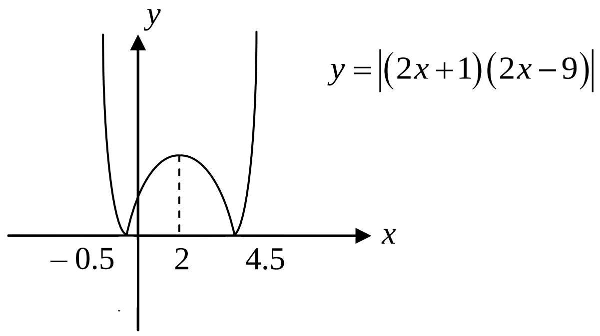
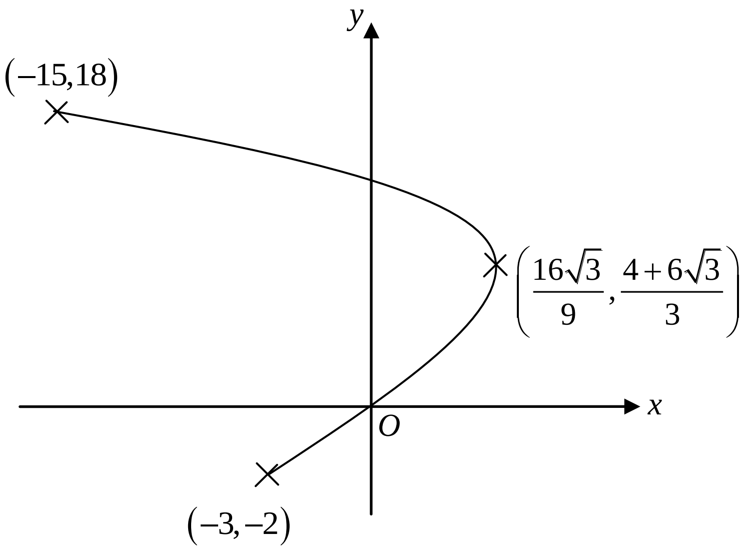
Qn	Solution
1	**Recurrence Relations**
(i)	$u_n = 1.05u_{n-1} - 40, \quad u_0 = 600, \quad n \geq 1$ $u_1 = 1.05u_0 - 40$ $u_2 = 1.05(1.05u_0 - 40) - 40$ $\quad = 1.05^2 u_0 - 40(1.05 + 1)$ $u_3 = 1.05\left[1.05^2 u_0 - 40(1.05+1)\right] - 40$ $\quad = 1.05^3 u_0 - 40(1.05^2 + 1.05 + 1)$ Thus, $u_n = 1.05^n u_0 - 40(1.05^{n-1} + \ldots + 1.05 + 1)$ $\quad = 1.05^n(60) - \frac{40(1.05^n - 1)}{1.05 - 1}$ $\quad = 1.05^n(600) - 800(1.05^n - 1)$ $\quad = 800 - 200(1.05^n)$
(ii)	At the start of 2020, $n = 8$. $u_8 = 800 - 200(1.05)^8 = 505 \ (3\text{s.f.})$ The predicted population is 505 000. The population will eventually be extinct in this lake. There will be no more fish in this lake.
(iii)	The proposed number to be harvested should be $0.05 \times 600\,000 = 30\,000$

Qn	Solution
2	**Functions**
(i)	 Least value of $a = -0.5$ Greatest value of $b = 2$
(ii)	Domain of f^{-1} is the range of f under the restricted domain. $\text{D}_{\text{f}^{-1}} = [0, 25]$ Under the restricted domain $\text{f}(x) = -(2x+1)(2x-9)$ Let $y = -\left[4x^2 - 16x - 9\right]$

	$y=-4\left[x^2-4x\right]+9$ $y=-4\left[(x-2)^2-4\right]+9$ $y=-4(x-2)^2+25$ $(x-2)^2=\dfrac{y-25}{-4}$ $x=2\pm\sqrt{\dfrac{25-y}{4}}$ Therefore $x=2-\dfrac{\sqrt{25-x}}{2}$ or $x=2+\dfrac{\sqrt{25-x}}{2}$ (reject $\because\ -0.5\le x\le 2$) Hence $f^{-1}(x)=2-\dfrac{\sqrt{25-x}}{2}$
(iii)	$D_f=[-0.5,2]\xrightarrow{f}R_f=[0,25]\xrightarrow{g}R_{gf}=[\ln 2,\ln 27]$

