of tangents and normals	Tutorial 5B Q4 (i) Tangent // x-axis $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=0$ (ii) Tangent // y–axis $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}$ undefined $\Rightarrow \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}=0$

No.	Skills	Examples of questions involving the skills
11.	Intersections of graphs in parametric form and cartesian form: find value of t and substitute into x and y to find coordinates	Tutorial 5B Q2(iii) Equation of tangent at P: $y+2=-\frac{4}{3}(x-2)$ Substituting $x=t^2-t,\ y=t^3+t$ into equation, $t^3+t+2=-\frac{4}{3}(t^2-t-2)$ $t^3+\frac{4}{3}t^2-\frac{1}{3}t-\frac{2}{3}=0$ Using GC, $t=-1$ (rejected $\because$ this t is for P) or $t=\frac{2}{3}$ Substituting $t=\frac{1}{2}$ into $x=t^2-t$, $y=t^3+t$, Hence $R(-\frac{2}{9},\frac{26}{27})$
12.	Expressing quantity in terms of a single variable in connected rates of change and local maxima and minima problems	Lecture Notes Example 22(i)   Given $V=\frac{1}{3}x^2h$ From diagram, $(2r)^2=x^2+x^2 \Rightarrow x^2=2r^2$ Also, $h^2=R^2-r^2 \Rightarrow r^2=R^2-h^2$ $\therefore\ V=\frac{1}{3}x^2h=\frac{2r^2}{3}h=\frac{2}{3}h(R^2-h^2)$ (shown)
13.	Rates of change: applying chain rule	Tutorial 5B Q5 $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi r^2(2r)=\frac{2}{3}\pi r^3$ (i) $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$ and $\frac{dl}{dt}=\frac{dl}{dr}\times\frac{dr}{dt}$ $\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt}$
14.	Using differentiation to find range of values of constants  	The equation of the curve C is $y=\frac{x^2-2ax-3a^2}{x+1}$, where a is a constant. Find $\frac{dy}{dx}$ and deduce that if C has two stationary points, then $-\frac{1}{3}<a<1$. Solution $y=\frac{x^2-2ax-3a^2}{x+1}$ $\frac{dy}{dx}=\frac{(x+1)(2x-2a)-(x^2-2ax-3a^2)}{(x+1)^2}$ $=\frac{2x^2-2ax+2x-2a-x^2+2ax+3a^2}{(x+1)^2}$ $=\frac{x^2+2x+(3a^2-2a)}{(x+1)^2}$ At stationary points, $\frac{dy}{dx}=0 \Rightarrow x^2+2x+(3a^2-2a)=0$ (*) Since C has 2 stationary points, (*) has 2 real distinct roots. $\therefore$ discriminant $=4-4(1)(3a^2-2a)>0$

<table>
<tr><th>No.</th><th>Skills</th><th>Examples of questions involving the skills</th></tr>
<tr><td></td><td></td><td>$3a^2-2a-1<0$
$(3a+1)(a-1)<0$
$-\frac{1}{3}<a<1$ (shown)</td></tr>
<tr><td>15.</td><td>Determine nature of stationary points analytically using 1st derivative test or 2nd derivative test, including use of 1st derivative test if 2nd derivative test is inconclusive</td><td><u>Lecture Notes Example 20</u>
$y=3x^5-x^3+5$
$\frac{dy}{dx}=15x^4-3x^2=3x^2\left(5x^2-1\right)$
$\frac{dy}{dx}=0 \Rightarrow x=0,\pm 0.447$
$\frac{d^2y}{dx^2}=60x^3-6x$
At $x=0.447$, $y=4.96$, $\frac{d^2y}{dx^2}=2.68>0$
$\therefore \left(0.447,\ 4.96\right)$ is a minimum point.
At $x=-0.447$, $y=5.04$, $\frac{d^2y}{dx^2}=-2.68<0$
$\therefore \left(-0.447,\ 5.04\right)$ is a maximum point.
At $x=0$, $y=5$, $\frac{d^2y}{dx^2}=0 \Rightarrow$ inconclusive
<u>Using 1st derivative test</u>
<table>
<tr><td>x</td><td>0^-</td><td>0</td><td>0^+</td></tr>
<tr><td>$\frac{dy}{dx}$</td><td>$-$</td><td>0</td><td>$-$</td></tr>
<tr><td></td><td>$\searrow$</td><td>$-$</td><td>$\searrow$</td></tr>
</table>
$\therefore \left(0,\ 5\right)$ is a point of inflexion.</td></tr>
<tr><td>16.</td><td>Sketching $y=\mathrm{f}'(x)$ given $y=\mathrm{f}(x)$ with horizontal/oblique asymptotes</td><td><u>Lecture Notes Example 26</u>


Solution

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Techniques of Differentiation

1. AJC13/C1Mid-year/Q1

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following, simplifying your answers.

(i) $y = \sin^{-1}(\mathrm{e}^{-\sqrt{x}})$ [3]

(ii) $y = \ln\left(\frac{1+\ln x}{x^x}\right)$ [4]

2. DHS13/ C1Mid-year/Q8

Differentiate the following with respect to x,

(a) $\sin^2 3x$ expressing your answer in terms of a single trigonometric function, [2]

(b) $\ln\left[(\sin x)(\cos^{-1} x)\right]$, [3]

(c) $x^{\log_3 x}$. [4]

3. MJC13/C1Mid-year/Q7

(a) Given $y = \ln\frac{\mathrm{e}^{x^2 y}}{x^2+1}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [4]

(b) A curve has parametric equations $x = 3u^2 - u$, $y = \tan^{-1} u$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of u. Hence find the range of values of u for which the curve is strictly increasing. [4]

4 PJC13/C1Mid-year/Q2

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}[\sec(\ln(3x^2-6))]$. [2]

(b) Given that $\cos^{-1}\sqrt{1-x^2} = \mathrm{e}^{-2\pi} + x + 2xy^2$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y, simplifying your answers. [4]

5. RI13/C1Mid-year/Q7

(a) A curve is given by the parametric equations

$$x = 2\theta - \sin 2\theta, \quad y = 3 - 2\cos^2\theta, \quad \text{for } 0 < \theta < \pi .$$

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \cot\theta$. [3]

(b) Given that $\mathrm{e}^{x^2} y = \pi^x \ln\pi$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y[(\ln\pi) - 2x]$. [4]

6. VJC13/C1Mid-Year/Q3

(a) Given that $y = \sin^{-1}(x^2)$ and $-1 < x < 1$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Deduce the set of values of x for which y decreases as x increases. [4]

(b) Given that $y = \ln\left[\tan(x+y)\right]$, show that $\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} = \left(1+\mathrm{e}^{2y}\right)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$. [3]

7. TJC20/MYA/Q5

(a) Given that $\ln y = x\ln\left[\mathrm{f}(x)\right]$, where $\mathrm{f}(x) > 0$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy\left[\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\right] + y\ln\left[\mathrm{f}(x)\right].$$

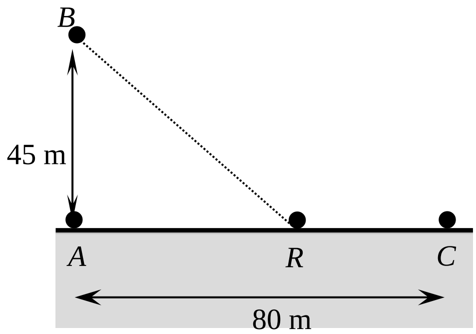
Hence find the derivative of $(1+2x)^x$ with respect to x. [3]

(b) **(i)** Given that $y = \tan^{-1}(x+y)$, show that $(x+y)^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 1$. [1]

(ii) Find the values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at the point $(1-\frac{\pi}{4}, \frac{\pi}{4})$. [4]

Maxima and Minima Problems

8. AJC14/C2Mid-year/Q11



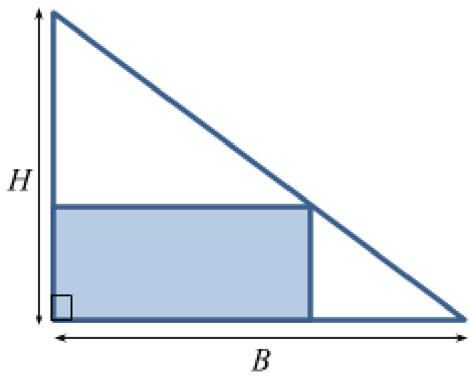
In the above diagram, not drawn to scale, John is in a canoe at point B which is 45 m from A, the nearest point on a straight shore AC. As part of an ironman competition, he needs to disembark from his canoe at point R and must then run along the shore to point C which is 80 m from A in order to complete the race. His rowing and running speeds are 2 m/s and 5 m/s respectively.

(i) If x denotes the distance, in metres, between A and R, and t denotes the time, in seconds, required to travel from B to C, show that $t = \frac{\sqrt{2025+x^2}}{2} + \frac{80-x}{5}$. [1]
(Assume that the time taken to disembark from the boat is negligible)

(ii) Find, by differentiation, the exact distance from A that John should disembark from his canoe for him to complete his race in the shortest time. [3]
(You do not need to justify that the time taken is the shortest.)

(iii) Another competitor, Alex managed to disembark from his canoe at point A when John was still at point B. The running speed of Alex is 1.5 m/s. Determine the range of values of x such that the difference in the time taken for Alex and John to reach point R is at most 5 seconds. [3]

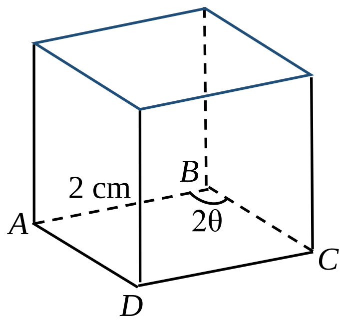
9. MJC14/C2Mid-year/Q3



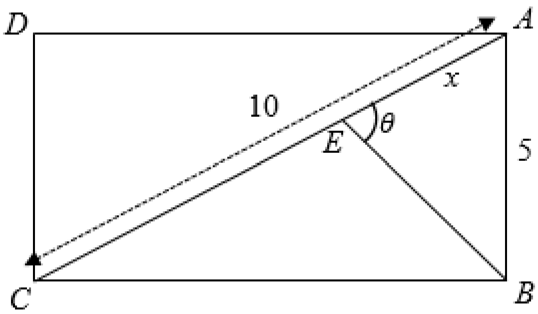
(i) A right-angled triangle has fixed base length B cm and fixed height H cm. A rectangle is inscribed in the triangle as shown in the diagram above.
Using differentiation, find the dimensions of the rectangle such that its area is at its maximum. [6]

(ii) Hence what can be said about the two unshaded triangles when the area of the inscribed rectangle is at its maximum? [1]

10. SRJC16/C2 MYE/II/4



(a) The base of a solid prism is a rhombus *ABCD*, where *AB* = 2 cm and $\angle ABC = 2\theta$ where $0 < \theta < \frac{\pi}{2}$, as shown in the figure. Given that the volume of the solid is 100 cm^3, find the value of θ when the total surface area of the prism is minimum. [6]

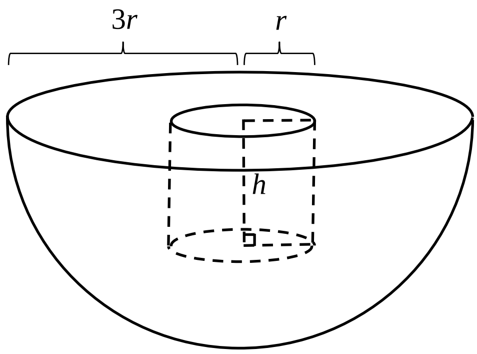


(b) *ABCD* is a rectangle where *AB* = 5 cm and *AC* = 10 cm. The point *E* is on *AC* where *AE* = *x* cm and angle *AEB* = θ radians. *E* is moving from *A* to *C* at the rate of 0.1 cm /s. Calculate, when *x* = 5, the rate at which the length of *BE* is changing. [3]

11. TJC16/C2 MYE/4

[It is given that the volume of a sphere of radius *r* is $\frac{4}{3}\pi r^3$ and the surface area of a sphere of radius *r* is $4\pi r^2$.]

