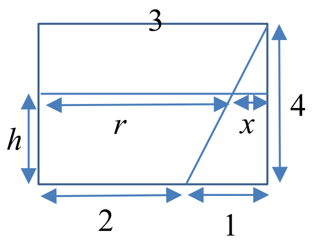
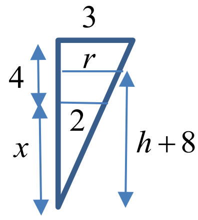
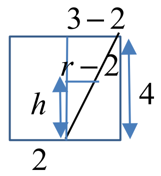
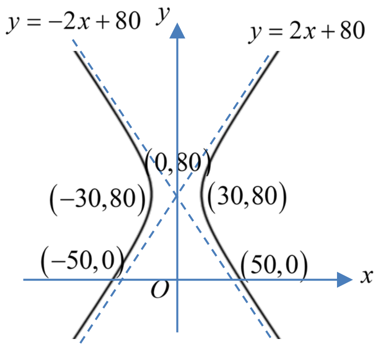
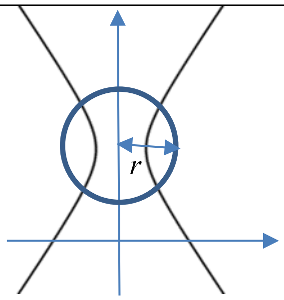
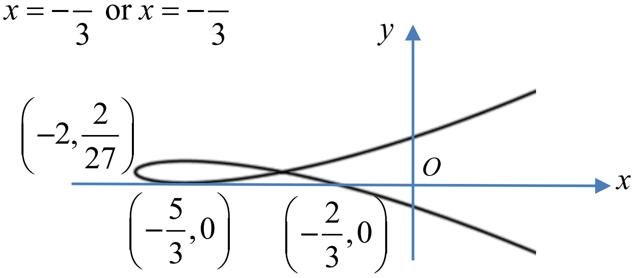
Qn	Solution	Comments
1i	$u_n = S_n - S_{n-1}$ $= \ln(n+1) - \ln(n-1+1)$ $= \ln\frac{n+1}{n} = \ln\left(1+\frac{1}{n}\right)$	
1ii	Observe that as $n \to \infty, \frac{1}{n} \to 0 \therefore u_n \to \ln 1 = 0.$ The sequence decreases and converges to 0.	
2	$\frac{10}{3-x} \geq x$ $\frac{10-3x+x^2}{3-x} \geq 0$ $\frac{\left(x-\frac{3}{2}\right)^2+\frac{31}{4}}{3-x} \geq 0$ Since $\left(x-\frac{3}{2}\right)^2+\frac{31}{4} > 0$ for all real x values, $\frac{1}{3-x} \geq 0$ $3-x > 0$ $x < 3$	If we conclude that the numerator has no real roots, (i.e. $b^2-4ac<0$, it is not sufficient to conclude .. $\frac{1}{3-x} \geq 0$ ..). What else do we need? $10-3x+x^2$ is an expression, not an equation. There are no roots for an expression. Is it correct to only state that $\left(x-\frac{3}{2}\right)^2+\frac{31}{4} \geq 0$? Which real number will be a solution to $\left(x-\frac{3}{2}\right)^2+\frac{31}{4} = 0$?
3i	$f(x) = x^3+ax^2+bx+c$ $f(1) = (1)^3+a(1)^2+b(1)+c = 8$ $a+b+c = 7$ (1) $f(2) = (2)^3+a(2)^2+b(2)+c = 12$ $4a+2b+c = 4$ (2) $f(3) = (3)^3+a(3)^2+b(3)+c = 25$ $9a+3b+c = -2$ (1) By GC, $a = -1.5$, $b = 1.5$ and $c = 7$	Recall factor / remainder theorem from O-level Additional Math. This is an assumed knowledge that you should have.
3ii	$f(x) = x^3-1.5x^2+1.5x+7$ $f'(x) = 3x^2-3x+1.5$ $= 3(x-0.5)^2-0.75+1.5$ $= 3(x-0.5)^2+0.75 > 0$ since $(x-0.5)^2 \geq 0$ Since $f'(x) > 0$, graph of f is a strictly increasing cubic graph which will only cut the x-axis once. Hence, there is only 1 real root for $f(x) = 0$. [In fact, since f(1) = 8 > 0 and f(–2) = –10 < 0, it suggests that there is a root between –2 and 1.]	"Show that the gradient of the curve is always positive. Hence, explain ..." Your explanation should make mention to the fact that $f'(x) > 0$

