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PRESBYTERIAN HIGH SCHOOL

ADDITIONAL MATHEMATICS **4049/02**
Paper 2

21 August 2023 Monday 2 hrs 15 min

MARKING SCHEME

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This question paper consists of **18** printed pages and **0** blank page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}ab\sin C$$

1 An object is heated in an oven until it reaches a temperature of X °C. It is then allowed to cool. Its temperature, θ °C, when it has cooled for time t minutes, is given by $\theta = 30 + 100(0.8)^{\frac{t}{6}}$.

(a) Find the value of X. [1]

$X = 30 + 100(0.8)^{\frac{0}{6}}$

$X = 130$ B1

(b) Find the value of θ when $t = 8$. [1]

$\theta = 30 + 100(0.8)^{\frac{8}{6}}$

$\theta = 104$ B1

(c) Find the value of t when $\theta = 95$. [3]

$95 = 30 + 100(0.8)^{\frac{t}{6}}$

$65 = 100(0.8)^{\frac{t}{6}}$

$(0.8)^{\frac{t}{6}} = 0.65$ M1

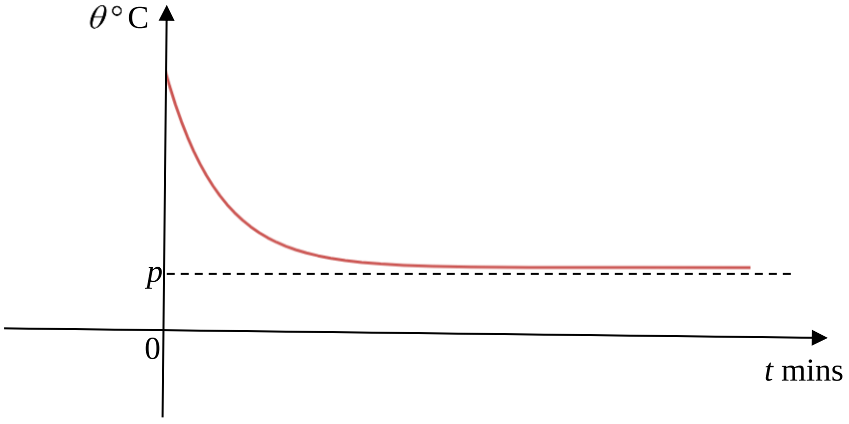
$\lg(0.8)^{\frac{t}{6}} = \lg 0.65$

$\frac{t}{6}\lg(0.8) = \lg 0.65$ M1

$t = \frac{6\lg 0.65}{\lg 0.8}$

$t = 11.6$ A1

(d) A sketch of the graph of θ against t is given below.

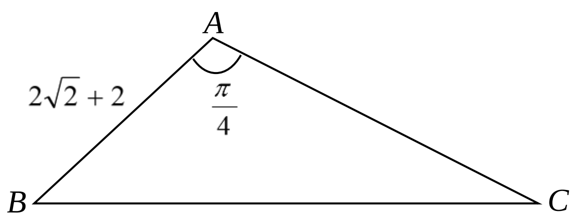


State the value of p. [1]

$p = 30$ B1

A calculator must not be used in this question.

2 **(a)** In the diagram, triangle ABC has an area of $\left(8\sqrt{2}+4\right)$ cm^2, angle $BAC = \dfrac{\pi}{4}$ radian and $AB = \left(2\sqrt{2}+2\right)$ cm. Find the length of AC, leaving your answer in the form $\left(p\sqrt{2}+q\right)$ cm, where p and q are integers. [5]



Area = $\dfrac{1}{2}\times AB\times AC\times \sin\angle BAC$

$8\sqrt{2}+4=\dfrac{1}{2}\left(2\sqrt{2}+2\right)(AC)\left(\dfrac{\sqrt{2}}{2}\right)$ M1

