RAFFLES INSTITUTION
H2 Mathematics (9758)
2024 Year 6

Term 3 Revision
Session 5: Vectors

Questions

1 MI Prelim 9758/2021/01/Q4

With respect to the origin *O*, the position vectors of the points *A*, *B* and *C* are **a**, **b** and **c** respectively. Point *C* lies on *AB* such that $AC:CB=1:2$. It is given that **a** is a unit vector and the length of *OB* is 2 units.

(i) Give a geometrical interpretation of $|\mathbf{a}\bullet\mathbf{c}|$. [1]

(ii) It is given that the angle *AOB* is $60°$. By considering $(2\mathbf{a}-\mathbf{b})\bullet(2\mathbf{a}-\mathbf{b})$, find $|2\mathbf{a}-\mathbf{b}|$. [3]

(iii) Find **c** in terms of **a** and **b**. [1]

(iv) Hence by considering cosine of angle *AOC* and cosine of angle *COB*, determine if the line segment *OC* bisects the angle *AOB*. [3]

	Suggested Solution
(i)	Since **a** is a unit vector, $\lvert\mathbf{a}\bullet\mathbf{c}\rvert$ is the length of projection of **c** onto **a**.
(ii)	$(2\mathbf{a}-\mathbf{b})\bullet(2\mathbf{a}-\mathbf{b})=4(\mathbf{a}\bullet\mathbf{a})-2(\mathbf{a}\bullet\mathbf{b})-2(\mathbf{b}\bullet\mathbf{a})+(\mathbf{b}\bullet\mathbf{b})$ $=4\lvert\mathbf{a}\rvert^2-4(\mathbf{a}\bullet\mathbf{b})+\lvert\mathbf{b}\rvert^2 \quad (\because\ \mathbf{b}\bullet\mathbf{a}=\mathbf{a}\bullet\mathbf{b})$ $=4(1)^2-4\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 60^\circ+(2)^2$ $=4-4(1)(2)\left(\frac{1}{2}\right)+4=4.$ $(2\mathbf{a}-\mathbf{b})\bullet(2\mathbf{a}-\mathbf{b})=\lvert 2\mathbf{a}-\mathbf{b}\rvert^2=4\Rightarrow\lvert 2\mathbf{a}-\mathbf{b}\rvert=2.$
(iii)	By Ratio Theorem, $\mathbf{c}=\frac{\mathbf{b}+2\mathbf{a}}{3}$.
(iv)	$\cos\angle AOC=\frac{\mathbf{a}\bullet\mathbf{c}}{\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert}$ $=\frac{\mathbf{a}\bullet\left(\frac{\mathbf{b}+2\mathbf{a}}{3}\right)}{\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert}$ [from **(iii)**] $=\frac{1}{3}\left(\frac{\mathbf{a}\bullet\mathbf{b}+2\mathbf{a}\bullet\mathbf{a}}{\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert}\right)$ $=\frac{1}{3}\left(\frac{\mathbf{a}\bullet\mathbf{b}+2}{\lvert\mathbf{c}\rvert}\right)$ [since $\mathbf{a}\bullet\mathbf{a}=\lvert\mathbf{a}\rvert^2=1^2$]

	$\cos\angle COB = \dfrac{\mathbf{b}\bullet\mathbf{c}}{\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert}$ $= \dfrac{\mathbf{b}\bullet\left(\dfrac{\mathbf{b}+2\mathbf{a}}{3}\right)}{\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert}$ [from **(iii)**] $= \dfrac{1}{3}\left(\dfrac{\mathbf{b}\bullet\mathbf{b}+2(\mathbf{a}\bullet\mathbf{b})}{\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert}\right)$ $= \dfrac{1}{3}\left(\dfrac{2^2+2(\mathbf{a}\bullet\mathbf{b})}{2\lvert\mathbf{c}\rvert}\right)$ [since $\mathbf{b}\bullet\mathbf{b} = \lvert\mathbf{b}\rvert^2 = 2^2$] $= \dfrac{1}{3}\left(\dfrac{2+\mathbf{a}\bullet\mathbf{b}}{\lvert\mathbf{c}\rvert}\right)$ Since $\cos\angle COB = \cos\angle AOC$, line OC bisects the angle AOB.