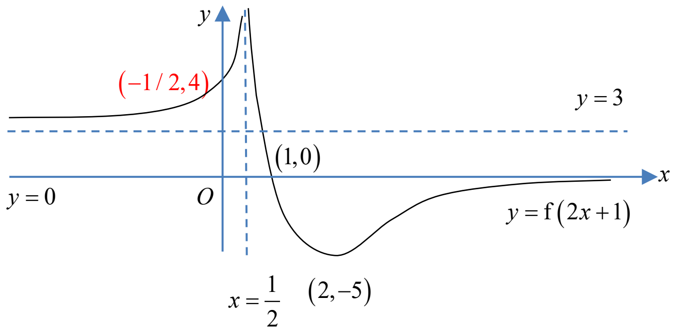
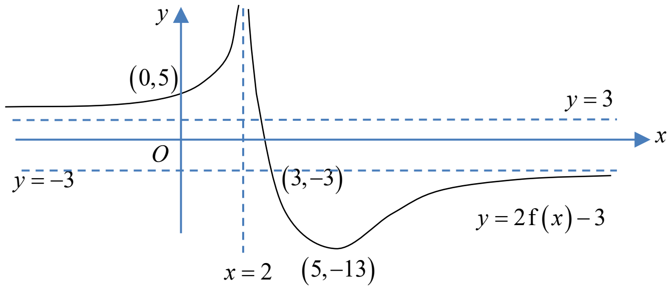
4ai	$\frac{1}{4}=\frac{x}{4-h}$ $x=\frac{4-h}{4}=1-\frac{h}{4}$ $r=3-x=2+\frac{h}{4}$		This is a show question. Clear working has to be shown. A clearly labelled diagram will aid in our explanation. Extra Note: There are no similar trapeziums in the figure. Having the same corresponding angles does not mean that the figures are similar. If the above statement is true, then **all** rectangles are similar to each other.
	<u>Alternative</u> $\frac{2}{x}=\frac{3}{x+4}\Rightarrow x=8$ $\frac{r}{h+8}=\frac{2}{8}$ $r=\frac{h+8}{4}=2+\frac{h}{4}$		
	<u>Alternative</u> $\frac{r-2}{h}=\frac{3-2}{4}$ $r=2+\frac{h}{4}$		
4aii	Volume of water, $V=\frac{1}{3}\pi\left(4+r^2+2r\right)h$ $V=\frac{1}{3}\pi\left(4+\left(2+\frac{h}{4}\right)^2+2\left(2+\frac{h}{4}\right)\right)h$ $=\frac{1}{3}\pi h\left(4+4+h+\frac{h^2}{16}+4+\frac{h}{2}\right)$ $=\frac{1}{3}\pi\left(\frac{h^3}{16}+\frac{3h^2}{2}+12h\right)$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}\pi\left(\frac{3h^2}{16}+3h+12\right)$ When $h=1$, $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$ $9=\frac{1}{3}\pi\left(\frac{3}{16}+3+12\right)\times\frac{\mathrm{d}h}{\mathrm{d}t}$ $=\frac{81\pi}{16}\times\frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{16}{9\pi}$ The rate of increase of the depth of water is $\frac{16}{9\pi}$ m min^{-1}.		We can use the given formula for volume of frustum of right circular cone, with base radius 2 m, and radius r m. You can only substitute constant values before differentiation. $V=\frac{1}{3}\pi\left(4+r^2+2r\right)h$ $\frac{\mathrm{d}V}{\mathrm{d}h}\neq\frac{1}{3}\pi\left(4+r^2+2r\right)$ <u>r is a variable too</u>. We have to use implicit differentiation and product rule if we want to differentiate it directly. (Refer to tutorial 9.2 Q5: $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$ $-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t}=0$) Write your answers properly. $\frac{16}{9\pi}\neq\frac{16}{9}\pi$

