H2 Mathematics 9758
Topic 10: DIFFERENTIATION TECHNIQUES
Tutorial Worksheets

1 By considering the derivative as a limit, show that the derivative of x^3 is $3x^2$.

[N00/I/4]

2 Differentiate each of the following with respect to x simplifying your answer.

(a) $\dfrac{x^2}{\sqrt{4-x^2}}$ **(b)** $\sqrt{1+\sqrt{x}}$ **(c)** $\left(\dfrac{x^3-1}{2x^3+1}\right)^4$

[Ans: **(a)** $\dfrac{x\left(8-x^2\right)}{\left(4-x^2\right)^{\frac{3}{2}}}$ **(b)** $\dfrac{1}{4\sqrt{x\left(1+\sqrt{x}\right)}}$ **(c)** $\dfrac{36x^2\left(x^3-1\right)^3}{\left(2x^3+1\right)^5}$]

3 Find the derivative with respect to x of

(a) $\cos x^\circ$, **(b)** $\cot(1-2x^2)$,

(c) $\tan^3(5x)$, **(d)** $\dfrac{\sec x}{1+\tan x}$.

[Ans: **(a)** $-\dfrac{\pi}{180}\sin x^\circ$ **(b)** $4x\,\text{cosec}^2\left(1-2x^2\right)$ **(c)** $15\tan^2(5x)\sec^2(5x)$ **(d)** $\dfrac{\sec x\left(\tan x-1\right)}{\left(1+\tan x\right)^2}$]

4 Find the derivative with respect to x of

(a) $y=\mathrm{e}^{1+\sin 3x}$ **(b)** $y=x^2\mathrm{e}^{\frac{1}{x}}$

(c) $y=\ln\left[\dfrac{1-x}{\sqrt{1+x^2}}\right]$ **(d)** $y=\dfrac{\ln(2x)}{x}$

(e) $y=\log_2\left(3x^4-\mathrm{e}^x\right)$ **(f)** $y=3^{\ln(\sin x)}$

[Ans: **(a)** $3\mathrm{e}^{1+\sin 3x}\cos 3x$ **(b)** $\mathrm{e}^{\frac{1}{x}}(2x-1)$ **(c)** $-\dfrac{1+x}{(1-x)(1+x^2)}$ **(d)** $\dfrac{1-\ln(2x)}{x^2}$ **(e)** $\dfrac{12x^3-\mathrm{e}^x}{\left(3x^4-\mathrm{e}^x\right)\ln 2}$ **(f)** $3^{\ln(\sin x)}\cot x\ln 3$]

5 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y for each of the following:

(a) $y^3 - 3x^2y + 2x^3 = 1$

(b) $(yx)^2 = x^2 2^x$

(c) $\mathrm{e}^{x+y} = \mathrm{e}^{2x} + y$

(d) $y^2 = x^2 + \sin xy$

[Ans: **(a)** $\frac{2x}{y+x}$ **(b)** $\frac{y}{2}\ln 2$ **(c)** $\frac{2\mathrm{e}^{2x} - \mathrm{e}^{x+y}}{\mathrm{e}^{x+y} - 1}$ **(d)** $\frac{2x + y\cos xy}{2y - x\cos xy}$]

6 Differentiate each of the following with respect to x:

(a) $\tan^{-1}\sqrt{x}$

(b) $5\sin^{-1}\left(\frac{x}{10}\right)$

(c) $\mathrm{e}^{\cos^{-1}2x}$

(d) $x\tan^{-1}(3x) - \ln\frac{1+9x^2}{1-9x^2}$

[Ans: **(a)** $\frac{1}{2\sqrt{x}(1+x)}$ **(b)** $\frac{5}{\sqrt{100-x^2}}$ **(c)** $-\frac{2\mathrm{e}^{\cos^{-1}2x}}{\sqrt{1-4x^2}}$ **(d)** $\tan^{-1}3x - \frac{15x}{1+9x^2} - \frac{18x}{1-9x^2}$]

7 Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following in terms of x and/or y:

(a) $y^3 = x\sin^{-1}x$

(b) $y = a^{2\log_a x}$

(c) $y = (\ln x)^x$

(d) $y = \sqrt[3]{\frac{\mathrm{e}^x(x+1)}{x^2+1}}$, $x > 0$

[Ans: **(a)** $\frac{1}{3y^2}\left(\sin^{-1}x + \frac{x}{\sqrt{1-x^2}}\right)$ **(b)** $2x$ **(c)** $y\ln(\ln x) + \frac{y}{\ln x}$

8 If $\ln y = \tan^{-1}t$, prove that $y\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (2t-1)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 0$.

