RAFFLES INSTITUTION
RAFFLES PROGRAMME 2023
YEAR 4 MATHEMATICS
TOPIC 2: REMAINDER & FACTOR THEOREMS AND PARTIAL FRACTIONS (MATH 1)

WORKSHEET 1

Name: ( ) Class: 4 ( ) Date:

WORKSHEET 1: POLYNOMIALS AND IDENTITIES
think! Add Math Textbook A Chapter 4 p.54

KEY UNDERSTANDING(S)

Students will understand that

- A polynomial $\mathrm{P}(x)$ can be written as $\mathrm{P}(x)=\mathrm{D}(x)\times\mathrm{Q}(x)+\mathrm{R}(x)$

LEARNER OUTCOMES

At the end of this worksheet, students will be able to

- Define a polynomial
- Write a polynomial in the form $\text{Dividend}=\text{Divisor}\times\text{Quotient}+\text{Remainder}$
- Distinguish between identities and equations.
- Use long division to divide polynomials

Blended Learning Online (SDL)
Students to access HeyMath! Lesson
Year 4 / Remainder and Factor Theorem / Polynomial Identities / Introduction [5:20]

(1) INTRODUCTION TO POLYNOMIAL

The table shows some examples of polynomials and non-polynomials in one variable.

Polynomials	Non-polynomials
x^2-4x+3	$x+1-\dfrac{1}{x}$
$2x^5+7x^4-4x^3+6x^2-x+1$	$-x^{\frac{1}{3}}+x^{-2}$
$\sqrt{2}x^4-\dfrac{5}{2}x$	$x^3-\sqrt{x}$
8	$\dfrac{1}{x^2}-\dfrac{2}{x^4}-\sqrt[3]{x}$

Compare the powers of x in polynomials with those in non-polynomials. What do you notice?

Why is the number '8' a polynomial?

(2) DEFINITIONS

A **polynomial in *x*** is an algebraic expression consisting of terms with **non-negative integral powers of *x*** only.

The general form of a polynomial in x is $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$, where the index n is a **non-negative integer** and the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are real numbers.

NOTE:

1. $a_n x^n, a_{n-1}x^{n-1}, a_{n-2}x^{n-2}, \ldots, a_2 x^2, a_1 x, a_0$ are known as the **terms** of the polynomial.
   $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are called the **coefficients** of the polynomial.
   a_0 is called the **constant term**.

2. The **degree** of a polynomial in x is the **highest power** of x that occurs in the polynomial.

3. Consider the polynomial $5x^3 + 7x^2 - 4x + 6$.
   $5x^3$ is called the ***leading term*** as it contains the highest power of x.
   The coefficient of the leading term is called the ***leading coefficient***.
   By convention, we usually arrange the terms of a polynomial in **descending powers** of the variable.

4. Commonly used notations for polynomials in x include $\mathrm{P}(x)$, $\mathrm{Q}(x)$, $\mathrm{f}(x)$ and $\mathrm{g}(x)$.

5. $\mathrm{P}(a)$ is the value of $\mathrm{P}(x)$ when $x = a$.
   For example, to find the value of the polynomial $\mathrm{P}(x) = 5x^3 + 7x^2 - 4x + 6$ when $x = 3$, we substitute $x = 3$ into $\mathrm{P}(x)$: $\mathrm{P}(3) = 5(3)^3 + 7(3)^2 - 4(3) + 6$
   $= 192$

6. Polynomials can be classified according to their degree and the number of terms.

Polynomial	**Degree**	**Name by degree**	**No. of terms**	**Name by no. of terms**
8				
$-7x-11$				
x^2-4x+3				
$-12x^3+x^2-x+5$				
$\sqrt{2}x^4-\frac{5}{2}x$				
$2x^5+7x^4-4x^3+6x^2-x+1$				

EG 1 Which of the following are polynomials?