Qn	Solution
3	**Curve Sketching & Differentiation (Parametric, Tan/Norm)**
(i)	$x=4t-t^3 \qquad y=t^2+3t$ $\dfrac{dx}{dt}=4-3t^2 \quad \dfrac{dy}{dt}=2t+3$ $\therefore \dfrac{dy}{dx}=\dfrac{2t+3}{4-3t^2}$ When $t=-1$, $x=-4-(-1)=-3$ $y=1+3(-1)=-2$ $\dfrac{dy}{dx}=\dfrac{2(-1)+3}{4-3(-1)^2}=1$ <u>Alternative method</u> Using GC, When $t=-1$, $x=-3$, $y=-2$, $\dfrac{dy}{dx}=1$
	Therefore, the equation of the tangent at $t=-1$ is $$y-(-2)=x-(-3)$$ $$y=x+1$$
(ii)	If the normal at P is perpendicular to the tangent at $t=-1$, then the tangent at P will be parallel to the tangent at $t=-1$. $\therefore \dfrac{dy}{dx}=\dfrac{2t+3}{4-3t^2}=1$

	$\Rightarrow 2t+3=4-3t^2$ $\Rightarrow 3t^2+2t-1=0$ $\Rightarrow (3t-1)(t+1)=0$ $\Rightarrow t=\frac{1}{3}$ or -1 (rejected) Therefore, at point P, the value of $t=\frac{1}{3}$.
(iii)	 At $t=-1$, from part (i) $x=-3$ $y=-2$ At $t=3$, $x=4(3)-3^3=-15$ $y=3^2+3(3)=18$ From part (i), $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2t+3}{4-3t^2}$ For the gradient to be undefined, denominator = 0. $\therefore 4-3t^2=0 \Rightarrow t=\sqrt{\frac{4}{3}}$ $(-\sqrt{\frac{4}{3}}$ is rejected $\because t\in[-1,3])$ At $t=\sqrt{\frac{4}{3}}=\frac{2\sqrt{3}}{3}$, $x=4\left(\frac{2\sqrt{3}}{3}\right)-\left(\frac{2\sqrt{3}}{3}\right)^3=\frac{16\sqrt{3}}{9}$ $y=\left(\frac{2\sqrt{3}}{3}\right)^2+3\left(\frac{2\sqrt{3}}{3}\right)=\frac{4+6\sqrt{3}}{3}$

Qn	Solution
4	**Area & Volume**
(i)	Area of $R=\int_0^a \left(e^{\frac{x}{2}}+1\right)^2 \mathrm{d}x$ $=\int_0^a \left(e^x+2e^{\frac{x}{2}}+1\right) \mathrm{d}x$ $=\left[e^x+4e^{\frac{x}{2}}+x\right]_0^a$ $=e^a+4e^{\frac{a}{2}}+a-5$
(ii)	$f^{-1}(x)=e^{\frac{x}{2}}+1$ $\pi\int_0^2 \left[f^{-1}(y)\right]^2 \mathrm{d}y=\pi\int_0^2 \left(e^{\frac{y}{2}}+1\right)^2 \mathrm{d}y$ $=\pi\left(e^2+4e+2-5\right)$ from **(i)** $=\pi\left(e^2+4e-3\right)$ It is the volume of revolution formed when the region bounded by the curve $y=2\ln(x-1)$, the x-axis, the y-axis, and the line y = 2 is rotated completely about the y-axis. <u>Alternatively</u> It is the volume of revolution formed when the region bounded by the curve $y=e^{\frac{x}{2}}+1$, the x-axis, the y-axis, and the line x = 2 is rotated completely about the x-axis.