5i	$\frac{1-2x}{\sqrt{4-x}}$ $=(1-2x)(4-x)^{-\frac{1}{2}}$ $=\frac{1}{2}(1-2x)\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$ $=\frac{1}{2}(1-2x)\left(1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\ldots\right)$ $=\frac{1}{2}(1-2x)\left(1+\frac{x}{8}+\frac{3}{128}x^2+\ldots\right)$ $=\frac{1}{2}\left(1-\frac{15}{8}x-\frac{29}{128}x^2+\ldots\right)$ $=\frac{1}{2}-\frac{15}{16}x-\frac{29}{256}x^2+\ldots$	When applying formula in binomial expansion, we can replace x with f(x), i.e. $(1+\mathrm{f}(x))^n$ $=1+n\,\mathrm{f}(x)$ $+\frac{n(n-1)}{2!}(\mathrm{f}(x))^2+\ldots$
5ii	Let $x=\frac{3}{4}$. $\frac{1-2\left(\frac{3}{4}\right)}{\sqrt{4-\frac{3}{4}}}\approx\frac{1}{2}-\frac{15}{16}\left(\frac{3}{4}\right)-\frac{29}{256}\left(\frac{3}{4}\right)^2$ $\frac{-\frac{1}{2}}{\sqrt{\frac{13}{4}}}\approx-\frac{1093}{4096}$ $-\frac{1}{\sqrt{13}}\approx-\frac{1093}{4096}$ $\sqrt{13}\approx\frac{4096}{1093}$ **<u>Alternative (if you rationalised the denominator)</u>** $-\frac{\sqrt{13}}{13}\approx-\frac{1093}{4096}\Rightarrow\sqrt{13}\approx\frac{14209}{4096}$	When asked to substitute $x=\frac{3}{4}$, we are required to do the substitution on both sides of the equation.
6i	Since the narrowest part is at height 80m, the centre of the hyperbola is at $(0,80)$. k = 80. At $(-50,0)$, $\frac{(-50)^2}{a^2}-\frac{(80)^2}{b^2}=1$ At $(-37.5,125)$, $\frac{(-37.5)^2}{a^2}-\frac{(45)^2}{b^2}=1$ By GC, $\frac{1}{a^2}=\frac{1}{900}, \frac{1}{b^2}=\frac{1}{3600}$ $a^2=900, b^2=3600$	This question can be solved as a system of linear equations with unknowns $\frac{1}{a^2}$ and $\frac{1}{b^2}$.

6ii		Recall the main features that we expect to see in a hyperbola.
6iii	$r=30$ 	A diagram would be useful to visualize this question. Observe that the centres of the hyperbola and circle are the same. We should capitalise on the graph drawn to deduce the value of r.
7i	$\frac{dx}{dt}=6t+2, \quad \frac{dy}{dt}=3t^2+2t$ $\frac{dy}{dx}=\frac{3t^2+2t}{6t+2}$ Since tangent is parallel to the y-axis, $\frac{dy}{dx}$ is undefined. $6t+2=0 \Rightarrow t=-\frac{1}{3}$ $x=3\left(-\frac{1}{3}\right)^2+2\left(-\frac{1}{3}\right)-\frac{5}{3}=-2$ $\therefore$ Equation of tangent is $x=-2$.	When a line is parallel to x-axis, $\frac{dy}{dx}=0$. When the line is parallel to y-axis, $\frac{dy}{dx}$ is undefined. A line that is parallel to y-axis has equation of the form $x=k$.
7ii	$x=3t^2+2t-\frac{5}{3}=3\left(t+\frac{1}{3}\right)^2-2\geq -2$ Least value of x is –2.	When we are considering $\frac{dy}{dx}=0$, we are finding stationary values of y. For those who considered $\frac{dx}{dt}=0$, you would need to check that x is a minimum.