9 If $y^2 + ay + b = x$ where a and b are constants, show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 = 0$.

10 For each of the following curves, find the gradient at the specified point:

(a) $x^3 + y^3 + 3xy - 1 = 0$ at the point $(2, -1)$

(b) $y^4 + x^2y^2 = 4a^3(x + 4a)$, where a is a constant, at the point $(a, 2a)$

[Ans: **(a)** -1 **(b)** $-\frac{1}{9}$]

11 N14/I/2

The curve C has equation $x^2y + xy^2 + 54 = 0$. Without using a calculator, find the coordinates of the point on C at which the gradient is -1, showing that there is only one such point.

[Ans: $(-3, -3)$]

12 It is given that x and y satisfy the equation $\tan^{-1} x + \tan^{-1} y + \tan^{-1}(xy) = \frac{7}{12}\pi$.

Find the value of y when $x = 1$.

(i) Express $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(xy)$ in terms of x, y and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) Show that, when $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3} - \frac{1}{2\sqrt{3}}$. [N00/I/11]

[Ans: $\frac{1}{\sqrt{3}}$ **(i)** $\frac{1}{1+(xy)^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)$]

13 Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

(a) $x = \frac{1}{1+t^2}$, $y = \frac{t}{1+t^2}$

(b) $x = \frac{1}{2}\left(\mathrm{e}^t - \mathrm{e}^{-t}\right)$, $y = \frac{1}{2}\left(\mathrm{e}^t + \mathrm{e}^{-t}\right)$

(c) $x = a\sec t$, $y = a\tan t$

(d) $x = \mathrm{e}^{3t}\cos 3t$, $y = \mathrm{e}^{3t}\sin 3t$

[Ans: **(a)** $\frac{t^2-1}{2t}$ **(b)** $\frac{\mathrm{e}^{2t}-1}{\mathrm{e}^{2t}+1}$ **(c)** $\operatorname{cosec} t$ **(d)** $\frac{\sin 3t + \cos 3t}{\cos 3t - \sin 3t}$]

14 Differentiate the following with respect to x:

(a) $\ln\left(x+\sqrt{x^2-4}\right)$ **(b)** $\sin^{-1}\left(\sqrt{1-x^4}\right)$ **(c)** $\left(x+x^2\right)^x$

[Ans: **(a)** $\dfrac{1}{\sqrt{x^2-4}}$ **(b)** $\dfrac{-2x}{\sqrt{1-x^4}}$ **(c)** $(x+x^2)^x\left(\dfrac{1+2x}{1+x}+\ln(x+x^2)\right)$]

15 Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y for the following equations:

(a) $\sin y+x=xy$ **(b)** $\ln(1+y)=\tan^{-1}x$ **(c)** $y=\sin(x+y)^2$

[Ans: **(a)** $\dfrac{y-1}{\cos y-x}$ **(b)** $\dfrac{1+y}{1+x^2}$ **(c)** $\dfrac{2(x+y)\cos(x+y)^2}{1-2(x+y)\cos(x+y)^2}$]

16 If $x^2+3xy-y^2=3$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point (1, 1). [Ans: -5, 78]

17 If $y=\mathrm{e}^{kt}\cos pt$, prove that $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}-2k\dfrac{\mathrm{d}y}{\mathrm{d}t}+\left(k^2+p^2\right)y=0$. If $\dfrac{\mathrm{d}y}{\mathrm{d}t}=2p$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}=3p$ when $t=\dfrac{3\pi}{2p}$, calculate k and prove that $p=\dfrac{9\pi}{8\ln 2}$.

[Ans: $k=\dfrac{3}{4}$]

18 Find, by the first principles, the first derivative of $\mathrm{f}(x)=\cos x$, given that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$.