2 DHS Prelim 9758/2021/02/Q5

The points A, B and R have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$ respectively.

(a) The point C has position vector $\frac{2}{7}\mathbf{a}-\frac{3}{7}\mathbf{b}$ and the point D is such that the origin O is the midpoint of the line segment CD. The point R lies on BD extended such that the ratio of BD to BR is $4:7$. Show that the points A, O and R are collinear and state the ratio of OA to OR. [4]

(b) It is given that the point R has position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and that $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$.

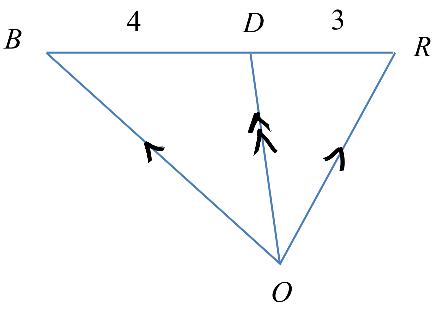
(i) Determine the exact area of the triangle AOB. [2]

(ii) Give the geometrical interpretation of the point R, given that $\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}) = 0$. [2]

(iii) Find the shortest distance between the point $(-8,-2,9)$ and the collection of all points R satisfying $\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}) = 0$. [2]

	Suggested Solution
(a)	$\overrightarrow{OD} = -\overrightarrow{OC}$ $\mathbf{d} = -\left(\dfrac{2}{7}\mathbf{a}-\dfrac{3}{7}\mathbf{b}\right)$ $= \dfrac{1}{7}(3\mathbf{b}-2\mathbf{a})$

By ratio theorem,



$$\overrightarrow{OD}=\frac{4\overrightarrow{OR}+3\overrightarrow{OB}}{7}$$

$$4\overrightarrow{OR}=7\overrightarrow{OD}-3\overrightarrow{OB}$$

$$\overrightarrow{OR}=\frac{1}{4}\left(7\left(\frac{3}{7}\mathbf{b}-\frac{2}{7}\mathbf{a}\right)-3\mathbf{b}\right)$$

$$=-\frac{1}{2}\mathbf{a}=-\frac{1}{2}\overrightarrow{OA}$$

Since $\overrightarrow{OR}$ is parallel to $\overrightarrow{OA}$ and O is a common point, the points A, O and R are collinear. (shown)

The ratio of AO to OR is 2:1

(b) (i) Area of triangle AOB

$$=\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}1\\3\\2\end{pmatrix}\times\begin{pmatrix}-1\\5\\3\end{pmatrix}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}-1\\-5\\8\end{pmatrix}\right|$$

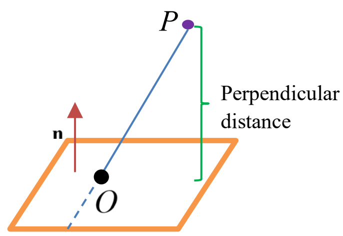
$$=\frac{1}{2}\sqrt{90}$$

$$=\frac{3}{2}\sqrt{10}$$

(ii) $\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})=0$ refers to the collection of all points on the plane that is perpendicular to $\mathbf{a}\times\mathbf{b}$ and containing the origin.

(iii) $\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})=0$

$$\mathbf{r}\cdot\begin{pmatrix}-1\\-5\\8\end{pmatrix}=0$$



Shortest distance from point to plane $=\dfrac{\left|\begin{pmatrix}-8\\-2\\9\end{pmatrix}\cdot\begin{pmatrix}-1\\-5\\8\end{pmatrix}\right|}{\left|\begin{pmatrix}-1\\-5\\8\end{pmatrix}\right|}=\dfrac{90}{\sqrt{90}}=\sqrt{90}$

3 MI Prelim 9758/2021/02/Q4

The plane p passes through the points with coordinates $(-k,\ 2,\ 5)$, $(0,\ 2,\ -1)$ and $\left(-\frac{1}{2},\ 3,\ -1\right)$, and the line l has equation $\frac{x+2}{-3}=y-2=\frac{z-4}{k}$, where k is a constant.