$8\sqrt{2}+4=\dfrac{1}{2}\left(2+\sqrt{2}\right)(AC)$

$16\sqrt{2}+8=\left(2+\sqrt{2}\right)(AC)$

$AC=\dfrac{16\sqrt{2}+8}{2+\sqrt{2}}$ M1

$=\dfrac{16\sqrt{2}+8}{2+\sqrt{2}}\times\dfrac{2-\sqrt{2}}{2-\sqrt{2}}$ M1

$=\dfrac{32\sqrt{2}-32+16-8\sqrt{2}}{4-2}$ M1

$=\dfrac{24\sqrt{2}-16}{2}$

$=12\sqrt{2}-8$ A1

(b) Find $\cos 75°$, giving your answer in the form $\dfrac{\sqrt{a}-\sqrt{b}}{4}$, where a and b are integers. [3]

$\cos 75° = \cos(30°+45°)$

$\cos 75° = \cos 30°\cos 45° - \sin 30°\sin 45°$ M1

$\cos 75° = \dfrac{\sqrt{3}}{2}\times\dfrac{\sqrt{2}}{2}-\dfrac{1}{2}\times\dfrac{\sqrt{2}}{2}$ M1

$\cos 75° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ A1

3 **(a)** Prove that $\text{cosec}2x - \cot 2x = \tan x$. [3]

$$\cos ec2x - \cot 2x = \frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x}$$

$$\cos ec2x - \cot 2x = \frac{1-\cos 2x}{\sin 2x}$$ M1

$$\cos ec2x - \cot 2x = \frac{2\sin^2 x}{2\sin x \cos x}$$ M1 for either formula

$$\cos ec2x - \cot 2x = \frac{\sin x}{\cos x}$$

$$\cos ec2x - \cot 2x = \tan x$$ AG1

(a) Hence solve $\text{cosec}2x - \cot 2x = 2\sec^2 x - 3$ for $0° \le x \le 360°$. [5]

$\cos ec2x - \cot 2x = 2\sec^2 x - 3$

$\tan x = 2\sec^2 x - 3$

$\tan x = 2\left(1+\tan^2 x\right) - 3$ M1

$\tan x = 2 + 2\tan^2 x - 3$

$2\tan^2 x - \tan x - 1 = 0$ M1

$(2\tan x + 1)(\tan x - 1) = 0$

$\tan x = -0.5$ or $\tan x = 1$ M1

Basic angle $= 26.6°$ or $= 45°$

$x = 180 - 26.6°, 360° - 26.6°$ $x = 45°, 180° + 45°$

$x = 45°, 153.4°, 225°, 333.4°$ A1, A1

4 **(a)** Solve $9^x + 5 = 2\left(3^{x+1}\right)$. [5]

$3^{2x} + 5 = 2\left(3^x \times 3\right)$

Let $u = 3^x$

$u^2 + 5 = 6u$ M1

$u^2 - 6u + 5 = 0$

$(u-1)(u-5) = 0$

$u = 1$ or $u = 5$ M1

$3^x = 1$ or $3^x = 5$

$x = 0$ $\quad x = \dfrac{\lg 5}{\lg 3}$ M1

$x = 0$ $\quad x = 1.46$ A1, A1

(b) Solve $2\log_4\left[\log_{100}(x^2+9)-\log_{100}x\right]=-1$. [5]