A paperweight is constructed from a hemisphere with radius $3r$ cm by removing a circular cylinder of radius r cm and height h cm as shown in the diagram below. The cost of anodizing the flat surfaces is $\$k$ per cm^2 while the cost of anodizing the curved surfaces is $\$2k$ per cm^2. As *r* and *h* vary, the total cost of anodizing the entire paperweight is a constant denoted by \$C.



(i) Show that the volume, V cm^3, of the paperweight is $V = \dfrac{117}{4}\pi r^3 - \dfrac{Cr}{4k}$. [3]

(ii) Find, in terms of C, as r varies, the cost of anodizing the flat surfaces when V is a minimum. [5]

12. TJC2017/Mid-year/7

(a) **(i)** Differentiate $\ln\left[\tan^{-1}(2x)\right]$ with respect to x. [2]

(ii) Given that $y = \operatorname{cosec} x \cot x$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\left(\operatorname{cosec}^3 x + y\cot x\right)$. [2]

(b) Given that $\mathrm{f}(x) = \dfrac{x}{2} - 4\ln x + \dfrac{15}{4}\ln(2x+1)$, show that $\mathrm{f}'(x) = \dfrac{x^2-4}{x(2x+1)}$. [2]

Hence show that the graph of $y = \mathrm{f}(x)$ has exactly one stationary point. [2]

Determine the nature of this stationary point without the use of a graphing calculator. [2]

13. TJC18/C1BT/5

A food company manufactures cans of instant soup. Each cylindrical can has base radius r cm and height h cm. The cans are made of thin metal sheets of negligible thickness. The production cost of the curved surface and the flat surface of a can is 3 cents and k cents per cm^2 respectively.

(i) Given that each can has a fixed capacity of V cm^3, show that the cost C, in cents, of producing each can is $\dfrac{6V}{r} + 2\pi k r^2$. [3]

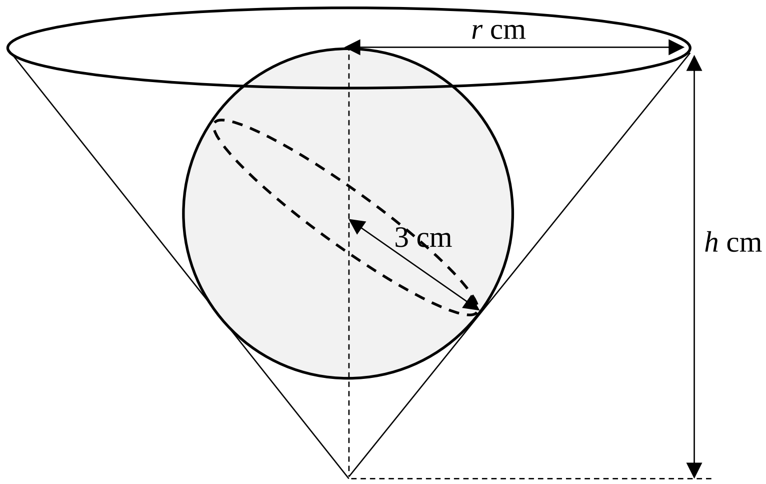
(ii) Using differentiation, show that $\dfrac{r}{h} = \dfrac{3}{2k}$ when the production cost of each can is minimised. [5]

The company intends to produce a bigger can with twice the capacity and the costs of the curved surface and flat surface of the bigger can are maintained at 3 cents and k cents per cm^2 respectively. A worker suggests that the ratio of base radius to height of the bigger can should be twice that of the smaller can in order to minimise the production cost.

(iii) Explain whether the worker is correct. [1]

14. PJC18/C1BT/7

[It is given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



A restaurant serves ice-cream in the shape of a sphere of radius 3 cm. The restaurant wants to make an open right conical-shaped container made of plastic of negligible thickness to contain the ice-cream. The container has radius r cm and vertical height h cm as shown in the diagram above. As r and h vary, the ice-cream is always in contact with the curved surface of the container and the highest point of the ice-cream is at the same level as the rim of the container.

Show that $r = \dfrac{3h}{\sqrt{h^2 - 6h}}$. Hence find, using differentiation, the minimum volume of the open conical-shaped container. [8]

Tangents & Normals

15. VJC14/C2Mid-year/Q9 (modified)

The curve C has parametric equations

$$x = t^2 - t, \quad y = t^2 + 4t + 8, \quad \text{for } t \in \mathbb{R}.$$

(i) Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ in terms of t. Hence find the coordinates of the minimum point on C. [You do not need to show that the stationary point is indeed a minimum point.] At the point A, the tangent to C is a vertical line. State the distance of this tangent from the y-axis. [4]

(ii) Sketch C, indicating the coordinates of the turning point and the intersections with the axes. [2]

It is given that the point P on C has parameter p.

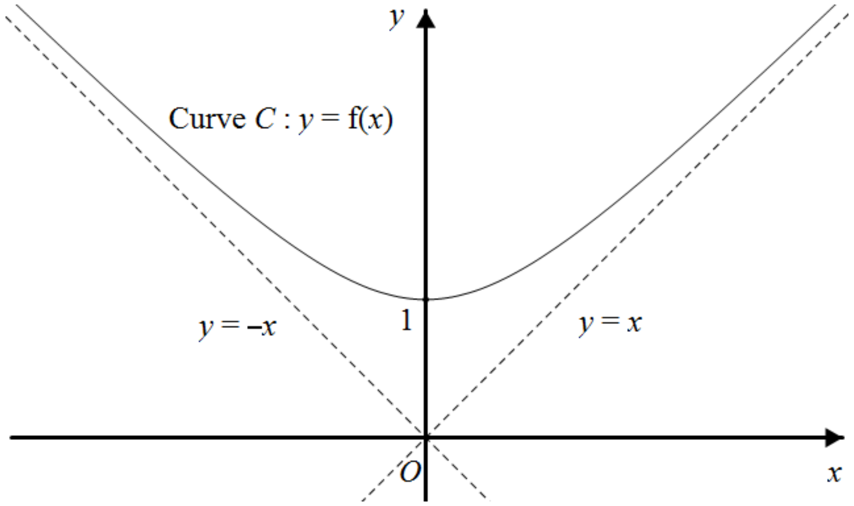
(iii) Show that the equation of the tangent at P is

$$(2p-1)y - (2p-1)\left(p^2 + 4p + 8\right) = (2p+4)\left(x - p^2 + p\right). \quad [1]$$

(iv) It is further given that the tangent at P passes through the origin. Find the possible coordinates of P, correct to 3 decimal places. [3]

16. CJC16/Prelim/II/3 (modified)

The diagram shows the graph of curve C represented by $y = \mathrm{f}(x)$, with oblique asymptotes $y = x$ and $y = -x$.



(a) On a separate diagram, sketch a graph of $y = \mathrm{f}'(x)$, clearly indicating the equation(s) of the asymptote(s) and axial-intercept(s). [2]

(b) The above curve C is represented by the parametric equations

$$x = \tan\theta, \quad y = \sec\theta, \text{ where } -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}.$$

(i) Show that the normal to the curve at point P, with coordinates $(\tan\theta,\ \sec\theta)$, for $0<\theta<\frac{\pi}{2}$, is given by $y=-x\operatorname{cosec}\theta+2\sec\theta$. [2]

(ii) The normal to the curve at point P intersects the x-axis at point N. Find the coordinates of the mid-point M of PN, in terms of θ. [2]

(iii) Taking O as the origin, show that the area of triangle OPN is $\tan\theta\sec\theta$. Point P moves along the curve such that the rate of change of its parameter θ with respect to time t is given by $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\cos\theta$. Find the exact rate of change of the area of triangle OPN when $\theta=\frac{\pi}{6}$. [4]

17. HCI 16/Prelim/I/7

The curve C has equation $y=\dfrac{x-2}{kx^2+x-2}$, where $k>1$.

(i) Find the equation of the tangent at the point A where C cuts the y-axis. [2]

(ii) Sketch C, giving the equations of asymptotes, the coordinates of turning points and axial intercepts in terms of k, if any. [4]

(iii) Find the equation of the normal at the point B where C cuts the x-axis. Leave your answer in terms of k. [2]

(iv) Hence show that the value of the area bounded by the tangent at A, the normal at B and both the x- and y-axes is more than $\frac{15}{8}$ square units. [2]

18. RI 16/Prelim/I/9

A curve C has parametric equations

$$x=t^2,\quad y=1+2t \quad \text{for } t>0.$$

(i) Sketch C. [2]

(ii) Find the equations of the tangent and the normal to C at the point $P(p^2,1+2p)$. [4]

The tangent and normal at P meet the y-axis at T and N respectively. Show that $\dfrac{PT^2}{TN}=p$. [4]

19. VJC 18/BT/8

A rocket is fired from the origin O. The path of the rocket, C, can be modelled by the parametric equations

$$x=\theta-\sin\theta,\ y=1-\cos\theta,$$

where $0\le\theta\le 2\pi$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\cot\frac{1}{2}\theta$ and find the equation of the tangent to C at the point where $\theta=\pi$. What can be said about the tangents of C as $\theta\to 0$ and $\theta\to 2\pi$? [5]

(ii) Sketch C, showing clearly the features of the rocket's path at the points where $\theta=0,\ \pi$ and 2π. [3]

An anti-rocket missile is launched from the point $\left(\frac{3\pi}{2},0\right)$ to intercept the rocket. This anti-rocket missile is programmed to follow a linear path to intercept the rocket in mid-air when its path is perpendicular to C.