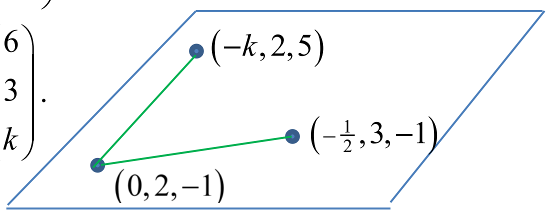
(i) Show that the cartesian equation of the plane is $6x+3y+kz=6-k$. [2]

(ii) Show that line l cannot be perpendicular to p. [2]

For the rest of this question, let $k=-2$.

(iii) Given that l meets p at point N, find the coordinates of N. [3]

(iv) Another plane π is parallel to the plane p. Given that the distance between p and π is 11 units, find the possible points of intersection between l and π. [3]

	Suggested Solution
(i)	Plane p is parallel to $\begin{pmatrix}0\\2\\-1\end{pmatrix}-\begin{pmatrix}-k\\2\\5\end{pmatrix}=\begin{pmatrix}k\\0\\-6\end{pmatrix}$ and $\begin{pmatrix}-\frac{1}{2}\\3\\-1\end{pmatrix}-\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$. Normal of p is parallel to $\begin{pmatrix}k\\0\\-6\end{pmatrix}\times\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}=\begin{pmatrix}6\\3\\k\end{pmatrix}$.  Hence equation of p is $\mathbf{r}\cdot\begin{pmatrix}6\\3\\k\end{pmatrix}=\begin{pmatrix}0\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}6\\3\\k\end{pmatrix}=6-k.$ Cartesian equation is $6x+3y+kz=6-k.$ (shown)
(ii)	$l:\ \frac{x+2}{-3}=y-2=\frac{z-4}{k}$ Let $\lambda=\frac{x+2}{-3}=y-2=\frac{z-4}{k}$ $x=-2-3\lambda$ $y=2+\lambda$ $z=4+k\lambda$ $l:\ \mathbf{r}=\begin{pmatrix}-2\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\k\end{pmatrix}$, $\lambda\in\mathbb{R}$

To show that l cannot be perpendicular to p :

Method 1: Show l is not perpendicular to a direction parallel to p.

Suppose l is perpendicular to p, then l is perpendicular to $\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$. [from **(i)**]

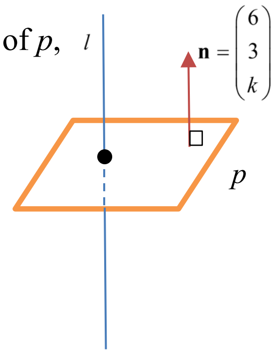
Since $\begin{pmatrix} -3 \\ 1 \\ k \end{pmatrix} \bullet \begin{pmatrix} -0.5 \\ 1 \\ 0 \end{pmatrix} = 1.5 + 1 = 2.5 \neq 0$, l is not perpendicular to $\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ l cannot be perpendicular to p. (shown)

Method 2: Show l is not parallel to normal of p.

Suppose l is perpendicular to p, then l is parallel to the normal of p,

i.e. $\begin{pmatrix} -3 \\ 1 \\ k \end{pmatrix} = t \begin{pmatrix} 6 \\ 3 \\ k \end{pmatrix}$, for some $t \in \mathbb{R}$.



$$\Rightarrow \begin{cases} -3 = 6t \\ 1 = 3t \\ k = tk \end{cases} \Rightarrow \begin{cases} t = -0.5 \\ t = \dfrac{1}{3} \\ t = 1 \end{cases}$$

Since there is no unique value of t, l is not parallel to the normal of p, i.e. l cannot be perpendicular to p. (shown)

(iii)

$l: \ \mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}, \ \lambda \in \mathbb{R}$

$p: \ \mathbf{r} \bullet \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = 8$

When l intersects p,

$$\left[\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \right] \bullet \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = 8$$

$$-12 + 6 - 8 + (3 - 18 + 4)\lambda = 8$$

$$-11\lambda = 22$$

$$\lambda = -2$$

position vector of $N = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix}$

$\therefore$ Coordinates of N are $(4, 0, 8)$.