$$2\log_4\left[\log_{100}(x^2+9)-\log_{100}x\right]=-1$$

$$\log_4\left[\log_{100}(x^2+9)-\log_{100}x\right]=-\frac{1}{2}$$

$$\left[\log_{100}(x^2+9)-\log_{100}x\right]=4^{-\frac{1}{2}}$$ M1

$$\log_{100}\frac{x^2+9}{x}=\frac{1}{2}$$ M1 quotient law

$$\frac{x^2+9}{x}=100^{\frac{1}{2}}$$

$$\frac{x^2+9}{x}=10$$ M1

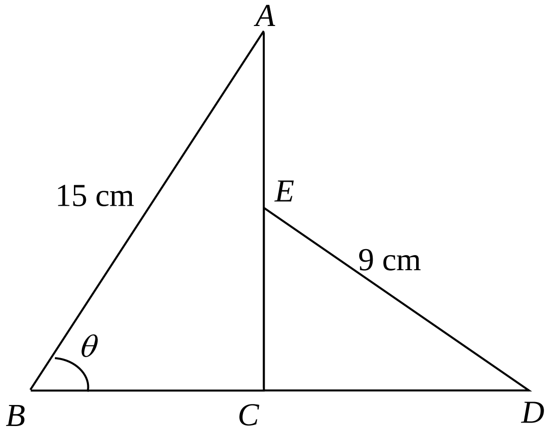
$$x^2+9=10x$$

$$x^2-10x+9=0$$ M1

$$(x-1)(x-9)=0$$

$x=1$ or $x=9$ A1

5 The diagram shows a quadrilateral *ABCDE* where triangle *ABC* is similar to triangle *DEC*. *AB* = 15 cm, *DE* = 9 cm, angle *ACD* = 90 ° and angle *ABC* is a variable angle θ, where 0 ° < θ < 90 °.



(a) Show that the perimeter, *P* cm, of the quadrilateral is given by $P = 24 + 24\sin\theta + 6\cos\theta$. [4]

In ΔABC, $\cos\theta = \dfrac{BC}{15}$ M1 either

$BC = 15\cos\theta$

$\sin\theta = \dfrac{AC}{15}$

$AC = 15\sin\theta$

In ΔDCE, $\cos\theta = \dfrac{EC}{9}$ M1either

$EC = 9\cos\theta$

$\sin\theta = \dfrac{DC}{9}$

$DC = 9\sin\theta$

Therefore $P = 15 + AE + 9 + DB$

$P = 24 + 15\sin\theta - 9\cos\theta + 9\sin\theta + 15\cos\theta$ M1

$P = 24 + 24\sin\theta + 6\cos\theta$ (shown) a.g. A1

(b) Express P in the form $R\sin(\theta+\alpha)+k$. [4]

$24\sin\theta+6\cos\theta=R\sin(\theta+\alpha)$

$R=\sqrt{6^2+24^2}=\sqrt{612}\ or\ 6\sqrt{17}$ M1

$\tan\alpha=\dfrac{6}{24}$ M1

$\alpha=14.0°$ M1

$P=\sqrt{612}\sin(\theta+14.0°)+24$ A1

(c) Find the value of θ when the perimeter is 38 cm. [2]

$24+\sqrt{612}\sin(\theta+14.03°)=38$

$\sqrt{612}\sin(\theta+14.03°)=14$ M1

$\sin(\theta+14.03°)=\dfrac{14}{\sqrt{612}}$

$\theta+14.03°=34.46°$

$\theta=20.4°$ (1 d.p.) A1

6 A piece of wire 60 cm long is bent to form the shape shown in the figure. This shape consists of a semi-circular arc, radius, *r* cm, and an equilateral triangle on the opposite ends of a rectangle of length 4*x* cm.

(a) Express *x* in term of *r*. [2]

$$2(2r)+2(4x)+\pi r=60 \qquad \text{M1}$$

$$4r+\pi r+8x=60$$

$$8x=60-4r-\pi r$$

$$x=\frac{60-4r-\pi r}{8} \qquad \text{A1}$$

(b) Hence show that the area enclosed, A cm^2, is given by

$$A=60r+r^2(\sqrt{3}-4-\frac{\pi}{2})$$ [3]

$$A=\frac{1}{2}\times 2r\times 2r\times \sin 60°+4x\times 2r+\frac{1}{2}\pi r^2$$ M1 for 2 areas, M2 for all 3 areas

$$A=2r^2\times\frac{\sqrt{3}}{2}+8r\times\frac{60-4r-\pi r}{8}+\frac{1}{2}\pi r^2$$

$$A=\sqrt{3}r^2+60r-4r^2-\pi r^2+\frac{1}{2}\pi r^2$$

$$A=60r+\sqrt{3}r^2-4r^2-\frac{1}{2}\pi r^2$$

$$A=60r+r^2(\sqrt{3}-4-\frac{\pi}{2}) \text{ (shown)} \qquad \text{AG1}$$

(c) Calculate the value of r for which A has a stationary value. Find this value of A and determine whether it is a maximum or a minimum. [5]