(iii) Find the coordinates of the point where the missile intercepts the rocket. [3]

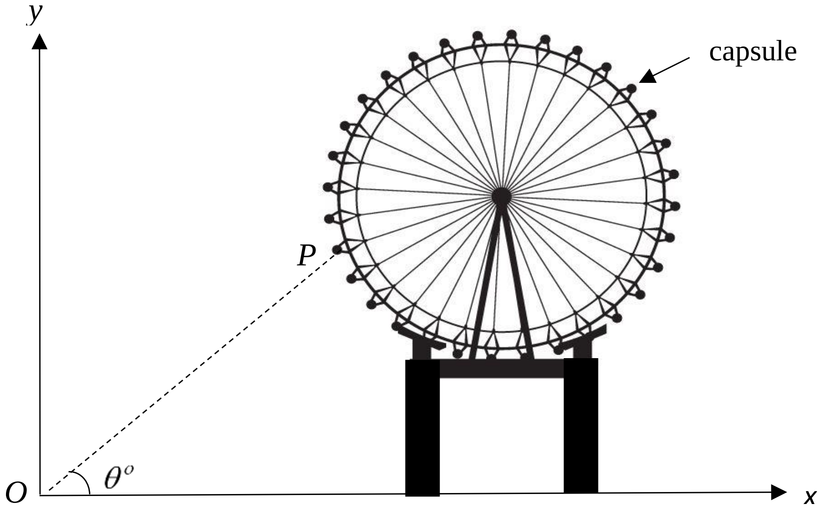
(iv) Find the equation that describes the path of the missile. Indicate the appropriate range of values of x for this equation. [2]

20. DHS 18/BT/9

The diagram below shows a ferris wheel and an observer positioned at origin *O*. A passenger *P* gets on a capsule and the path travelled by *P* can be modelled by a curve *C* with parametric equations

$$x = 75\sin(\tfrac{1}{15}\pi t) + 120, \quad y = -75\cos(\tfrac{1}{15}\pi t) + 165, \quad \text{for } 0 \le t \le 30,$$

where *x* and *y* refer to the horizontal and vertical displacement of *P* with respect to *O* respectively, and *t* refers to the elapsed time (in minutes) from his initial position.
You may assume the capsule to be of negligible size.



Ferris Wheel

(a) (i) Using differentiation, find the equation of the tangent to *C* at *P* where $t = p$, in terms of π. [4]

(ii) The angle of elevation, θ° for the observer to view *P* is the angle that the line *OP* makes with the positive *x*-axis. Using the tangent found earlier, or otherwise, find the largest value of θ. [4]

(b) An object is projected from *O* at $t = 0$ and its motion can be modelled with the equation $y = -\frac{1}{320}(x-200)^2 + 125$, where *x* and *y* refer to the horizontal and vertical displacement of the object with respect to *O*.

(i) Find the coordinates of the points where the object crosses *C*. [4]

(ii) The horizontal displacement *x* of the object at time *t* is given by $x = 8t$. Explain whether the object will hit *P*. [2]

Rates of Change

21. ACJC14/C2Mid-year/Q3

A circular cylinder is inscribed in a sphere with radius 26 cm so that all points on the circumference of the two circular ends are on the surface of the sphere at all times. Show that the relationship between the radius *r* cm and the height *h cm* of the cylinder is given by $4r^2 + h^2 = 2704$. [1]

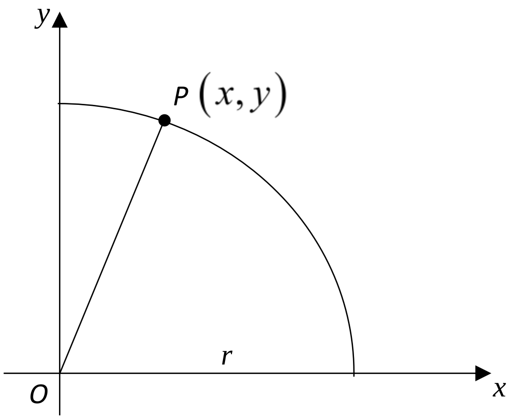
At a certain instant, the radius of the cylinder is 24 cm and is decreasing at the rate of 0.5 cm s^{-1}.

(i) Find the rate at which the height is changing at that instant. [3]

(ii) Find the value of the radius of the cylinder when its curved surface area is a maximum. [4]

[You need not establish that the resulting value of the area is a maximum.]

22. NYJC 16/Prelim/I/5

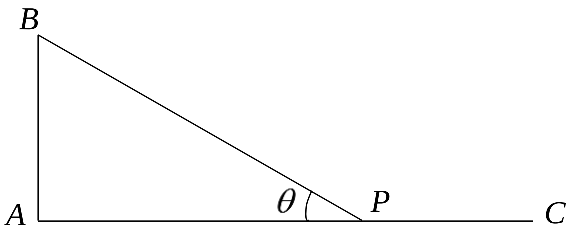


A particle P moves along the curve with equation $x^2 + y^2 = r^2$, where $x \geq 0, y \geq 0,$ and r is a constant. By letting $m = \tan\left(\sin^{-1} \frac{y}{r}\right)$, find an expression for $\frac{\mathrm{d}m}{\mathrm{d}y}$ in terms of y and r. Given that the rate of change of y with respect to time t is 0.1% of r, show that

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \left(\frac{r}{10\sqrt{r^2 - y^2}}\right)^3.$$

State the geometrical meaning of $\frac{\mathrm{d}m}{\mathrm{d}t}$. [7]

23. TJC 16/Prelim/I/5



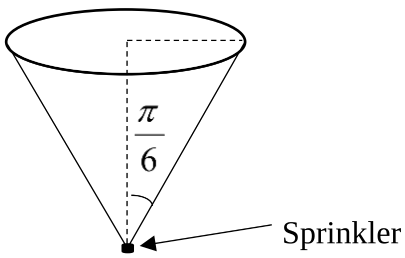
In the diagram, A and C are fixed points 500 m apart on horizontal ground. Initially, a drone is at point A and an observer is standing at point C. The drone starts to ascend vertically at a steady rate of $3\,\mathrm{m\,s^{-1}}$ as the observer starts to walk towards A with a steady speed of $4\,\mathrm{ms^{-1}}$. At time t, the drone is at point B and the observer is at point P. Given that the angle APB is θ radians, show that $\theta = \tan^{-1}\left(\frac{3t}{500-4t}\right)$. [2]

(i) Find $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ in terms of t. [2]

(ii) Using differentiation, find the time t when the rate of change of θ is maximum.[4]

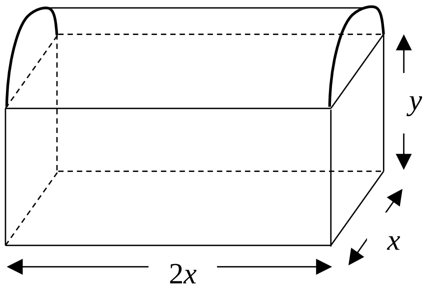
24. HCI 17/BT1/8

(a) [It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



A conical water tank of semi-vertical angle $\frac{\pi}{6}$, which is held with its axis vertical and vertex downwards (see diagram above), is installed at the centre of the roof of a greenhouse. It has a sprinkler attached to the vertex where it is programmed to spray water once a day.
At the start of each day, the water tank will be filled to the brim with $5000\pi\ \text{cm}^3$ of water. When the sprinkler is in operation, water runs through the sprinkler at a constant rate of $0.9\pi\ \text{cm}^3\text{s}^{-1}$. Find the rate at which the radius of the water surface is decreasing 25 minutes after the sprinkler is in operation, leaving your answer correct to 4 significant figures.
[You may assume that the volume of the sprinkler is negligible.] [5]

(b)



A greenhouse, as shown in the diagram above, is made up of three parts.

- The roof is modelled by the curved surface of a semi-circular prism of diameter x m and length $2x$ m as well as 2 semi-circles of diameter x m at the two ends.
- The walls are modelled by the lateral surface of a cuboid of length $2x$ m, breadth x m and height y m.
- The floor is modelled by a rectangular surface of length $2x$ m and breadth x m.

It is known that the costs of constructing the walls, the flooring and the roof of the greenhouse is $\$k$ per m^2, $\$0.5k$ per m^2 and $\$4k$ per m^2 respectively.

(i) Given that the volume of the greenhouse is a fixed value V m^3, show that the cost C of building the greenhouse is given by

$$C = \frac{3kV}{x} + \frac{17}{4}k\pi x^2 + kx^2.$$ [3]

(ii) Using differentiation, find the exact value of x in terms of V such that the cost of building the greenhouse is a minimum. [4]

Answers

1. **(i)** $\frac{dy}{dx} = \frac{-e^{-\sqrt{x}}}{2\sqrt{x}\sqrt{1-e^{-2\sqrt{x}}}}$ **(ii)** $\frac{dy}{dx} = \frac{1}{x(1+\ln x)} - 1 - \ln x$

2. **(a)** $3\sin 6x$ **(b)** $\cot x - \frac{1}{(\cos^{-1} x)\sqrt{1-x^2}}$ **(c)** $\frac{2}{\ln 3} x^{(\log_3 x)-1} \ln x$

3. **(a)** $\frac{dy}{dx} = \frac{2x^3y + 2xy - 2x}{1-x^4}$ or $\frac{2x(x^2y + y - 1)}{1-x^4}$

(b) $\frac{dy}{dx} = \frac{1}{(1+u^2)(6u-1)}$; $u > \frac{1}{6}$

4. **(a)** $\sec(\ln(3x^2-6))\tan(\ln(3x^2-6))\frac{2x}{x^2-2}$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{4xy}\left\{\dfrac{x}{|x|\sqrt{(1-x^2)}}-1-2y^2\right\}$

6. **(a)** $\{\, x\in\mathbb{R}: -1<x<0 \,\}$

7. **(a)** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=xy\left(\dfrac{2}{1+2x}\right)+(1+2x)^x\ln(1+2x)$ **(b)(ii)** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1$; $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-4$

8. **(ii)** Exact distance from A to R = $\dfrac{90}{\sqrt{21}}$ **(iii)** $35.5\le x\le 69.8$.

9. **(i)** $x=\dfrac{B}{2}$ $y=\dfrac{H}{2}$ **(ii)** The two unshaded triangles are congruent

10. **(a)** $\theta=\dfrac{\pi}{4}$, **(b)** $\dfrac{1}{20}$cm/s

11. **(ii)** $\dfrac{C}{39}$

12. **(a)(i)** $\dfrac{2}{\left(1+4x^2\right)\left[\tan^{-1}\left(2x\right)\right]}$; **(b)** minimum point

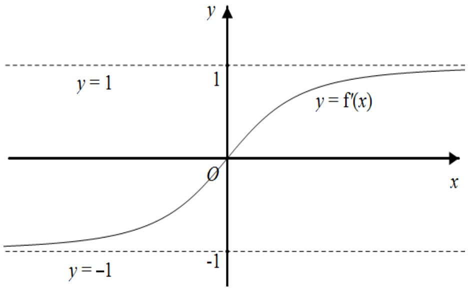
13. **(ii)** C is minimum when $\dfrac{r}{h}=\dfrac{3}{2k}$.