(a) $x+\frac{1}{x}$ (b) $(2x-1)^3-9$ (c) $(ex)^2-\sqrt{5}$

(d) $\sqrt{x+1}-2x$ (e) $2x^4y-5xy+3y^2+x-1$ (f) $\lg x+3x^2$

EG 2 Given that $\mathrm{P}(x)=2x^4-x^2+3x-7$ is a polynomial in x, determine the following:

(a) Coefficient of x^4 = _____

(b) Coefficient of x^2 = _____

(c) Degree of $\mathrm{P}(x)$ = _____

(d) $\mathrm{P}(2)$ = ___________________________

(e) $\mathrm{P}(-1)$ = __________________________

(3) IDENTITIES

If two polynomials $\mathrm{P}(x) = Ax^3 + Bx^2 + Cx + D$ and $\mathrm{Q}(x) = 2x^3 + 3x^2 - x + 4$ are **equivalent**, then the equation $\mathrm{P}(x) = \mathrm{Q}(x)$ is true for all real values of x, i.e. $A = 2$, $B = 3$, $C = -1$ and $D = 4$. We call this equation an **identity**.

We use the symbol '$\equiv$' to denote an identity, i.e. $\mathbf{P}(\boldsymbol{x}) \equiv \mathbf{Q}(\boldsymbol{x})$.

EG 3 Find the values of A, B and C if $4x^2 + 3x - 7 \equiv A(x-1)(x+3) + B(x-1) + C$

Method 1: Compare coefficients

Method 2: Substitute suitable values of *x*

Why can we arbitrarily choose any value of x for substitution?

Method 3: Combination of Method 1 and Method 2

Steps:

1. Compare coefficients of highest power terms
2. Substitute factor value(s) of x
3. Compare constant terms (or sub. $x = 0$)
4. Substitute other value(s) of x or compare coefficients of other powers of x

Computational Thinking - Opportunities for Algorithmic Thinking

The method of finding unknown constants allow opportunities for algorithmic thinking:

- ***Write out the general approach as a sequence of steps***
- ***Work out the steps to find unknown constants in identity***

EG 4 Given that $x^3 + 2x^2 + x + 5 \equiv (x-1)(Ax^2 + Bx + C) + D$, find the values of A, B, C and D.

EG 5 Given $4x^3 - 6x^2 + 1 \equiv (x-2)(x+1)\mathrm{Q}(x) + ax + b$, where $\mathrm{Q}(x)$ is a polynomial, find the values of a and b.

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Year 4 / Remainder and Factor Theorem / Polynomial Identities / Examples 1 to 3 [9:43] to consolidate learning before doing Homework 1.

HOMEWORK 1

LEVEL 1

1. Given that $3x^3 + 5x^2 - 4x - 5 \equiv (x-1)(x+A)(Bx+2) + C$, find the values of A, B and C.

   [Ans: $A = 2, B = 3, C = -1$]

2. *think! Add Math Textbook A p.60 Ex 4A Q6c*
   Given that $2(Ax+1)(x-1)(x-B) + C = 2x^3 + 6x^2 - 2x + 8$ for all values of x, find the values of A, B and C.

   [Ans: $A = 1, B = -3, C = 14$]

3. *think! Add Math Textbook A p.60 Ex 4A Q10*
   The expression $(ax+b)(x-1) + c(x^2+5)$ is equal to 18 for all values of x. find the values of a, b and c.

   [Ans: $a = -3, b = -3, c = 3$]

LEVEL 2

1. *think! Add Math Textbook A p.60 Ex 4A Q6d*
   Given that $2x^4 - 13x^3 + 19x^2 + 5x + 1 = (x-4)(Ax+1)(x^2 + Bx + 1) + C$ for all values of x, find the values of A, B and C.