Qn	Solution
5	**Vectors**
(i)	l_1: $\mathbf{r}=\begin{pmatrix}4\\5\\-6\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\3\end{pmatrix}, \lambda\in\mathbb{R}$ p_1: $x+2y-3z=4$ p_1: $\mathbf{r}\cdot\begin{pmatrix}1\\2\\-3\end{pmatrix}=4$ Since $\begin{pmatrix}-1\\-2\\3\end{pmatrix}=-\begin{pmatrix}1\\2\\-3\end{pmatrix}$, the line l_1 is parallel to n_1. The line l_1 is perpendicular to the plane p_1.

$$\begin{pmatrix} 4-\lambda \\ 5-2\lambda \\ -6+3\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 4$$

$4-\lambda+10-4\lambda+18-9\lambda=4$

$28=14\lambda$

$\lambda=2$

Coordinates of foot of perpendicular is $(2,1,0)$.

(ii)

$$l_{2}: \mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ \mu \in \mathbb{R}$$

$$\mathbf{n}_2 = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

$$p_2: \ \mathbf{r} \cdot \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

p_2: $x+4y+3z=2$ (shown)

Since $\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, l_1 and l_2 are not parallel and since $\begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = 6 \neq 2$, l_1 is not on p_2, the two lines are on different planes, Hence, they are skew lines.

(iii)

p_1: $x+2y-3z=4$

p_2: $x+4y+3z=2$

Using GC, $l_{3}: \mathbf{r} = \begin{pmatrix} 6 \\ -1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 9 \\ -3 \\ 1 \end{pmatrix}, \ \gamma \in \mathbb{R}$

(iv)

p_3: $x-2y+7z-\beta-\alpha(2x+2y-z+1)=0$.

$$p_3: \ \mathbf{r} \cdot \begin{pmatrix} 1-2\alpha \\ -2-2\alpha \\ 7+\alpha \end{pmatrix} = \alpha+\beta$$

Given that the three planes have no point in common,

$$\begin{pmatrix} 1-2\alpha \\ -2-2\alpha \\ 7+\alpha \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -3 \\ 1 \end{pmatrix} = 0 \qquad \begin{pmatrix} 6 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1-2\alpha \\ -2-2\alpha \\ 7+\alpha \end{pmatrix} \neq \alpha+\beta$$

$9-18\alpha+6+6\alpha+7+\alpha=0$ $\qquad$ $6-12\alpha+2+2\alpha \neq \alpha+\beta$

$11\alpha=22$ $\qquad$ $8-11\alpha \neq \beta$

$\alpha=2$ $\qquad$ $\beta \neq -14$

$\therefore \ \alpha=2$ and $\beta \neq -14$

Qn	Solution
6	**Sampling Methods**
(i)	Quota Sampling
(ii)	The sample obtained is **not representative** of the population because the **proportion of students selected from each stratum is different**. OR The sample obtained is **non-random** because **every student does not have an equal chance of being selected**.
(iii)	Stratified sampling Divide the population of 100 into four non-overlapping strata according to levels and gender. The students from each stratum are randomly selected using simple random sampling with the sample size being proportional to the relative size of each stratum,

	Males	Females
JC 1	$\frac{480}{1600}\times 100 = 30$	$\frac{560}{1600}\times 100 = 35$
JC 2	$\frac{240}{1600}\times 100 = 15$	$\frac{320}{1600}\times 100 = 20$

Qn	Solution
7	**Hypothesis Testing**
(i)	For a one-tailed test, we are testing whether there is a definite increase or definite decrease in a population parameter. As for a two-tailed test, we are merely testing for a change in a population parameter.
(ii)	See below

$\bar{x} = \frac{1279}{50} + 30 = 55.58$; $\sum x = 55.58 \times 50 = 2779$

$s^2 = \frac{1}{49}\left[155233 - \frac{2779^2}{50}\right] = 15.84040816$

Let μ denote the population mean weight of the boys.

$H_0: \mu = m$

$H_1: \mu > m$

Since n is large, by CLT: $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ approximately.

Test statistic: $Z = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim N(0,1)$

Level of Significance: 5%

Critical Region: Reject H_0 if z-value > 1.6449

i.e. $\frac{55.58 - m}{\sqrt{15.840/50}} > 1.6449$

$m < 54.654$

$m < 54.7$ (3s.f.)