(iii) Since the costs of producing the curved and flat surfaces remain unchanged, from (ii), the ratio $\dfrac{r}{h}=\dfrac{3}{2k}$ is independent of the volume of the can. Hence the worker's suggestion is incorrect.

14. $V=\dfrac{3\pi h^2}{h-6}$; $h=12$; 226 (3s.f.) or 72π

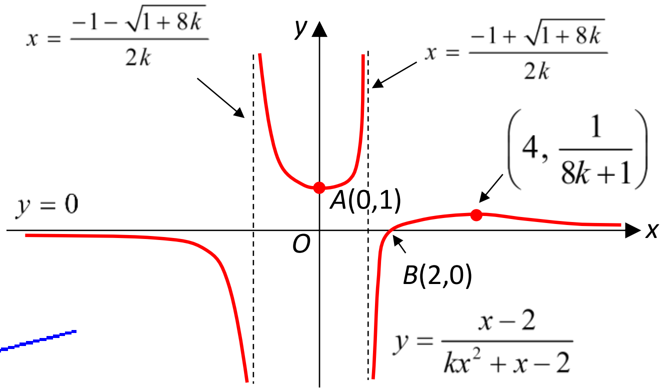
15. **(i)** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2t+4}{2t-1}$, $(6,\ 4)$, $\dfrac{1}{4}$ unit **(iv)** $(-0.246, 9.952)$ and $(16.886, 6.688)$

16. **(a)**



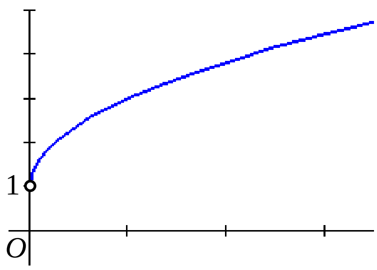
(b)(ii) $\left(\tfrac{3}{2}\tan\theta,\ \tfrac{1}{2}\sec\theta\right)$ **(iii)** $\dfrac{\mathrm{d}A}{\mathrm{d}t}=\dfrac{5}{3}$

17. **(i)** $y=1$; **(ii)**



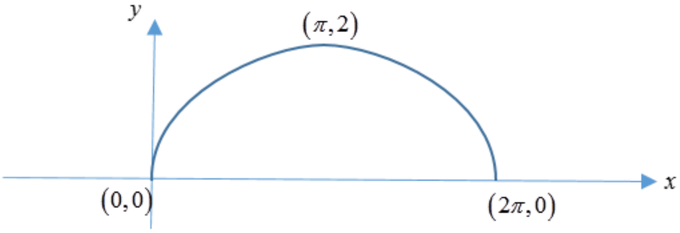
(iii) $y=-4kx+8k$

18.



(ii) $y = \frac{1}{p}x + 1 + p$; $y = -px + p^3 + 1 + 2p$

19. **(i)** y = 2 **(ii)**



(iii) Coordinate of interception: $(\frac{3\pi}{2} + 1, 1)$

(iv) $y = x - \dfrac{3\pi}{2}$ (or $y = 1.00x - 4.71$)

20. **(a)(i)** $y = \tan\left(\dfrac{\pi p}{15}\right)\left(x - \left(75\sin\left(\dfrac{\pi p}{15}\right) + 120\right)\right) - 75\cos\left(\dfrac{\pi p}{15}\right) + 165$ **(ii)** 75.5° (1 d.p.)

(b)(i) (98.9, 93.0) and (183, 124) **(ii)** Will not hit P

21. **(i)** $\dfrac{dh}{dt} = 2.4\text{cms}^{-1}$ **(ii)** $r = \sqrt{338}$ cm

22. $\dfrac{r^2}{\left(r^2 - y^2\right)^{\frac{3}{2}}}$; $\dfrac{dm}{dt}$ is the rate of change of the gradient of the line OP

23. **(i)** $\dfrac{60}{t^2 - 160t + 10000}$ *or* $\dfrac{1500}{9t^2 + (500 - 4t)^2}$ **(ii)** 80

24. **(a)** $0.001520 \text{ cm s}^{-1}$; **(b)(ii)** $x = \sqrt[3]{\dfrac{6V}{17\pi + 4}}$

6A

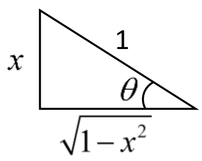
Techniques of Integration

Skill Set

s/n	Skills	Examples of questions involving the skills
1.	Recognise the form $[\mathrm{f}(x)]^n\ \mathrm{f}'(x)$ where $n \neq -1$, and use the standard result $\int [\mathrm{f}(x)]^n\ \mathrm{f}'(x)\,\mathrm{d}x = \frac{1}{(n+1)}[\mathrm{f}(x)]^{n+1} + C$ to evaluate the integral.	• Direct application of standard result Lecture notes Example 1(ii): $\int \frac{(\ln x-1)}{x}\,\mathrm{d}x = \int (\ln x-1)^1\left(\frac{1}{x}\right)\mathrm{d}x = \frac{1}{2}(\ln x-1)^2 + C$ • Rewrite function in $[\mathrm{f}(x)]^n\ \mathrm{f}'(x)$ form Tutorial 6A Q1(a)(ii): $\int x(x^2+1)^3\,\mathrm{d}x = \frac{1}{2}\int 2x(x^2+1)^3\,\mathrm{d}x = \frac{1}{8}(x^2+1)^4 + C$ Lecture notes Example 1(iii): $\int \frac{t^2}{\sqrt{2t^3-1}}\,\mathrm{d}t = \frac{1}{6}\int (2t^3-1)^{-\frac{1}{2}}(6t^2)\,\mathrm{d}t$ $= \frac{1}{6}\frac{(2t^3-1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$ $= \frac{1}{3}(2t^3-1)^{\frac{1}{2}} + C$
2.	Recognise the form $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$ and use the standard result $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln\lvert \mathrm{f}(x)\rvert + C$ to evaluate the integral.	• Direct application of standard result Lecture notes Example 2(iii): $\int \frac{1}{t\ln t}\,\mathrm{d}t = \int \frac{\frac{1}{t}}{\ln t}\,\mathrm{d}t = \ln\lvert\ln t\rvert + C$ • Rewrite function in $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$ form Lecture notes Example 2(i): $\int \frac{1}{1-3x}\,\mathrm{d}x = -\frac{1}{3}\int \frac{-3}{1-3x}\,\mathrm{d}x = -\frac{1}{3}\ln\lvert 1-3x\rvert + C$ Lecture notes Example 4(i): $\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x$ $= -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x$ $= -\ln\lvert\cos x\rvert + C$ Tutorial 6A Q3(j):

s/n	Skills	Examples of questions involving the skills
		$\int \frac{1}{2+\mathrm{e}^{-x}}\,\mathrm{d}x = \int \frac{1}{2+\frac{1}{\mathrm{e}^x}}\,\mathrm{d}x$ $= \int \frac{\mathrm{e}^x}{2\mathrm{e}^x+1}\,\mathrm{d}x$ $= \frac{1}{2}\int \frac{2\mathrm{e}^x}{2\mathrm{e}^x+1}\,\mathrm{d}x$ $= \frac{1}{2}\ln(2\mathrm{e}^x+1)+C$
3.	Recognise the form $\mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$ and $\mathrm{f}'(x)a^{\mathrm{f}(x)}$, and use the standard results $\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}\,\mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)}+C$ and $\int \mathrm{f}'(x)a^{\mathrm{f}(x)}\,\mathrm{d}x = \frac{1}{\ln a}a^{\mathrm{f}(x)}+C$ respectively to evaluate the integral.	• Rewrite function in $\mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$ or $\mathrm{f}'(x)a^{\mathrm{f}(x)}$ form Lecture notes Example 5(ii): $\int x\mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{1}{-2}\int(-2x)\mathrm{e}^{-x^2}\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x^2}+C$ Lecture notes Example 5(iii): $\int 3^{1-x}\,\mathrm{d}x = -\int(-1)3^{1-x}\,\mathrm{d}x = -\frac{1}{\ln 3}3^{1-x}+C$
4.	Use trigonometry identities.	• Sine and Cosine double-angle formulae Tutorial 6A Q1(d)(i): $\int \sin^2 x\,\mathrm{d}x = \int \frac{1-\cos 2x}{2}\,\mathrm{d}x = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$ Tutorial 6A Q1(d)(ii): $\int 3\cos 3x \sin 3x\,\mathrm{d}x = \frac{3}{2}\int \sin 6x\,\mathrm{d}x = -\frac{3}{2}\left(\frac{\cos 6x}{6}\right)+C$ $= -\frac{1}{4}\cos 6x + C$ Further example: $\int \frac{1}{1+\cos x}\,\mathrm{d}x = \int \frac{1}{2\cos^2\frac{x}{2}}\,\mathrm{d}x = \frac{1}{2}\int \sec^2\frac{x}{2}\,\mathrm{d}x$ $= \tan\frac{x}{2}+C$ • Sum-to-product (or factor) formulae Tutorial 6A Q1(d)(iii): $\int 7\sin 4x\cos 3x\,\mathrm{d}x = \frac{7}{2}\int 2\sin 4x \cos 3x\,\mathrm{d}x$ $= \frac{7}{2}\int \sin 7x + \sin x\,\mathrm{d}x$ $= -\frac{1}{2}\cos 7x - \frac{7}{2}\cos x + C$ • Identity $1+\tan^2 x = \sec^2 x$ Example: $\int \tan^2 2\theta\,\mathrm{d}\theta = \int(\sec^2 2\theta - 1)\,\mathrm{d}\theta$ $= \frac{1}{2}\tan 2\theta - \theta + C$