   [Ans: $A = 2, B = -3, C = 5$]

2. Given $5x^3 + 6x^2 - x + 2 = (x+1)(x-2)\mathrm{Q}(x) + px + q$ for all values of x, find the values of p and q.

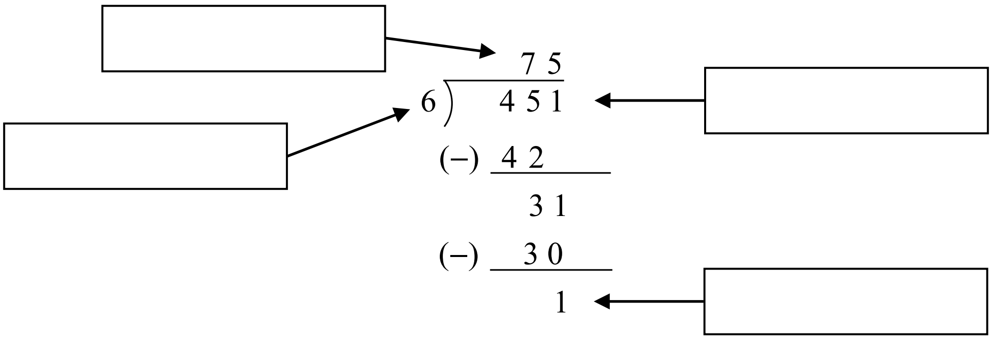
   [Ans: $p = 20, q = 24$]

3. Given that $2x^4+4x^3+Ax^2-11x+16\equiv\left(Bx^2-3\right)\left(x^2+2x+C\right)+Dx+7$, find the values of A, B, C and D.

[Ans: $A=-9$, $B=2$, $C=-3$, $D=-5$]

(4) LONG DIVISION OF POLYNOMIALS

Let us recall long division of positive integers, e.g. divide 451 by 6:



We can express the dividend 451 in terms of the divisor, quotient and remainder as follows:

$$451 \quad = \quad 6 \quad \times \quad 75 \quad + \quad 1$$

Dividend = Divisor × Quotient + Remainder

In general,

if a polynomial $\mathrm{P}(x)$ is divided by another polynomial $\mathrm{D}(x)$, then

$$\underset{\text{Dividend}}{\underset{\uparrow}{\mathbf{P}(x)}} \equiv \underset{\text{Divisor}}{\underset{\uparrow}{\mathbf{D}(x)}} \times \underset{\text{Quotient}}{\underset{\uparrow}{\mathbf{Q}(x)}} + \underset{\text{Remainder}}{\underset{\uparrow}{\mathbf{R}(x)}}$$

and **degree of $\mathbf{R}(x)$ < degree of $\mathbf{D}(x)$**.

EG 6 By long division, find the quotient and remainder for each of the following divisions. Hence, express the polynomial in the form $\text{Dividend} \equiv \text{Divisor} \times \text{Quotient} + \text{Remainder}$.

(a) $\left(x^3 + 7x^2 - 4x\right) \div (x+3)$

(b) $\left(4x^3 + 4x^2 - 6x + 7\right) \div (2x-1)$

(c) $\left(3x^4+6x^3+2x+6\right)\div\left(x^2+1\right)$

Note: For missing terms in the dividend, you can leave a space or include the zero term ($0x^2$) as a placeholder.

From the above examples, we can deduce the following rules for division of polynomials.

1. Degree of the dividend $\mathrm{P}(x)$ _______ Degree of the divisor $D(x)$

2. Degree of the remainder $\mathrm{R}(x)$ _______ Degree of the divisor $D(x)$; otherwise the division is incomplete.

What is the quotient and remainder when $8x^2+12x-1$ is divided by $2x^2+3x+1$?

Note: In addition to long division, Synthetic Division may also be used. Refer to FOR YOUR INTEREST at Section 5.

EG 7

(i) By long division, find the quotient and remainder when $3x^3 - 4x^2 - 5x + 1$ is divided by $3x - 1$.

(ii) Write the polynomial $3x^3 - 4x^2 - 5x + 1$ in the form $\text{Dividend} \equiv \text{Divisor} \times \text{Quotient} + \text{Remainder}$.

(iii) Hence, factorise $3x^3 - 4x^2 - 5x + 2$ completely.