7iii	When $y=0,\ t^3+t^2=0$ $t=0$ or $t=-1$ $x=-\frac{5}{3}$ or $x=-\frac{2}{3}$ 	Read the question: There is no mention about the range of values of t that can be used, so on the GC, we have to adjust the T_{min}. We need to zoom in appropriately to see the shape of the graph. Although there is no mention of $x=-2$ in this part of the question, it is implied that we need this value from the earlier parts in the question.
8	$y=\tan\left(\frac{1}{2}x\right)$ $\frac{dy}{dx}=\frac{1}{2}\sec^2\left(\frac{1}{2}x\right)$ $\frac{d^2y}{dx^2}=\frac{1}{2}(2)\sec\left(\frac{1}{2}x\right)\left(\frac{1}{2}\sec\left(\frac{1}{2}x\right)\tan\left(\frac{1}{2}x\right)\right)$ $=\frac{1}{2}\sec^2\left(\frac{1}{2}x\right)\tan\left(\frac{1}{2}x\right)=y\frac{dy}{dx}$	
8i	$\frac{d^3y}{dx^3}=\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}$ When $x=0,\ y=0,\ \frac{dy}{dx}=\frac{1}{2},\ \frac{d^2y}{dx^2}=0,\ \frac{d^3y}{dx^3}=\frac{1}{4}$ $y\approx\frac{1}{2}x+\frac{\frac{1}{4}}{3!}x^3=\frac{1}{2}x+\frac{1}{24}x^3$	We have to differentiate the given expression in the question. The formula for Maclaurin expansion is $f(x)$ $=f(0)+xf'(0)$ $+\frac{x^2}{2!}f''(0)+\ldots$ There is no need to replace x with $\frac{1}{2}x$ here. (compare with Q5i)
8ii	$\int\tan\left(\frac{1}{2}x\right)dx=2\ln\left\|\sec\left(\frac{1}{2}x\right)\right\|+C$ Since $-\pi<x<\pi,\ -\frac{\pi}{2}<\frac{x}{2}<\frac{\pi}{2}$. $\cos\left(\frac{1}{2}x\right)>0\Rightarrow\sec\left(\frac{1}{2}x\right)>0$ Hence $\int\tan\left(\frac{1}{2}x\right)dx=2\ln\left(\sec\left(\frac{1}{2}x\right)\right)+C$	Do not forget to divide by $\frac{d}{dx}\left(\frac{1}{2}x\right)=\frac{1}{2}$, a constant. We need to remove the modulus notation if it is possible to do so in the question. We do not integrate the result from (i) here, question did not ask for a maclurin series, or an approximation.

8iii	$2\ln\left(\sec\left(\frac{1}{2}x\right)\right)\approx\int\left(\frac{1}{2}x+\frac{1}{24}x^3\right)\mathrm{d}x=\frac{1}{4}x^2+\frac{1}{96}x^4+C$ $\ln\left(\sec\left(\frac{1}{2}x\right)\right)\approx\frac{1}{8}x^2+\frac{1}{192}x^4+\frac{C}{2}$ When $x=0$, $\frac{C}{2}=0\Rightarrow C=0$ $\ln\left(\sec\left(\frac{1}{2}x\right)\right)=\frac{1}{8}x^2+\frac{1}{192}x^4+\ldots$	There are many methods to solve this question. However, if it is a hence, or deduce question, this will be the only acceptable method. We will need to find the value of the arbitrary constant as well. Note for those who used series from MF26, You have to check range of values of x for valid expansion.
9ai	Consider $(a+\mathrm{i}b)^2=3-4\mathrm{i}$ $a^2-b^2+2ab\mathrm{i}=3-4\mathrm{i}$ Comparing the real and imaginary parts $a^2-b^2=3$ and $2ab=-4$ $\Rightarrow b=\frac{-2}{a}$ $a^2-\frac{4}{a^2}=3$ $a^4-3a^2-4=0$ $(a^2-4)(a^2+1)=0$ $a=\pm2$ (Since a is a real number) When $a=2,\ b=-1,$ When $a=-2,\ b=1,$ Hence the 2 roots are $2-\mathrm{i}$ or $-2+\mathrm{i}$	
9aii	$z^2-4\mathrm{i}z+4\mathrm{i}-7=0$ $z=\frac{4\mathrm{i}\pm\sqrt{(-4\mathrm{i})^2-4(4\mathrm{i}-7)}}{2}$ $z=\frac{4\mathrm{i}\pm\sqrt{-16+28-16\mathrm{i}}}{2}$ $z=2\mathrm{i}\pm\sqrt{3-4\mathrm{i}}$ $z=2+\mathrm{i}$ or $z=-2+3\mathrm{i}$	
9b	$\left(1+2z^{2n}\right)^*=1+2\left(z^*\right)^{2n}$ $\frac{2z^n}{1+2z^{2n}}=\frac{2z^n\left(1+2\left(z^{2n}\right)^*\right)}{\left(1+2z^{2n}\right)\left(1+2\left(z^{2n}\right)^*\right)}=\frac{2z^n\left(1+2\left(z^*\right)^n\left(z^*\right)^n\right)}{\left(1+2z^{2n}\right)\left(1+2\left(z^{2n}\right)^*\right)}$ $=\frac{2z^n+4\|z\|^{2n}\left(z^*\right)^n}{\|1+2z^{2n}\|^2}=\frac{2z^n+2\left(z^n\right)^*}{\|1+2z^{2n}\|^2}=\frac{4\,\mathrm{Re}\left(z^n\right)}{\|1+2z^{2n}\|^2}=k\ ,\ k\in\mathbb{R}$	