s/n	Skills	Examples of questions involving the skills
5.	Simplify improper rational function by performing long division first before attempting integration.	Tutorial 6A Q3(b): $\int \frac{x^3+x^2-x+5}{x^2+x-2}\mathrm{d}x = \int x + \frac{x+5}{x^2+x-2}\mathrm{d}x$
6.	Use partial fractions to simplify proper rational functions with linear factors in the denominator.	Tutorial 6A Q3(b): $\int \frac{x^3+x^2-x+5}{x^2+x-2}\mathrm{d}x = \int x + \frac{x+5}{x^2+x-2}\mathrm{d}x$ $= \int x + \frac{x+5}{(x+2)(x-1)}\mathrm{d}x$ $= \int x + \frac{-1}{x+2} + \frac{2}{x-1}\mathrm{d}x$
7.	For rational functions of the form $\frac{1}{px^2+qx+r}$, $p<0$ or $p>0$; and $\frac{1}{\sqrt{px^2+qx+r}}$, $p<0$; complete the square for the quadratic expression in the denominator before applying standard integration results found in MF26.	Tutorial 6A Q3(d): $\int \frac{1}{2y^2+4y+5}\mathrm{d}y = \frac{1}{2}\int \frac{1}{y^2+2y+\frac{5}{2}}\mathrm{d}y$ $= \frac{1}{2}\int \frac{1}{(y+1)^2+\frac{3}{2}}\mathrm{d}y$ Tutorial 6A Q3(f): $\int \frac{1}{\sqrt{3-t^2+2t}}\mathrm{d}t = \int \frac{1}{\sqrt{2^2-(t-1)^2}}\mathrm{d}t$
8.	For rational functions of the form $\frac{sx+t}{px^2+qx+r}$ [but not $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$], rewrite the function as $\frac{sx+t}{px^2+qx+r} = A\cdot\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(px^2+qx+r\right)}{px^2+qx+r} + B\cdot\frac{1}{px^2+qx+r}$ before applying standard integration results.	Lecture notes Example 14: $\int \frac{x-1}{x^2+x+1}\mathrm{d}x = \int \frac{\frac{1}{2}(2x+1)}{x^2+x+1} - \frac{\frac{3}{2}}{x^2+x+1}\mathrm{d}x$
9.	For integration involving the use of a substitution, change the original variable to the new variable using differentiation and direct substitution before integration. Remember to express final answer in terms of the original variable.	Lecture notes Example 16: $u = 3x+1$ $\frac{\mathrm{d}u}{\mathrm{d}x} = 3$ $\int \frac{x}{(3x+1)^3}\,\mathrm{d}x = \int \frac{\frac{1}{3}(u-1)}{u^3}\left(\frac{1}{3}\mathrm{d}u\right)$ $= \frac{1}{9}\int u^{-2} - u^{-3}\mathrm{d}u$ $= \frac{1}{9}\left(-\frac{1}{u} + \frac{1}{2u^2}\right) + C$ $= \frac{1}{9}\left(-\frac{1}{3x+1} + \frac{1}{2(3x+1)^2}\right) + C$

s/n	Skills	Examples of questions involving the skills
10.	For substitution involving trigonometry, draw a right angled triangle to obtain other simple trigonometric functions sine, cosine or tangent. This is useful in expressing the answer in terms of the original variable.	Lecture notes Example 17: $x=\sin\theta$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\cos\theta$ $\therefore\ \int\sqrt{1-x^2}\,\mathrm{d}x$ $=\dfrac{1}{2}\int\cos 2\theta+1\,\mathrm{d}\theta$ $=\dfrac{1}{4}\sin 2\theta+\dfrac{1}{2}\theta+C$ $=\dfrac{1}{2}\sin\theta\cos\theta+\dfrac{1}{2}\theta+C$ $=\dfrac{1}{2}x\sqrt{1-x^2}+\dfrac{1}{2}\sin^{-1}x+C$ 
11.	For integration of definite integrals by substitution, change the limits from values of the original variable to the corresponding values of the new variable.	Tutorial 6A Q4(c): $x=2\sin\theta$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=2\cos\theta$ When $x=1,\ 2\sin\theta=1\Rightarrow\theta=\dfrac{\pi}{6}$ When $x=\sqrt{3},\ 2\sin\theta=\sqrt{3}\Rightarrow\theta=\dfrac{\pi}{3}$ $\int_1^{\sqrt{3}}\dfrac{x}{\sqrt{4-x^2}}\,\mathrm{d}x=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\dfrac{2\sin\theta}{\sqrt{4-4\sin^2\theta}}2\cos\theta\,\mathrm{d}\theta$
12.	For integration of a single function that cannot be integrated directly, let $v'=1$ and apply integration by parts.	Examples include: $\int 1\cdot\tan^{-1}x\,\mathrm{d}x=x\tan^{-1}x-\int\dfrac{x}{1+x^2}\,\mathrm{d}x$ $\int 1\cdot\cos^{-1}x\,\mathrm{d}x=x\cos^{-1}x-\int x\left(-\dfrac{1}{\sqrt{1-x^2}}\right)\mathrm{d}x$ $\int 1\cdot\sin^{-1}x\,\mathrm{d}x=x\sin^{-1}x-\int x\left(\dfrac{1}{\sqrt{1-x^2}}\right)\mathrm{d}x$ $\int 1\cdot\ln x\,\mathrm{d}x=x\ln x-\int x\left(\dfrac{1}{x}\right)\mathrm{d}x$
13.	The same integral reappearing on the RHS after integration by parts	Tutorial 6A Q3(g): $\int e^x\sin x\,\mathrm{d}x$ $=e^x\sin x-\int e^x\cos x\,\mathrm{d}x$ $=e^x\sin x-(e^x\cos x+\int e^x\sin x\,\mathrm{d}x)$ $\therefore\ 2\int e^x\sin x\,\mathrm{d}x=e^x\sin x-e^x\cos x+C'$ $\int e^x\sin x\,\mathrm{d}x=\dfrac{1}{2}e^x(\sin x-\cos x)+C$

1. **TJC14/C1Mid-year/Q9**

(a) Find $\int (\cos 2x \sin 7x)\, \mathrm{d}x$. [2]

(b) Find $\int \frac{4x-6}{x^2+6x}\, \mathrm{d}x$. [4]

(c) Using integration by parts, find $\int \mathrm{e}^x \tan^{-1}(\mathrm{e}^x)\, \mathrm{d}x$. [3]

2. **AJC14/C2Mid-yearP1/Q1**

(a) Find $\int \sqrt{\operatorname{cosec} 2x - \sin 2x}\, \mathrm{d}x$. [3]

(b) Find $\int \ln(4+x^2)\, \mathrm{d}x$. [3]

3. **DHS16/C1MYE/2**

(a) Find $\int x\left[(1-3x^2)^5 + \mathrm{e}^{x^2+1}\right]\mathrm{d}x$. [3]

(b) Find $\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}(x^2)$. Hence, or otherwise, find the exact value of

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\, \mathrm{d}x.$$ [3]

4. **SAJC2021/C2BT1/2**

(a) **(i)** Differentiate e^{x^2} with respect to x. [1]

(ii) Hence, find the exact value of $\int_0^2 x^3 \mathrm{e}^{x^2}\, \mathrm{d}x$. [3]

(b) Find $\int \frac{x+1}{x^2-6x+13}\, \mathrm{d}x$. [3]

(c) **(i)** Find $\int \mathrm{e}^x \cos x\, \mathrm{d}x$. [3]

(ii) Hence find the possible values of $\int_0^{n\pi} \mathrm{e}^x \cos x\, \mathrm{d}x$, given that n is a positive integer. [3]

5. **MI16/PU2Promo/2**

Find $\int x\mathrm{e}^x\, \mathrm{d}x$. Hence find the exact value of $\int_{-2}^{1} |x|\, \mathrm{e}^x\, \mathrm{d}x$. [5]

6. **NYJC14/C2Mid-yearP1/Q2**

Given that p is a positive constant, find the exact value of k such that

$$\int_{\frac{1}{p}}^{\frac{\sqrt{3}}{p}} \frac{1}{p^2x^2+1}\, \mathrm{d}x = k\int_0^{\frac{3}{p}} |1-px|\, \mathrm{d}x.$$ [5]

7. **DHS17/C2BT1/1**

(a) Find $\int \sin\left(\frac{3}{2}x\right)\cos\left(\frac{1}{2}x\right)\ \mathrm{d}x$. [2]

(b) Using the substitution $y=\dfrac{1}{x}$, find the exact value of $\displaystyle\int_{2}^{\frac{4}{\sqrt{3}}}\frac{1}{x\sqrt{x^2-4}}\,\mathrm{d}x$. [4]

8. **SRJC14/C2Mid-yearP1/Q12**

(a) Use the substitution x = 2sec θ to find $\displaystyle\int\frac{1}{x^3\sqrt{x^2-4}}\,\mathrm{d}x$. [4]

(b) **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\left(\tan^{-1}x\right)^2\right]$. [1]

(ii) Find $\displaystyle\int\frac{x^3+x+1}{x^2+1}\,\mathrm{d}x$. [2]

(iii) Hence, find $\displaystyle\int\frac{\left(x^3+x+1\right)\tan^{-1}x}{x^2+1}\mathrm{d}x$. [4]

9. **PJC14/C2Mid-yearP2/Q1**

(a) By using substitution $x=a\cos\theta$, show that

$$\int\frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x=-\frac{x}{2}\left(\sqrt{a^2-x^2}\right)-\frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right)+C,$$

where C is an arbitrary constant. [5]

(b) **(i)** State the derivative of $\mathrm{e}^{\cos(x)}$. [1]

(ii) Find $\int\sin(2x)\mathrm{e}^{\cos x}\ \mathrm{d}x$. [4]

10. **RI17/C2BT2/3**

(i) Show that $u+\dfrac{1}{u}=2\sec x$, where $u=\sec x+\tan x$. [2]

(ii) Use the substitution $u=\sec x+\tan x$ to find the exact value of

$$\int_0^{\frac{\pi}{6}}\frac{\sec^2 x}{\left(\sec x+\tan x\right)^3}\ \mathrm{d}x.$$ [4]

11. **VJC17/C2CT1/1**

Determine the exact value of $\displaystyle\int_0^{\frac{\pi}{6}}x\cos 2x\ \mathrm{d}x$. Hence evaluate $\displaystyle\int_0^{\frac{\pi}{6}}x\sin^2 x\ \mathrm{d}x$, leaving your answer in exact form. [6]

Answers

1(a)	$-\frac{1}{2}\left(\frac{\cos 9x}{9}+\frac{\cos 5x}{5}\right)+C$	**1(b)**	$2\ln\left\|x^2+6x\right\|-3\ln\left\|\frac{x}{x+6}\right\|+C$ or $-\ln\left\|x\right\|+5\ln\left\|x+6\right\|+C$
1(c)	$e^x\tan^{-1}\left(e^x\right)-\frac{1}{2}\ln\left(e^{2x}+1\right)+C$		
2(a)	$\sqrt{\sin 2x}+C$	**2(b)**	$x\ln\left(4+x^2\right)-2x+4\tan^{-1}\frac{x}{2}+C$
3(a)	$-\frac{(1-3x^2)^6}{36}+\frac{1}{2}e^{x^2+1}+C$	**3(b)**	$\frac{\pi}{12}$
4 (i)	$2xe^{x^2}$	**5**	xe^x-e^x+c; $2-3e^{-2}$
6	$k=\frac{\pi}{30}$		
7(a)	$-\frac{\cos(2x)}{4}-\frac{\cos(x)}{2}+C$	**7(b)**	$\frac{\pi}{12}$
8(a)	$\frac{\sqrt{x^2-4}}{8x^2}+\frac{1}{16}\cos^{-1}\left(\frac{2}{x}\right)+C$	**8(bi)**	$\frac{2\tan^{-1}x}{1+x^2}$
8(bii)	$\frac{x^2}{2}+\tan^{-1}x+C$	**8(biii)**	$\frac{x^2}{2}\tan^{-1}x+\frac{1}{2}\left(\tan^{-1}x\right)^2-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+C$
9(bi)	$-\sin(x)e^{\cos(x)}$	**9(bii)**	$-2\left[\cos(x)e^{\cos x}-e^{\cos x}\right]+C$
10(ii)	$\frac{5}{18}$	**11**	$\frac{\sqrt{3}\pi}{24}-\frac{1}{8}$; $\frac{\pi^2}{144}-\frac{\sqrt{3}\pi}{48}+\frac{1}{16}$

HWA CHONG INSTITUTION
2021 JC1 BLOCK TEST H2 MATH
DURATION: 3 HOURS

1 It is given that

$$y = \frac{x^2 + 3x + 1}{x - 2}, \quad x \in \mathbb{R}, x \neq 2.$$

Using an algebraic method, find the set of values that y cannot take and leave your answer in exact form. [4]

2 A curve C has equation

$$\frac{xy - y^2}{(x+1)^2} = x, \text{ where } x \neq -1.$$

Find the equation of the tangent to C which is parallel to the y-axis. [5]

3 The curve C_1 has equation

$$y = ax + b + \frac{b}{ax + b},$$

where a is a negative constant and b is a positive constant.