$$\frac{dA}{dr} = 60 + 2r(\sqrt{3} - 4 - \frac{\pi}{2})$$ M1

$$\frac{dA}{dr} = 0 \Rightarrow 60 + 2r(\sqrt{3} - 4 - \frac{\pi}{2}) = 0$$ M1

$$2r(\sqrt{3} - 4 - \frac{\pi}{2}) = -60$$

$$r = \frac{-60}{2(\sqrt{3} - 4 - \frac{\pi}{2})}$$

$r = 7.82$ cm A1

$$A = 60(7.815) + (7.815)^2(\sqrt{3} - 4 - \frac{\pi}{2})$$

$A = 234\ cm^2$ A1

$$\frac{d^2A}{dr^2} = 2(\sqrt{3} - 4 - \frac{\pi}{2}) < 0$$

Therefore, the area is maximum

} A1

7 The equation of the curve is $y=(2x+1)\left(\sqrt{x-3}\right)$.

(a) Show that $\frac{dy}{dx}$ can be written in the form $\frac{6x-11}{2\sqrt{x-3}}$. [4]

$$\frac{dy}{dx}=(2x+1)\times\frac{1}{2}(x-3)^{-\frac{1}{2}}+(x-3)^{\frac{1}{2}}(2)$$ M1, M1

$$\frac{dy}{dx}=\frac{1}{2}(x-3)^{-\frac{1}{2}}\left((2x+1)+4(x-3)\right)$$

$$\frac{dy}{dx}=\frac{1}{2}(x-3)^{-\frac{1}{2}}\left(2x+1+4x-12\right)$$ M1

$$\frac{dy}{dx}=\frac{1}{2}(x-3)^{-\frac{1}{2}}\left(6x-11\right)$$

$$\frac{dy}{dx}=\frac{6x-11}{2\sqrt{x-3}}$$ AG1

(b) A particle moves along the curve in such a way that the *x*-coordinate is increasing at a constant rate of 3 units per second. Find the rate of change of *y* when *x* = 7. [2]

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{6(7)-11}{2\sqrt{7-3}}\times 3$$ M1

$$\frac{dy}{dt}=23.25 \quad \text{unit/s}$$ A1

(c) Use the result from **(a)** to evaluate $\int_4^7 \frac{3(6x-11)}{\sqrt{x-3}}\,dx$. [4]

$$\int_4^7 \frac{(6x-11)}{2\sqrt{x-3}}\,dx = \left[(2x+1)\sqrt{x-3}\right]_4^7$$ M1

$$6\int_4^7 \frac{(6x-11)}{2\sqrt{x-3}}\,dx = 6\left[(2x+1)\sqrt{x-3}\right]_4^7$$ M1

$$\int_4^7 \frac{3(6x-11)}{\sqrt{x-3}}\,dx = 6\left[(2(7)+1)\sqrt{7-3}-(2(4)+1)\sqrt{4-3}\right]$$ M1

$$\int_4^7 \frac{3(6x-11)}{\sqrt{x-3}}\,dx = 126$$ A1

8 **(a)** Factorise $x^3 - 27k^3$ as a product of a linear and a quadratic factor. [2]

$(x-3k)\left(x^2+3kx+9k^2\right)$ B1, B1

(b) Factorise $x^2-(3k-1)x-3k$. [1]

$(x+1)(x-3k)$ B1

(c) The equation $x^3 - 27k^3 = x^2-(3k-1)x-3k$ has only 1 real root. Find the set of values of the constant *k*. [6]