10a i		A translation of 1 unit in the –ve x-direction followed by a stretch parallel to x-axis by factor ½ , y-axis invariant. **OR** Stretch parallel to x-axis by factor ½ , y-axis invariant followed by a translation of **<u>½ unit</u>** in the –ve x-direction.
10a ii		A stretch parallel to y-axis by factor 2, followed by a translation of 3 units in the –ve y-direction; **OR** A translation of **<u>3/2 units</u>** in the –ve y-direction followed by a stretch parallel to y-axis by factor 2.
10b	Let r and h be the radius and the height of the cylinder respectively. Fixed vol. $p = \pi r^2 h \Rightarrow h = \dfrac{p}{\pi r^2}$ Surface area, S $= 2\pi r^2 + 2\pi r h$ $= 2\pi r^2 + 2\pi r\left(\dfrac{p}{\pi r^2}\right)$ $= 2\pi r^2 + \dfrac{2p}{r}$ $\dfrac{dS}{dr} = 4\pi r - \dfrac{2p}{r^2}$ For min. S, $\dfrac{dS}{dr} = 4\pi r - \dfrac{2p}{r^2} = 0 \quad \Rightarrow \quad r = \left(\dfrac{p}{2\pi}\right)^{\frac{1}{3}}$ $\dfrac{d^2S}{dr^2} = 4\pi + \dfrac{4p}{r^3} > 0$ since r and p are positive. $\therefore$ S is minimum when $r = \left(\dfrac{p}{2\pi}\right)^{\frac{1}{3}}$ cm.	

11a	$\int \frac{\sin 3x}{2-\cos 3x}\,dx = \frac{1}{3}\int \frac{3\sin 3x}{2-\cos 3x}\,dx$ $= \frac{1}{3}\ln\lvert 2-\cos 3x\rvert + C$ $= \frac{1}{3}\ln(2-\cos 3x) + C$ since $-1<\cos 3x<1 \Rightarrow 2-\cos 3x>0$	
11b	$\int \frac{e^x}{(1+2e^x)^3}\,dx = \frac{1}{2}\int 2e^x(1+2e^x)^{-3}\,dx$ $= -\frac{1}{4}(1+2e^x)^{-2} + C$	
11c	$\int \frac{2-x}{4+x^2}\,dx$ $= \int \frac{2}{4+x^2}\,dx + \int \frac{-x}{4+x^2}\,dx$ $= 2\int \frac{1}{2^2+x^2}\,dx + \left(-\frac{1}{2}\right)\int \frac{2x}{4+x^2}\,dx$ $= 2\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right) - \frac{1}{2}\ln\lvert 4+x^2\rvert + C$ $= \tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\ln\lvert 4+x^2\rvert + C$, $= \tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\ln(4+x^2) + C$, since $4+x^2>0$	
11d	$\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2 y}}\sec^2 y\,dy$ $= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\sec^2 y}}\sec^2 y\,dy$ $= \int_0^{\frac{\pi}{4}} \sec y\,dy$ $= \left[\ln\lvert \sec y+\tan y\rvert\right]_0^{\frac{\pi}{4}}$ $= \ln\left\lvert \sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right\rvert - \ln\lvert \sec 0+\tan 0\rvert$ $= \ln(\sqrt{2}+1)$	

11e

$$\int x^2 e^{x^3}\ \mathrm{d}x = \frac{1}{3}\int 3x^2 e^{x^3}\ \mathrm{d}x = \frac{1}{3}e^{x^3} + C$$

$$\int x^5 e^{x^3}\ \mathrm{d}x = \int x^3\left(x^2 e^{x^3}\right)\mathrm{d}x$$

$$= x^3\left(\frac{e^{x^3}}{3}\right) - \int\left(x^2 e^{x^3}\right)\mathrm{d}x$$

$$= \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C$$

12 ai

n	Year	\$ received at end of Year
1	2016	\$50
2	2017	\$50 + \$50
3	2018	\$50 + \$50 + \$50

Total amount received at the end of 20 years

$$= \$\left[\frac{20}{2}\left(2(50)+19(50)\right)\right]$$

$$= \$10500$$

Read question carefully to comprehend what is given.
Mr Reech holds 200 units of the bond at the end of the 20th year, **NOT** 2100 like what some of us thought.
We are asked to calculate the **total interest** (or coupons) received over 20 years.
Every year the interest increases by \$50 due to the extra 10 bonds that is purchased. So we should deal with an AP with first term 50 and common difference 50.