Qn	Solution
8	**PnC, Probability**

(i)

Method 1

Required Probability

$$=\frac{2!\times 6!}{7!}=\frac{2}{7}$$

Method 2

Required Probability

$$=\frac{5!\times 6\times 2}{7!}=\frac{2}{7}$$

(ii)

Let A be the event that no two women are seated next to each other.
Let B be the event that couple is seated together.

Method 1

Required probability

$$=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$$

$$=\frac{\left(\frac{3!\times 3!}{7!}\right)\times 2}{\left(\frac{2}{7}\right)}=\frac{1}{20}$$

Method 2

Required probability

$$=\frac{\mathrm{n}(A\cap B)}{\mathrm{n}(B)}$$

$$=\frac{2!\times 3!\times 3!}{2!\times 6!}$$

$$=\frac{1}{20}$$

Method 1

From (ii) $\mathrm{P}(A\mid B)=\frac{1}{20}$.

$$\mathrm{P}(A)=\frac{3!\times 4!}{7!}$$

$$=\frac{1}{35}$$

Since $\mathrm{P}(A\mid B)\neq \mathrm{P}(A)$,

the events are not independent.

Method 2

$$\mathrm{P}(A)=\frac{3!\times 4!}{7!}=\frac{1}{35}$$

From (i) $\mathrm{P}(B)=\frac{2}{7}$, (ii) $\mathrm{P}(A\cap B)=\frac{1}{70}$.

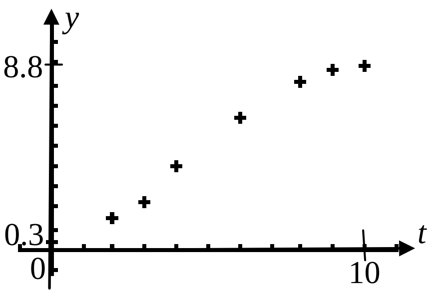
$$\mathrm{P}(A)\mathrm{P}(B)=\left(\frac{1}{35}\right)\left(\frac{2}{7}\right)=\frac{2}{245}$$

Since $\mathrm{P}(A\cap B)\neq \mathrm{P}(A)\mathrm{P}(B)$,

the events are not independent.

Qn	Solution
9	**Correlation & Regression**

(i)



(ii)

Using GC, r = 0.989.
Using regression line of y on t, $y=0.041899+0.94916t$

When $t=7$, $y=6.6860=6.69\,\text{cm}^2$ (3s.f.)
Since $t=7$ is within the data range and r = 0.989 is close to 1, the answer is reliable.

(iii)

When $t=80$, $y=75.974=76.0\ \text{cm}^2$ (3s.f)

	Note that 76.0cm^2 > 64cm^2 (the total area of the slice of bread.) The regression line may not be suitable as it is impossible for the bread to keep growing mould. Also, from the scatter diagram, it shows that as t increases (after 8 days, the mould starts to grow at a decreasing rate. Hence, a linear model may not be appropriate.

Qn	Solution
10	**Normal Distribution & Sampling**
(i)	Let X be the time taken for Miss Lau to wrap a large hamper (in minutes). $X \sim N\left(18, 4^2\right)$ Let Y be the time taken for Miss Lau to wrap a small hamper (in minutes). $Y \sim N\left(10, \sigma^2\right)$ <u>Method 1</u> $P(X \le 11) = P(Y \le 6.5)$ Using G.C., finding the intersection of the 2 graphs, $\sigma = 2$. <u>Method 2</u> $P(X \le 11) = P(Y \le 6.5)$ $0.040059 = P(Z \le \frac{6.5-10}{\sigma})$ $\frac{6.5-10}{\sigma} = -1.7500$ $\sigma = \frac{6.5-10}{-1.7500}$ $= 2$ (Shown)
(ii)	$Y_1 + Y_2 \sim N\left(20, 8\right)$ $Y_1 + Y_2 - X \sim N\left(2, 24\right)$ Required probability $= P(Y_1 + Y_2 \le X)$ $= P(Y_1 + Y_2 - X \le 0)$ $= 0.34155$ $= 0.342$ (to 3 s.f.)
(iii)	Let T be the mean time taken for Miss Lau to wrap a hamper (in minutes). $T = \frac{X_1 + X_2 + Y_1 + Y_2 + ... + Y_n}{2+n}$ $\text{E}(T) = \frac{2(18) + n(10)}{2+n}$ $\text{Var}(T) = \frac{2(4^2) + n(2^2)}{\left(2+n\right)^2}$ $\therefore T \sim N\left(\frac{36+10n}{2+n}, \frac{32+4n}{\left(2+n\right)^2}\right)$ When $n = 8$, $\text{P}(T \le 12) = 0.69146 < 0.7$ When $n = 9$, $\text{P}(T \le 12) = 0.76657 > 0.7$ $\therefore$ Least $n = 9$.