(iv)	**Method 1**:

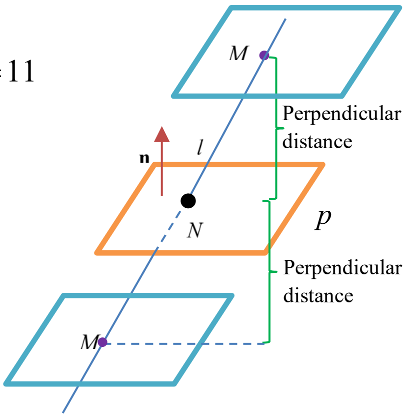
Let the point of intersection of l and π be M.

Since M lies on l, $\overrightarrow{OM} = \begin{pmatrix}-2\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}-3\\1\\-2\end{pmatrix}$ for some $\lambda \in \mathbb{R}$

$$\overrightarrow{MN} = \begin{pmatrix}4\\0\\8\end{pmatrix} - \left[\begin{pmatrix}-2\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}-3\\1\\-2\end{pmatrix}\right] = \begin{pmatrix}6\\-2\\4\end{pmatrix} - \lambda\begin{pmatrix}-3\\1\\-2\end{pmatrix}$$

Distance from M to $p = 11$

$$\left|\left[\begin{pmatrix}6\\-2\\4\end{pmatrix} - \lambda\begin{pmatrix}-3\\1\\-2\end{pmatrix}\right] \bullet \frac{1}{\sqrt{6^2+3^2+(-2)^2}}\begin{pmatrix}6\\3\\-2\end{pmatrix}\right| = 11$$



$$\frac{1}{7}\left|36-6-8+(18-3-4)\lambda\right| = 11$$

$$|22+11\lambda| = 77$$

$$22+11\lambda = 77 \text{ or } -77$$

$$11\lambda = 55 \text{ or } -99$$

$$\lambda = 5 \text{ or } -9$$

$$\overrightarrow{OM} = \begin{pmatrix}-17\\7\\-6\end{pmatrix} \text{ or } \begin{pmatrix}25\\-7\\22\end{pmatrix}$$

Hence, possible points of intersections between l and π are $(-17, 7, -6)$ and $(25, -7, 22)$.

Method 2: Distance between two planes

$$p: \mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix} = 8$$

Let the equation of π be $\mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix} = a$

Distance between p and $\pi = \left|\frac{8-a}{\sqrt{6^2+3^2+(-2)^2}}\right| = \left|\frac{8-a}{7}\right|$

$$\left|\frac{8-a}{7}\right| = 11$$

$$|8-a| = 77$$

$$8-a = 77 \text{ or } -77$$

$$a = -69 \text{ or } 85$$

	Hence possible equations of π are $\mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=-69$ or $\mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=85$ To find the point of intersection between l and π : l: $\mathbf{r}=\begin{pmatrix}-2\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\-2\end{pmatrix}$, $\lambda\in\mathbb{R}$ For π : $\mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=-69$, $\left[\begin{pmatrix}-2\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\-2\end{pmatrix}\right]\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=-69$ $-12+6-8+\left(-18+3+4\right)\lambda=-69$ $-11\lambda=-55$ $\lambda=5$ $\overrightarrow{OM}=\begin{pmatrix}-17\\7\\-6\end{pmatrix}$ Similarly, for π : $\mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=85$, $\left[\begin{pmatrix}-2\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\-2\end{pmatrix}\right]\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=85$ $-12+6-8+\left(-18+3+4\right)\lambda=85$ $-11\lambda=99$ $\lambda=-9$ $\overrightarrow{OM}=\begin{pmatrix}25\\-7\\22\end{pmatrix}$. Hence, possible points of intersections between l and π are $(-17, 7, -6)$ and $(25, -7, 22)$.