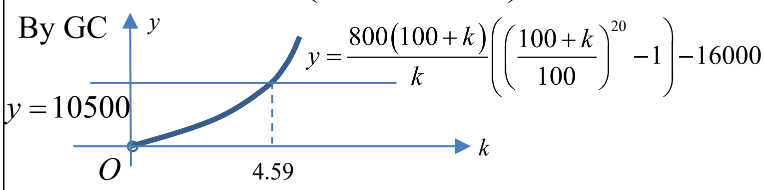
12a ii

n	Year	\$ beginning of Yr	\$ end of Yr
1	2016	\$800	$800\left(1+\frac{k}{100}\right)$
2	2017	$800\left(1+\frac{k}{100}\right)+800$	$800\left(1+\frac{k}{100}\right)+800\left(1+\frac{k}{100}\right)^2$
3	2018	$+800\left(1+\frac{k}{100}\right)^2+$ $800\left(1+\frac{k}{100}\right)+800$	$800\left(1+\frac{k}{100}\right)+800\left(1+\frac{k}{100}\right)^2$ $+800\left(1+\frac{k}{100}\right)^3$

Amount of money in account at the end of n years

It is incorrect to write $1+\frac{k}{100}$ as $1.01k$ as $1.01k = k + 0.01k$.

This is a show question. We need to be very clear in our steps leading to the **given** answer.

	$=800\left(1+\frac{k}{100}\right)+800\left(1+\frac{k}{100}\right)^2+\ldots800\left(1+\frac{k}{100}\right)^n$ $=\frac{800\left(1+\frac{k}{100}\right)\left(1-\left(1+\frac{k}{100}\right)^n\right)}{1-\left(1+\frac{k}{100}\right)}$ $=\frac{800\left(\frac{100+k}{100}\right)\left(\left(\frac{100+k}{100}\right)^n-1\right)}{\frac{k}{100}}$ $=\frac{800(100+k)}{k}\left(\left(\frac{100+k}{100}\right)^n-1\right)$	
12a iii	$10500>\frac{800(100+k)}{k}\left(\left(\frac{100+k}{100}\right)^{20}-1\right)-800(20)$ By GC  $0<k<4.59$	Read carefully. We want the **<u>total interest in (i)</u>** to be **<u>more than the total interest in (ii)</u>**. Use all the given answers and n = 20 to form this simple inequality. Learn to recognise when an equation cannot be solved algebraically. Do not waste time trying to simplify it. Immediately use GC to sketch relevant graphs and look for point of intersection.

<table>
<tr>
<td>12b</td>
<td>
<table>
<tr><td>n</td><td>Year</td><td>Interest paid</td></tr>
<tr><td>1</td><td>2016</td><td>100,000(0.02) = 2000</td></tr>
<tr><td>2</td><td>2017</td><td>2000(0.5) = 1000</td></tr>
<tr><td>3</td><td>2018</td><td>$2000(0.5)^2 = 500$</td></tr>
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Maximum theoretical interest in dollars
$= \frac{2000}{1-0.5} = 4000$
Total interest (in dollars) collected over n years
$= \frac{2000\left(1-0.5^n\right)}{1-0.5} = 4000\left(1-0.5^n\right)$

$4000 - 4000\left(1-0.5^n\right) \le 20$
$4000(0.5)^n \le 20$
$(0.5)^n \le \frac{1}{200}$
$n\ln(0.5) \le \ln\left(\frac{1}{200}\right)$
$n \ge \frac{\ln\left(\frac{1}{200}\right)}{\ln(0.5)}$
$n \ge 7.64$

Minimum number of years = 8
</td>
<td>What is the maximum theoretical interest payable by the bank? Use the interest collected at end of 2016 and a common ratio of ½ to calculate.
Next try to find an n that is large enough for total interest collected over n years to be reaching maximum theoretical interest such that the difference cannot be more than $20.
Note that ln (0.5) is negative and hence we need to switch the sign of the inequality when we divide both sides by ln (0.5).
This question can also be solved using either graph or table (Contrast with 12aiii, which cannot be solved using table method)</td>
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<tr>
<td colspan="3">General comments from Teachers

There is a need for better time management during tests and exams. Many did not have time to attempt Q12. The ability to recognize when GC can be used to skip tedious algebraic working is an advantage during assessments.
Need more attempts at long contextual questions to improve comprehension of question.
Timed practice is a good exercise for training yourself to work under pressure.

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