(i) Find the x-coordinate(s) of the stationary point(s) of C_1 in terms of a and b. (You do not need to show the nature of the stationary point(s).) [2]

(ii) Given that $0 < b < 1$, show that $\sqrt{b} - b > 0$. [1]

(iii) Sketch C_1, where $0 < b < 1$, giving the equation(s) of any asymptote(s), coordinate(s) of any point(s) where C_1 crosses the axes and x-coordinate(s) of any turning point(s) in terms of a and b. [3]

4 A sequence $a_0,\ a_1,\ a_2,\ \ldots$ is such that $a_{n+1}=a_n+ka_{n-1}$, where k is a non-zero real constant and $n\in\mathbb{Z}^+$.

(i) Given that $a_0=2$, $a_1=7$ and $a_2=11$, find k . [2]

It is known that the nth term of this sequence is given by

$$a_n=A\left(2^n\right)+B\left(-1\right)^n+C\ ,$$

where A, B and C are constants.

(ii) Find A, B and C. [3]

(iii) Find $\sum_{r=1}^{n}a_r$ in terms of n. [3]

5 The function f is given by

$$\mathrm{f}\left(x\right)=\begin{cases}5\cos\left(\dfrac{\pi x}{6}-\dfrac{\pi}{2}\right)-2, & \text{where } 0\le x<3,\\ 3\mathrm{e}^{|3-x|} , & \text{where } 3\le x<6.\end{cases}$$

and that $\mathrm{f}\left(x\right)=\mathrm{f}\left(x+6\right)$ for all real values of x.

(i) Sketch the graph of $y=\mathrm{f}\left(x\right)$ for $-4\le x<10$, giving the exact coordinates of the end-points and y-intercepts. [4]

(ii) It is given that f is restricted to a domain where $0\le x<6$. Define f^{-1} in similar form. [4]

6 The function h is defined by

$$\mathrm{h}:x\mapsto\sqrt{x}+\sqrt{4-x}\ \text{ for }\ x\in\mathbb{R},\quad q<x\le 4.$$

(i) Given that h^{-1} exist, state the smallest value of q . [1]

Use the value of q found in part **(i)** for the rest of the question.

(ii) Sketch on the same diagram the graphs of $y=\mathrm{h}\left(x\right)$, $y=\mathrm{h}^{-1}\left(x\right)$ and $y=\mathrm{hh}^{-1}\left(x\right)$, showing clearly the relationship between the three graphs. [4]

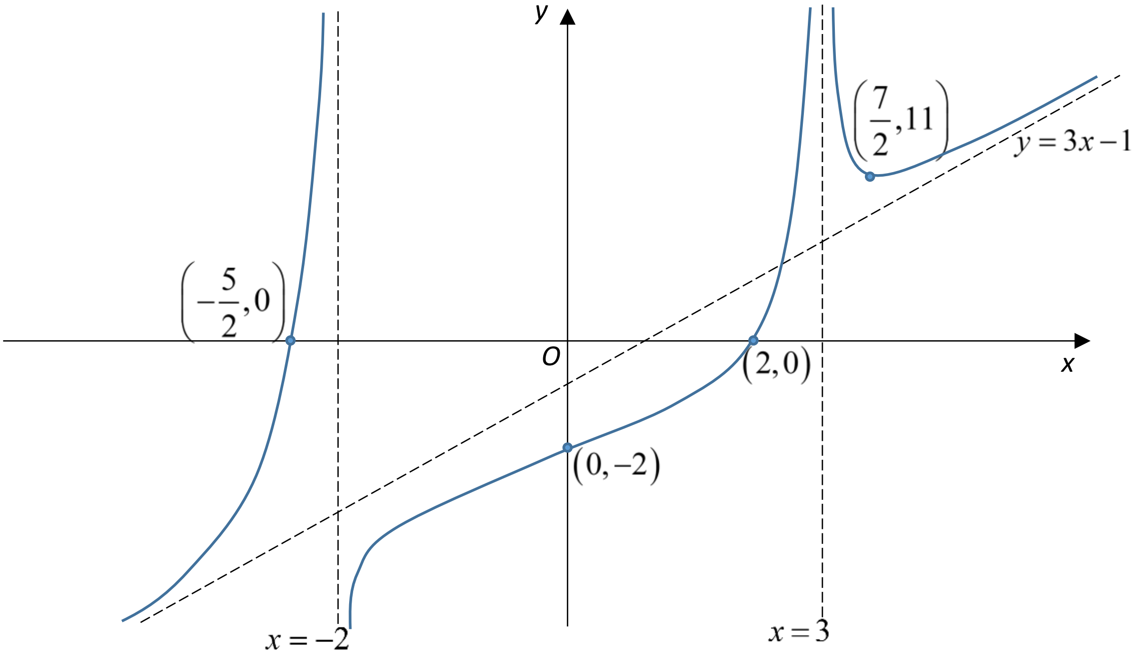
(iii) Find the set of values of x for which $\mathrm{h}^{-1}\mathrm{h}\left(x\right)=\mathrm{hh}^{-1}\left(x\right)$. [1]

The function g is defined by

$$\mathrm{g}: x \mapsto \frac{\ln x}{x} \quad \text{for } x \in \mathbb{R}^{+}.$$

(iv) Find the exact coordinates of the stationary point of the graph of g. Show that gh exists and find the exact range of gh. [4]

7 **(a)** The diagram shows the graph of $y = \mathrm{f}(x)$. The curve has a minimum point at $\left(\frac{7}{2}, 11\right)$ and axial intercepts at $\left(-\frac{5}{2}, 0\right)$, $(2,0)$ and $(0,-2)$. The curve also has vertical asymptotes $x=-2$, $x=3$ and an oblique asymptote $y=3x-1$.



Sketch, on separate diagrams, labelling clearly the coordinates of any axial intercepts (where applicable), turning points and equations of any asymptotes, the graphs of

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \mathrm{f}(2-x)$. [3]

(b) Describe a sequence of transformations that transform the graph of $y = \dfrac{1}{x^2+4x+3}$ onto the graph of $y = \dfrac{3x^2-4}{x^2-1}$. [4]

8 **(i)** Show that $\dfrac{6}{(r-2)(r)(r+1)}$ can be expressed in the form

$$\frac{A}{r-2}+\frac{B}{r}+\frac{C}{r+1},$$

where A, B and C are constants to be determined. [2]

(ii) Hence find $\displaystyle\sum_{r=3}^{N}\frac{1}{(r-2)(r)(r+1)}$. [3]

(iii) Give a reason why the series in part **(ii)** converges, and state its limit. [2]

(iv) Use your result in part **(ii)** to express $\displaystyle\sum_{r=10}^{2N}\frac{1}{(r-3)(r-1)(r)}$ in terms of N. [4]

9 A curve C_2 has parametric equations

$$x=\cos t,\quad y=\sin t+\cos t,\ \text{ for } 0\le t<2\pi .$$

(i) Express y as a single trigonometric function and hence find the exact set of values of y for which C_2 exists. [2]

(ii) Find the exact coordinates of the points where C_2 crosses the axes. [3]

(iii) Show that the Cartesian equation of C_2 is $y^2-2xy+2x^2=1$. [2]

(iv) Sketch C_2, stating the *y*-coordinates of the turning points and coordinates of the axial intercepts. [2]

(v) Another curve C_3 has equation

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1,$$

where *a*, *b*, *h* and *k* are constants with $a>0$ and $b>0$.

The tangent to the curve C_3 at $\left(0,\sqrt{2}\right)$ is parallel to the *y*-axis. The equations of the two asymptotes of C_3 are $y=\sqrt{2}x$ and $y=-\sqrt{2}x+2\sqrt{2}$. Find the values of *a*, *b*, *h* and *k*. [2]

(vi) Using parts **(iv)** and **(v)**, explain how C_3 could be used to determine the number of real roots of the equation

$$\left[b(\cos t-h)\right]^2-\left[a(\sin t+\cos t-k)\right]^2=(ab)^2$$

where $0\le t<2\pi$. [1]

10 Following a malaria outbreak in the country of Mutapa, a healthcare company, Agadez, was engaged by the government to provide vaccines to the citizens of Mutapa. Agadez claimed that they can vaccinate a total of $1\,300\,000-1\,300\,000(0.9)^m$ citizens by the m^{th} week of 2021.

(i) Find an expression for a_m, the number of citizens that Agadez would be able to vaccinate in the m^{th} week. Show that a_m follows a geometric progression and state its common ratio. [4]

(ii) Find the number of citizens Agadez would be able to vaccinate in total if the vaccination programme had no end date. [2]

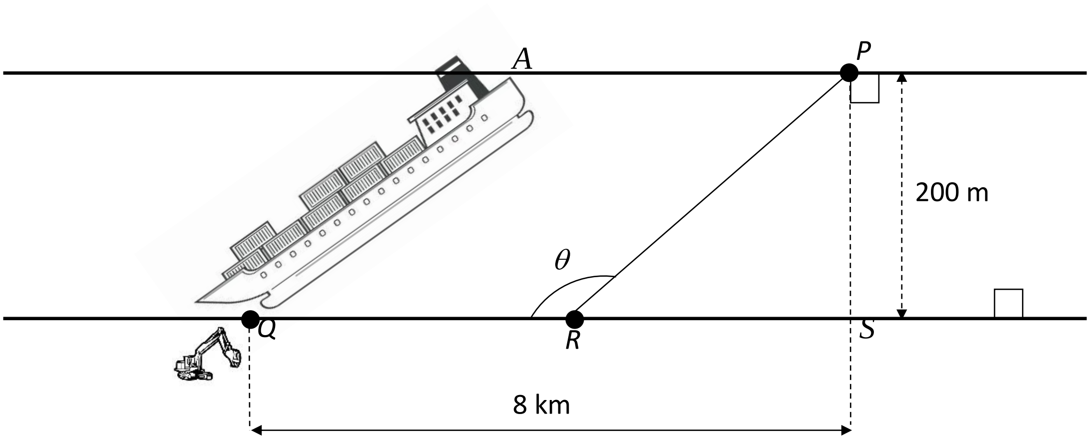
At the start of 2021, a second healthcare company, Butua, was also engaged to provide vaccines to the citizens of Mutapa from week 1 of 2021. Butua started with 60 nurses and would recruit 8 nurses each week (ie. they had 68 nurses by the 2^{nd} week of 2021). Each nurse worked 5 days a week and could vaccinate 24 citizens a day.