4 NYJC Prelim 9758/2021/01/Q10

One day, Eddie came home from a birthday party and brought back a helium filled balloon. After playing with it, he accidentally released the balloon at the point (1, 2, 3) and it floated vertically upwards at a speed of 1 unit per second. t seconds later, a sudden gust of wind caused the balloon to move in the direction of $\mathbf{i}+4\mathbf{j}+6\mathbf{k}$.

You may assume that $z=0$ refers to the horizontal ground.

(i) Find the angle in which the balloon has changed in direction after the gust of wind blew it away. [3]

(ii) Find the Cartesian equation of the plane that the balloon is moving along. [3]

(iii) Given that the balloon eventually stayed at the point (2, 6, 12) on the ceiling, find the time t when the gust of wind blew the balloon away. [3]

Eddie decides to shoot the balloon down with his catapult.

(iv) Assume he was holding his catapult at (3, 2, 1) initially and he walked along the path parallel to $2\mathbf{i}+\mathbf{j}$. Find the position vector of the point where he should place his catapult so that the distance between his catapult and the balloon is at its minimum. Hence find this distance. [4]

	Suggested Solution
(i)	Let θ be the required angle. $\cos\theta=\dfrac{\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\4\\6\end{pmatrix}}{\sqrt{1}\sqrt{1^2+4^2+6^2}}=\dfrac{6}{\sqrt{53}}$ $\theta=34.5^o$ (1 d.p)
(ii)	$\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}0\\0\\1\end{pmatrix}+s\begin{pmatrix}1\\4\\6\end{pmatrix}$ $\mathbf{n}=\begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}1\\4\\6\end{pmatrix}=\begin{pmatrix}-4\\1\\0\end{pmatrix}$ $\mathbf{r}\bullet\begin{pmatrix}-4\\1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}-4\\1\\0\end{pmatrix}=-2$ $-4x+y=-2$

(iii)

$$\begin{pmatrix}2\\6\\12\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}0\\0\\1\end{pmatrix}+s\begin{pmatrix}1\\4\\6\end{pmatrix}$$

$2=1+s\Rightarrow s=1$

$12=3+t+6(1)\Rightarrow t=3$

(iv)

$l:\ \mathbf{r}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}$, $\lambda\in\mathbb{R}$ where l represents the path of the catapult

Let the final position of the balloon be X and the position of the catapult to achieve minimum distance between catapult and balloon be F.

Since F lies on l, $\overrightarrow{OF}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}$ and

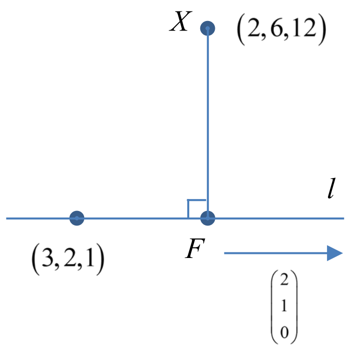
$$\overrightarrow{XF}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}-\begin{pmatrix}2\\6\\12\end{pmatrix}=\begin{pmatrix}1\\-4\\-11\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}$$

For XF to be minimum, $\overrightarrow{XF}\cdot\begin{pmatrix}2\\1\\0\end{pmatrix}=0$

$$\left[\begin{pmatrix}1\\-4\\-11\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\0\end{pmatrix}=0$$

$$2-4+(4+1)\lambda=0$$

$$\lambda=\frac{2}{5}$$



$$\overrightarrow{OF}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\frac{2}{5}\begin{pmatrix}2\\1\\0\end{pmatrix}=\frac{1}{5}\begin{pmatrix}19\\12\\5\end{pmatrix}$$

$$\overrightarrow{XF}=\begin{pmatrix}1\\-4\\-11\end{pmatrix}+\frac{2}{5}\begin{pmatrix}2\\1\\0\end{pmatrix}=\frac{1}{5}\begin{pmatrix}9\\-18\\-55\end{pmatrix}$$

$$XF=\sqrt{\left(\frac{9}{5}\right)^2+\left(\frac{-18}{5}\right)^2+(-11)^2}=11.7\text{ (3 s.f)}$$

$\therefore$ He should place the catapult at the point with position vector $\frac{1}{5}\begin{pmatrix}19\\12\\5\end{pmatrix}$ and the distance is 11.7 units.