EG 8 Given that $2x^3 - x^2 - 2x + 3 = (Ax-3)(x-1)(x+2) + B(x-1) + C$ for all values of x, evaluate A, B and C. Hence, deduce the quotient and the remainder when $2x^3 - x^2 - 2x + 3$ is divided by $x^2 + x - 2$,

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Students to access HeyMath! Lesson
Year 4 / Remainder and Factor Theorem / Dividing Polynomials / Example 1 & 2 [5:39] and Activity 1 & 2 to consolidate learning before doing Homework 2.

HOMEWORK 2

LEVEL 1

1. Find the quotient and remainder for each of the following.

   (a) $(3x^3 - 2x + 5) \div (x+2)$ [Ans: $3x^2 - 6x + 10$; -15]

   (b) $(12x^3 + 12x^2 + 7x + 1) \div (2x+1)$ [Ans: $6x^2 + 3x + 2$; -1]

   (c) $(2x^3 - 4x^2 + 7x - 5) \div (2x^2 - x - 1)$ [Ans: $x - \frac{3}{2}$; $\frac{13}{2}x - \frac{13}{2}$]

2. Given that $2x^3 - 7x^2 + 7x - 5 \equiv A(x-1)^3 + Bx(x-1) + C$, find the values of A, B and C. Hence, find the remainder when $2x^3 - 7x^2 + 7x - 5$ is divided by $x - 1$.

   [Ans: $A = 2, B = -1, C = -3$; -3]

3. Given that $3x^3 + x^2 - 2x - 3 \equiv (Ax+B)(x-1)(x+2) + C(x-1) + D$, find the values of A, B, C and D. Hence, or otherwise, find the remainder when $3x^3 + x^2 - 2x - 3$ is divided by $x^2 + x - 2$,

   [Ans: $A = 3, B = -2, C = 6, D = -1$; $6x - 7$]

LEVEL 2

1. By long division, find the quotient and remainder for each of the following divisions. Hence, write the polynomial in the form $\text{Dividend} \equiv \text{Divisor} \times \text{Quotient} + \text{Remainder}$.

   (a) $\left(2x^4 + 5x^3 - 7x\right) \div \left(x^2 - 2\right)$ [Ans: $6x^2 + 3x + 2$; -1]

   (b) $\left(x^3 + 4x^2 + 3x + 7\right) \div (1 + x)^2$ [Ans: $x + 2$; $-2x + 5$]

2. Find the values of A, B and C if $2x^3+3x^2-14x-5 \equiv (2x+A)(x^2+4x+3)+Bx+C$. Hence, determine the remainder when $2x^3+3x^2-14x-5$ is divided by x^2+4x+3.

[Ans: $A=-5, B=0, C=10$; 10]

3. Given that $3x^3-2x^2+x-4 \equiv A(x-1)+B(x-1)(x+1)+Cx(x^2-1)+D$, find the values of A, B, C and D. Hence, find the quotient when $3x^3-2x^2+x-4$ is divided by $x-1$,

[Ans: $A=4, B=-2, C=3, D=-2$; $3x^2+x+2$]

(5) FOR YOUR INTEREST

(5.1) SYNTHETIC DIVISION FOR LINEAR DIVISOR

Synthetic division is performed with less effort than the long division method. The requirements for synthetic division are:

- the divisor must be linear
- the leading coefficient in the divisor should be also equal to one

The basic technique for synthetic division is:

Bring down, multiply and add, multiply and add, multiply and add, …

(A) Linear Divisor of the form *x* + *b*

Example: Find the quotient and remainder for when $7x^3+4x+8$ is divided by $x+2$.

Steps to perform synthetic division:

1. Write down the coefficients of the dividend in descending powers. Use zero(s) to fill in the missing term(s).

2. Set the divisor to zero to obtain the number to put in the division box. In the example, since divisor is $x+2$, we put -2 in the division box.