(iii) Find in terms of n, the number of citizens that Butua would be able to vaccinate in the n^{th} week. Hence, find the number of citizens vaccinated by Butua in the 20^{th} week. Find also the total number of citizens Butua vaccinated by the 20^{th} week.

[5]

(iv) It is given that Agadez and Butua started to provide vaccines to the citizens of Mutapa on the same particular 1^{st} week of 2021 and the number of citizens vaccinated weekly by both healthcare companies were compared. In which week would Butua first vaccinate more citizens than Agadez? [2]

11 (a) The diagram below shows a cargo ship blocking the Swee Canal of width 200 m. *PS* is the width of the Swee Canal and the banks of the Swee Canal, *AP* and *QS*, are parallel to each other. A worker standing at *P* wishes to cross the canal to the digger at *Q*, which is 8 km from *S*. In order to reach *Q* in the shortest possible time, the worker decides to hand paddle across the canal to *R* such that *PR* makes an angle of θ radians with *QR* as shown in the diagram below. He will then complete the rest of the journey by walking along *RQ*. The worker hand paddles at a speed of 2.4 km/hour and walks at a speed of 4 km/hour.



(i) Show that the total time taken in hours by the worker to travel from P to Q can be expressed as

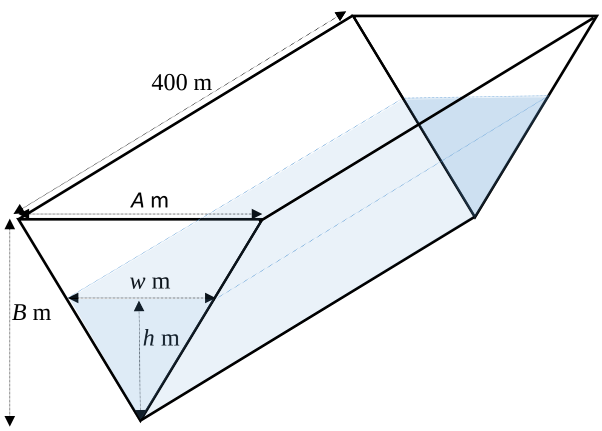
$$\alpha \operatorname{cosec}\theta + \beta \cot\theta + \gamma\ ,$$

where $\frac{\pi}{2} < \theta < \pi$ and α, β and γ are constants to be determined. [3]

(ii) Hence use differentiation to find the earliest time the worker would arrive at Q if he departs from P at 8.20 am. [4]

(iii) State an assumption, in the context of the question, necessary for your calculation in parts **(i)** and **(ii)**. [1]

(b) The hull of the cargo ship is assumed be have the shape of an inverted isosceles triangular prism with length 400 m and negligible thickness. The inverted isosceles triangle has fixed width and fixed height of A m and B m respectively as shown in the diagram below.



(i) The hull of the cargo ship is initially empty. After some time, water starts to enter the hull of the cargo ship at a constant rate of $10 \text{ m}^3/\text{s}$. At time t seconds after the start, the width and height of the water in the cargo ship are w m and h m respectively. Find an expression for the volume of water in the hull of the cargo ship in terms of h, A and B at time t seconds. [2]
[The volume of a prism is $\text{base area} \times \text{height}$.]

(ii) Find the rate of change of h after 30 minutes, leaving your answer in the form of $\frac{1}{120\sqrt{k}}\sqrt{\frac{B}{A}}$, where k is an integer to be determined. [3]

MOCK PAPER

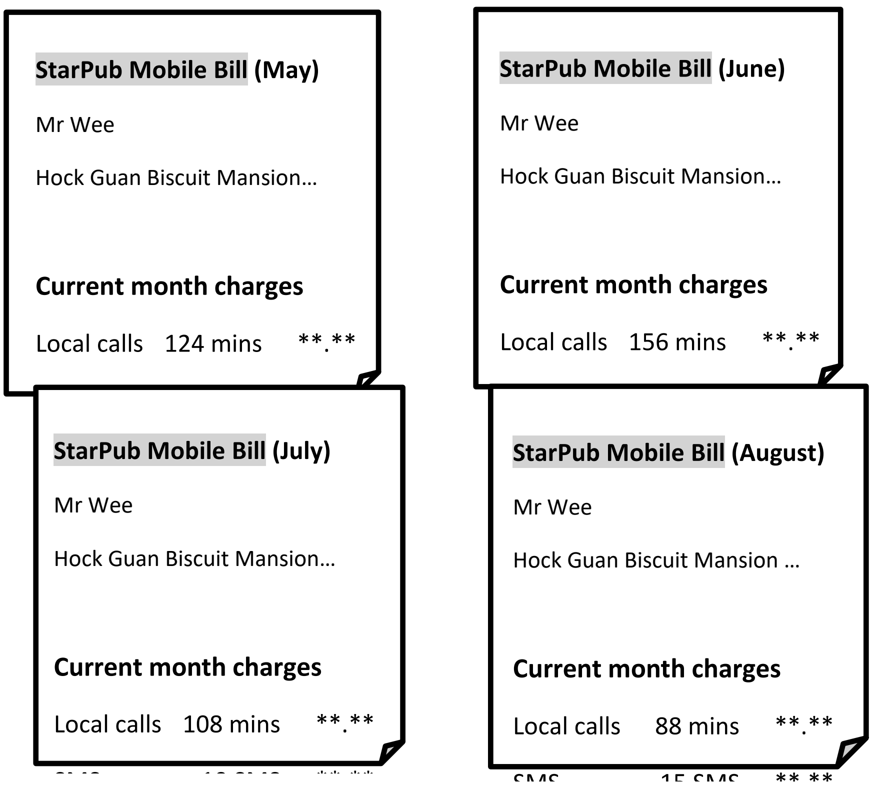
Duration: 3 HOURS

1 Using an algebraic method, solve the inequality $\dfrac{5}{1-x} \le 6x^2+7x+5$ leaving your answers in exact form. [3]

2 A function is defined as $\mathrm{f}(x)=3x^2-6x-1$. By completing the square, describe a sequence of transformations that transforms the graph of $y=x^2$ onto the graph of $y=\mathrm{f}(x)$. [3]

3 A telecommunication company, StarPub charges their mobile prepaid card users based on the usage for local calls, text messaging and data. Local calls are charged per minute, while text messaging is charged per SMS sent and data usage is charged per megabytes (MB) used or part thereof. For instance, a data usage in the interval of $(1000, 1001]$ MB will be charged the amount for the usage of 1001 MB of data.

Below are the monthly mobile prepaid card bill statements for Mr Wee from May to August 2020.



StarPub Mobile Bill (May)

Mr Wee

Hock Guan Biscuit Mansion...

Current month charges

Local calls 124 mins **.**

StarPub Mobile Bill (June)

Mr Wee

Hock Guan Biscuit Mansion...

Current month charges

Local calls 156 mins **.**

StarPub Mobile Bill (July)

Mr Wee

Hock Guan Biscuit Mansion...

Current month charges

Local calls 108 mins **.**

StarPub Mobile Bill (August)

Mr Wee

Hock Guan Biscuit Mansion ...

Current month charges

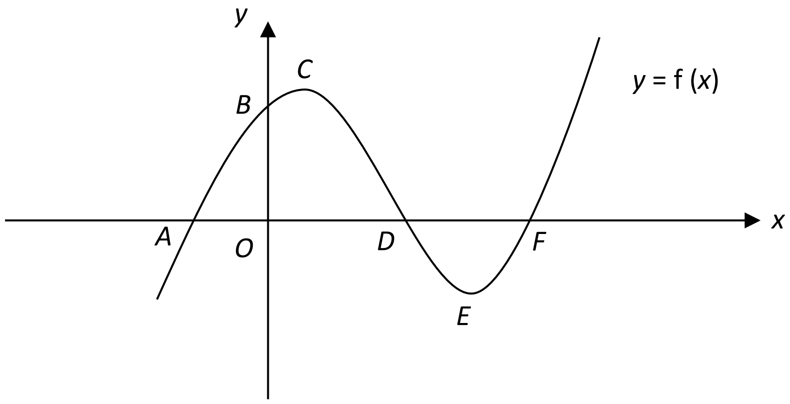
Local calls 88 mins **.**

It is known that the unit cost for local calls and text messaging remained unchanged from May to August 2020. The unit cost for data usage remained unchanged for May and June 2020 but was increased by 8% with effect from 1 July 2020.

(i) Find the unit cost for local calls, text messaging and data usage for June 2020 in dollars, giving your answers correct to the nearest 3 decimal places. [2]

(ii) The data usage for August 2020 printed on the bill has faded away. Using your answers in part **(i)**, find the data usage for August 2020 to the nearest MB. [2]

4 The diagram below shows the curve $y = \mathrm{f}(x)$. The curve cuts the x–axis at $A(-1.5,\ 0)$, $D(3,\ 0)$ and $F(5,\ 0)$ as well as the y–axis at $B(0,\ 2.5)$. It has a maximum point at $C(1,\ 3)$ and a minimum point at $E(4,\ -1.5)$.



Sketch, on separate diagrams, the following graphs. State clearly the equations of any asymptotes and the coordinates of the points corresponding to A, B, C, D, E and F (if any).

(i) $y = \mathrm{f}'(x)$, [2]

(ii) $y = \mathrm{f}(|x|)$, [2]

(iii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

(iv) Deduce the number of distinct positive solutions for the equation of $\mathrm{f}(|x|) = \dfrac{1}{\mathrm{f}(x)}$. [1]

5 **(i)** Using the method of differences, show that

$$\sum_{r=2}^{n} \frac{1}{(r-1)(r+2)} = \frac{1}{3}\left[A + \frac{B}{n} + \frac{C}{n+1} + \frac{D}{n+2}\right]$$

where A, B, C and D are constants to be determined. [4]

(ii) Using the result in part **(i)**, deduce the exact value of $\displaystyle\sum_{r=2}^{\infty} \frac{1}{r(r+3)}$.