$x^3 - 27k^3 = x^2-(3k-1)x-3k$

$(x-3k)\left(x^2+3kx+9k^2\right)=(x+1)(x-3k)$ M1

$(x-3k)\left(x^2+3kx+9k^2\right)-(x+1)(x-3k)=0$

$(x-3k)\left(x^2+(3k-1)x+9k^2-1\right)=0$ M1

Since only 1 real root

$(3k-1)^2-4(9k^2-1)<0$ M1

$9k^2-6k+1-36k^2+4<0$

$-27k^2-6k+5<0$ M1

$27k^2+6k-5>0$

$(9k+5)(3k-1)>0$ M1 for $-\frac{5}{9}$ *and* $\frac{1}{3}$ seen

$k<-\frac{5}{9}$ *or* $k>\frac{1}{3}$ A1

9 The equation of the circle, C, is $x^2 + y^2 - 6x + 10y - 66 = 0$.

(a) Find the coordinates of the centre of C and the radius of C. [4]

Centre $= \left(\frac{-6}{-2}, \frac{10}{-2}\right)$ M1

Centre is $(3, -5)$ A1

Radius $= \sqrt{(3)^2 + (-5)^2 - (-66)}$ M1

$= 10$ units A1

(b) Write down an equation of a vertical tangent to the circle. [1]

$x = -7$ or $x = 13$ B1

The point *A*(-5, 1) lies on the circle.

(c) Find the equation of the tangent to the circle at point *A*. [3]

$m = \frac{1-(-5)}{-5-(3)}$ M1

$m_{AB} = -\frac{3}{4}$

Gradient of tangent $m_{AB} = -\frac{1}{-\frac{3}{4}} = \frac{4}{3}$

$y - 1 = \frac{4}{3}(x - (-5))$ M1

$3y - 3 = 4x + 20$

$3y = 4x + 23$ A1

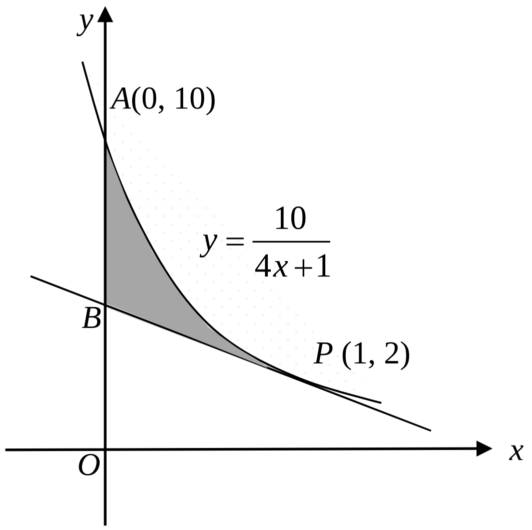
(d) *AB* is the diameter of the circle and *P* is the point (0, 6). Explain why the angle *APB* is an acute angle. [2]

Distance of *P* from centre $\sqrt{(3-0)^2 + (-5-6)^2} = \sqrt{130} > 10$ M1

P is outside the circle, angle *APB* is an acute angle A1

10 The diagram shows part of the curve $y = \dfrac{10}{4x+1}$ intersecting the y-axis at $A(0, 10)$.

The tangent to the curve at the point $P(1, 2)$ intersects the y-axis at B.



(a) Show the coordinates of B is (0 , 3.6). [4]

$y = \dfrac{10}{4x+1} = 10(4x+1)^{-1}$

$\dfrac{dy}{dx} = -10(4x+1)^{-2}(4)$ M1

$\dfrac{dy}{dx} = -40(4x+1)^{-2}$

When $x = 1$ $\dfrac{dy}{dx} = -40(4(1)+1)^{-2}$

$\dfrac{dy}{dx} = -1.6$ M1

$\dfrac{y-2}{0-1} = -1.6$ M1

$y - 2 = 1.6$

$y = 3.6$

Coordinate of B is (0, 3.6) AG1

(b) Find the **exact** area of the shaded region. [5]

$$Area = \int_0^1 \frac{10}{4x+1}\,dx - \frac{1}{2}(3.6+2)(1)$$ M1, M1

$$Area = \left[\frac{10\ln(4x+1)}{4}\right]_0^1 - 2.8$$ M1

$$Area = \left[\frac{10\ln(4+1)}{4} - \frac{10\ln(1)}{4}\right] - 2.8$$ M1

$$Area = \frac{5}{2}\ln 5 - 2.8 \text{ unit}^2$$ A1