3. Bring the leading coefficient straight down once the setup is done.

$$\begin{array}{r|rrrr} -2 & 7 & 0 & 4 & 8 \\ & & & & \\ \hline & 7 & & & \end{array}$$

4. Multiply the number in the division box with the leading coefficient and write the outcome below the coefficient of the 2nd term.

$$\begin{array}{r|rrrr} -2 & 7 & 0 & 4 & 8 \\ & & -14 & & \\ \hline & 7 & & & \end{array}$$

multiply

5. Add the two numbers together and write the outcome at the bottom of the column.

$$\begin{array}{r|rrrr} -2 & 7 & 0 & 4 & 8 \\ & & -14 & & \\ \hline & 7 & -14 & & \end{array}$$

add

6. Repeat Steps 4 and 5 for the rest of the coefficients.

$$\begin{array}{r|rrrr} & 7 & 0 & 4 & 8 \\ -2 & & -14 & 28 & -64 \\ \hline & 7 & -14 & 32 & -56 \end{array}$$

7. The last sum is the remainder and those before it form the coefficients of the terms in the quotient, which starts with one degree lower than the dividend.

   Result:

   $7x^3 + 4x + 8 = (x+2)(7x^2 - 14x + 32) - 56$

   Hence, the quotient is $7x^2 - 14x + 32$ and the remainder is -56.

Practice A

Use synthetic division to find the quotient and remainder for each of the following divisions.

(a) $(8x^3 - 4x^2 + 7x - 160) \div (x - 7)$

(b) $(7 - 3x^3) \div (x - 3)$

[Ans: (a) quotient $= 8x^2 + 52x + 371$, remainder $= 2437$
(b) quotient $= -3x^2 - 9x - 27$, remainder $= -74$]

<u>(B) Linear Divisor of the form *ax* + *b*</u>

Example: Find the quotient and remainder for when $4x^3+4x^2-6x+7$ is divided by $2x-1$.

Let $2x-1=0$

$$x=\frac{1}{2}$$

$$\begin{array}{c|cccc} & 4 & 4 & -6 & 7 \\ \frac{1}{2} & & 2 & 3 & -\frac{3}{2} \\ \hline & 4 & 6 & -3 & \frac{11}{2} \end{array}$$

Result:

$$\begin{aligned} 4x^3+4x^2-6x+7 &= \left(x-\frac{1}{2}\right)\left(4x^2+6x-3\right)+\frac{11}{2} \\ &= (2x-1)\left(2x^2+3x-\frac{3}{2}\right)+\frac{11}{2} \end{aligned}$$

Hence, the quotient is $2x^2+3x-\frac{3}{2}$ and the remainder is $5\frac{1}{2}$.

Note: ***In general, for synthetic division, when the divisor is ax + b, the quotient is obtained by dividing all the coefficients by a. It can be illustrated as follows:***

$$\begin{aligned} \mathrm{f}(x) &\equiv \left(x+\frac{b}{a}\right)\mathrm{Q}(x)+\mathrm{R}(x) \\ &= a\left(x+\frac{b}{a}\right)\frac{\mathrm{Q}(x)}{a}+\mathrm{R}(x) \\ &= (ax+b)\frac{\mathrm{Q}(x)}{a}+\mathrm{R}(x) \end{aligned}$$

Hence, the quotient is $\frac{\mathrm{Q}(x)}{a}$ and the remainder is still $\mathrm{R}(x)$.

Practice B

Use synthetic division to find the quotient and remainder for each of the following divisions.

(a) $\left(4x^3+3x^2-5\right)\div(2x+1)$

(b) $\left(12x^4-7x^3+4x^2+2x-2\right)\div(3x-1)$

[Ans: (a) quotient $=2x^2+\frac{1}{2}x-\frac{1}{4}$, remainder $=-4\frac{3}{4}$

(b) quotient $=4x^3-x^2+x+1$, remainder $=-1$]

(5.2) CALCULATORS FOR POLYNOMIAL DIVISION

Here are some calculators to help you verify your answers for polynomial division. Scan the respective QR codes to start exploring.

Synthetic Division Calculator

https://calculator-online.net/synthetic-division-calculator/

Polynomial Long Division Calculator

https://calculator-online.net/polynomial-long-division-calculator/