Hence show that $\dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \ldots > \dfrac{13}{36}$. [4]

6 **Do not use a calculator in answering this question.**

Show that $\dfrac{x-1}{x^2-2x+3}-\dfrac{2-2x}{2x-x^2+1}=\dfrac{(1-x)(x^2-2x+7)}{(x^2-2x+3)(x^2-2x-1)}$. [1]

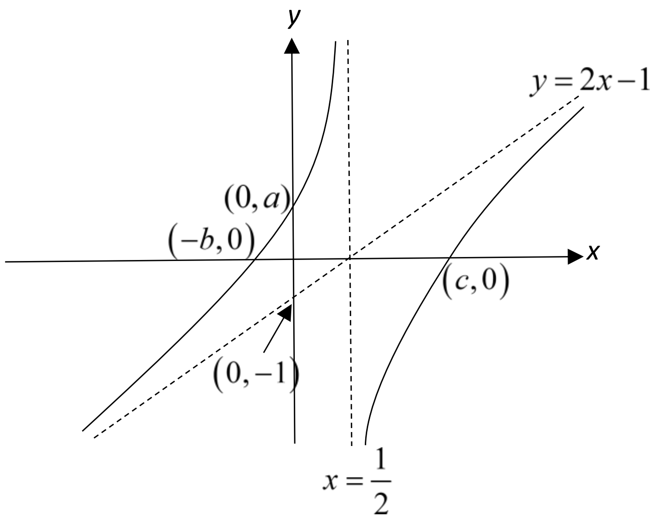
Solve the inequality $\dfrac{x-1}{x^2-2x+3}\geq\dfrac{2-2x}{2x-x^2+1}$. [3]

Hence solve the following exactly.

(i) $\dfrac{2x-1}{4x^2-4x+3}\geq\dfrac{2-4x}{4x-4x^2+1}$. [1]

(ii) $\dfrac{\tan x}{\tan^2 x+2}\geq\dfrac{-2\tan x}{2-\tan^2 x}$ for $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$. [3]

7 The diagram shows the graph of $y=\mathrm{f}(x)$. It has asymptotes $x=\dfrac{1}{2}$ and $y=2x-1$ and it passes through the points $(0,\ a)$, $(-b,\ 0)$ and $(c,0)$, where a, b and c are positive constants.



Function g is defined by

$$\mathrm{g}(x)=\mathrm{f}(x),\qquad x>k,$$

where k is a constant.

(i) State the minimum value of k for the function g^{-1} to exist. [1]

In the rest of the question, let k be the value stated in **(i)**.

(ii) On the same diagram, sketch the graphs of g and g^{-1}, stating, in terms of a, b and/or c, the coordinates of the points of intersection with the axes, and the equations of the asymptotes. [3]

(iii) By considering the respective ranges, show that it is not possible for the range of gg^{-1} and the range of $\mathrm{g}^{-1}\mathrm{g}$ to be the same. [2]

Given that f is defined by

$$f : x \mapsto \frac{4x^2 - 4x - a}{2x - 1}, \quad x \in \mathbb{R}, x \neq \frac{1}{2}.$$

(iv) Find, in terms of a, the value(s) of x when $g(x) = g^{-1}(x)$. [3]

(v) It is given that $h(x) = g(x)$. For $a = 0$, state a possible domain of h such that the range of hh^{-1} is the same as the range of $h^{-1}h$. [1]

8 **(a)** Find $\int \sin px \sin qx \, dx$, where p and q are real constants. [2]

(b) Find the exact value of $\int_{-\ln 2}^{0} \left(\frac{e^{2x}}{e^{2x}+1} \right) dx$, giving your answer as a single logarithm. [2]

(c) Find $\int \frac{x}{\sqrt{8 - 2x - x^2}} \, dx$. [3]

(d) Find $\frac{d}{dx}\left(\tan^3 x\right)$. [1]

Hence find $\int \sec^4 x \, dx$. [2]

9 The function f is defined by

$$f : x \mapsto x^2 - 4x + e^x, \; x \geq a$$

where a is a positive integer.

(i) State the least value of a for the inverse function of f to exist. Hence find the value of $f^{-1}(1)$. You may leave your answer correct to 3 decimal places. [3]

Use the least value of a found in part **(i)** for the remaining parts of the question.

(ii) Sketch, on the same diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing the graphical relationship between the two graphs. [3]

(iii) Explain why the solution(s) to the equation $f(x) = f^{-1}(x)$ can be obtained by solving the equation $x^2 - 5x + e^x = 0$. [1]

(iv) It is given that the gradient of the tangent to the curve with equation $y = f^{-1}(x)$ is $\frac{1}{3}$ at the point with $x = m$. Find the value of m, giving your answer correct to 3 decimal places. [3]

10 A curve C has parametric equations

$$x = \sin 2t, \quad y = \cos^2 t + 1, \quad 0 \le t \le \frac{\pi}{2}.$$

(i) Sketch C, showing clearly the axial-intercepts and the vertex. [3]

(ii) Express $\frac{dy}{dx}$ in the form $a \tan bt$, where a and b are constants to be determined. [2]

(iii) The tangents to the points P and Q on C are such that these two tangents meet at the point $\left(2, \frac{3}{2}\right)$. Find the coordinates of P and Q, giving your answers correct to 3 decimal places. [4]

(iv) Find the Cartesian equation of C. [3]

11 **(a)** Given that U_n is a linear polynomial in terms of n, explain why $\{U_n\}$ is an arithmetic sequence. [2]

(b) Edward plans to take up a study loan of $30000 with a local bank. For this type of loan, the bank will only start to compound the interest the year after Edward's graduation. The bank compounds interest on 1 Jan, at a rate of 5% per annum on the outstanding amount on 31 Dec of the preceding year.

Edward is going to graduate in 2024. He plans to make a repayment of $200 at the end of every month, starting from Jan 2025.

(i) Show that the outstanding loan amount is $28155 on 31 Dec 2026. [1]

(ii) Taking 2025 as the first year, show that the outstanding loan amount at the end of the nth year is $6000\left[8 - 3(1.05)^n\right]$. [3]

(iii) Find the year and the month in which Edward will finish servicing his loan. Hence determine the total interest he paid, giving your answer to the nearest cent. [4]

(iv) Edward intends to finish servicing the loan in at most 60 monthly repayments. Find the minimum monthly repayment needed, correct to the nearest dollar. [2]

12 [It is given that the volume of a circular cone with base radius r and vertical height h is $\frac{1}{3}\pi r^2 h$.]

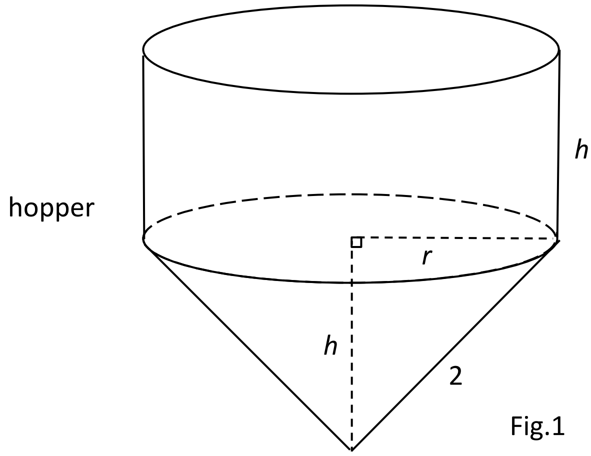


Fig.1

A hopper consists of an open cylinder of height h m joined to an open cone of radius r m and height h m (see Fig.1). The slant edge of the cone has a fixed length of 2 m.

(i) As r and h vary, use differentiation to find the exact maximum volume of the hopper. [6]

A coffee manufacturer uses such a hopper that has a cone of radius 1.2 m. For each batch, the coffee drink is initially filled to the brim of the hopper and mixed thoroughly before it is transferred to the packing chamber at a constant rate of 0.2 m^3/s.

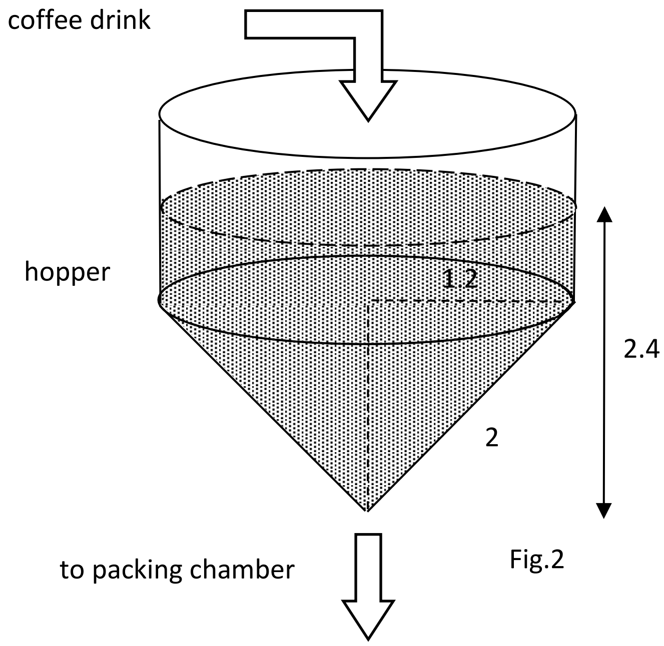


Fig.2

(ii) At a particular point in time, the height of the coffee drink in the cylindrical section is
2.4 m from the bottom of the cone (see Fig.2 shaded region). Find the rate at which the height of the coffee drink is changing at this instant. [2]

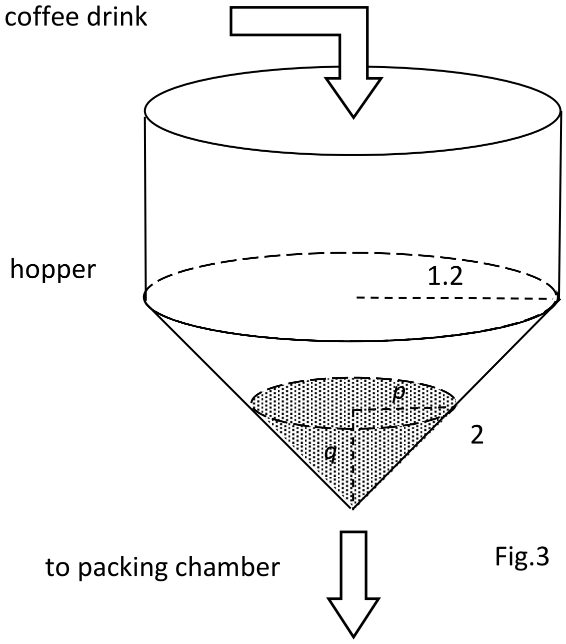


Fig.3

(iii) After some time has passed in the transfer process, there remains some coffee drink in the conical section with radius p m and height q m (see Fig.3 shaded region). Find p in terms of q and hence calculate the rate of decrease of q when $q = 0.5$. [4]