Qn	Solution
11	**Poisson Distribution**
	Each baby delivered is independent of one another and the average rate of number of babies delivered in a month is constant for every month.
(i)	Let C be the number of babies delivered in a month $C \sim \text{Po}(5)$ $P(C > 10) = 1 - P(C \le 10)$ $= 0.013695 = 0.0137\ (3\text{ s.f.})$
(ii)	$P(C \le n) < 0.95$ Using GC, $P(C \le 8) = 0.93191 < 0.95$ $P(C \le 9) = 0.96817 > 0.95$ Therefore largest value of n is 8
(iii)	Since n = 50 is large, by Central Limit Theorem, $\overline{C} \sim \text{N}(5, \frac{5}{50})$ approximately $P(4 < \overline{C} < 6) = 0.998\ (3\text{ s.f.})$
(iv)	Let X be the number of babies delivered in a month in the second village $X \sim \text{Po}(15)$ $X + Y \sim \text{Po}(20)$ $P(X + Y \le 11) = 0.0214\ (3\text{ s.f.})$

Qn	Solution
12	**Binomial Distribution**
(a)	$X \sim \text{B}(n, p)$ $\frac{p_{k+1}}{p_k} = \frac{\binom{n}{k+1} p^{k+1}(1-p)^{n-k-1}}{\binom{n}{k} p^k (1-p)^{n-k}}, \ k = 0, 1, 2, ..., n-1.$ $= \frac{\frac{n!}{(k+1)!(n-k-1)!} p}{\frac{n!}{k!(n-k)!}(1-p)}$ $= \frac{k!(n-k)!\,p}{(k+1)!(n-k-1)!(1-p)} = \frac{(n-k)p}{(k+1)(1-p)}$ (shown) When n = 10 and $p = \frac{1}{3}$, If $p_k > p_{k+1}$, then $\frac{p_{k+1}}{p_k} < 1$,

	$\therefore \dfrac{\frac{1}{3}(10-k)}{\frac{2}{3}(k+1)} < 1$ $10-k < 2(k+1)$ $3k > 8$ $k > \dfrac{8}{3}$ Thus $k =$ 3, 4, …., 9. So $p_3 > p_4 > ... > p_{10}$ Conversely, if $p_k < p_{k+1}$, then $k < \dfrac{8}{3}$. Thus k = 0, 1, 2 . So, $p_0 < p_1 < p_2 < p_3$. Since p_3 is the greatest, therefore the most probable number of successes is 3.
(b) (i)	Let X be the no. of adults, out of 8, having some knowledge of a foreign language. $X \sim \text{B}(8, 0.3)$ $\text{P}(X \le 2) = 0.552$ (3 s.f.)
(ii)	Let Y be the no. of adults, out of 400, having some knowledge of a foreign language. $Y \sim \text{B}(400,\ 0.3)$ Since n = 400 is large, $np = 120 > 5$ and $nq = 280 > 5$, Y ~ N(120 , 280×0.3) i.e Y ~ N(120, 84) approximately. $\text{P}(Y < n) \ge 0.9$ $\text{P}(Y < n - 0.5) \ge 0.9$ using continuity correction From GC, When $n = 132$, $\text{P}(Y < n - 0.5) = 0.89522 < 0.9$ When $n = 133$, $\text{P}(Y < n - 0.5) = 0.91369 > 0.9$ Least value of n =133
	Let T be the no. of adults, out of 400, having some knowledge of the particular foreign language. $T \sim \text{B}(400,\ 0.01)$ Since n = 400 is large and $np = 4 < 5$, T ~ Po(4) approximately. $\text{P}(T \le 4) =$ 0.629 (3 s.f.)