Differentiation:

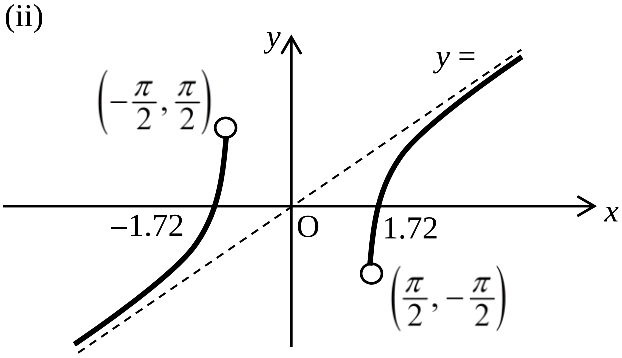
(Classified to 3 key sub-topics
Tangents and Normals/ Rate of Change & Maxima Problems / Maclaurin's series)

Solutions

	Tangents & Normals
1	$\frac{d}{dx}\left[\cos^{-1}\left(\frac{1}{x^2}\right)\right]=\frac{-1}{\sqrt{1-\left(\frac{1}{x^2}\right)^2}}\left(-\frac{2}{x^3}\right)$ $=\frac{2}{x\sqrt{x^4-1}}$
	$x=\ln(\sin t)\ \Rightarrow \frac{dx}{dt}=\frac{\cos t}{\sin t}$ $y=\cot t\ \Rightarrow \frac{dy}{dt}=-\text{cosec}^2 t$ Therefore $\frac{dy}{dx}=-\frac{1}{\sin^2 t}\times\frac{\sin t}{\cos t}=-\frac{2}{2\sin t\cos t}=-\frac{2}{\sin 2t}$ (proved)
	Gradient of normal $=\frac{1}{2}\sin 2t=\frac{1}{2}$ $\Rightarrow \sin 2t=1$ $\Rightarrow 2t=\frac{\pi}{2}\Rightarrow t=\frac{\pi}{4}$ Therefore $x=\ln\left(\sin\frac{\pi}{4}\right)=\ln\left(\frac{\sqrt{2}}{2}\right)=-\frac{1}{2}\ln 2$ $\therefore y=1$ Equation of normal: $y-1=\frac{1}{2}\left(x-\left(-\frac{1}{2}\ln 2\right)\right)\Rightarrow y=\frac{1}{2}\left(x+\frac{1}{2}\ln(2)\right)+1$
	$PQ = RQ$ $\Rightarrow$ normal at R is the reflection of the above normal in the x-axis $\Rightarrow$ equation of normal at R is $\Rightarrow y=-\left(\frac{1}{2}\left(x+\frac{1}{2}\ln(2)\right)+1\right)$ $\Rightarrow y=-\frac{1}{2}\left(x+\frac{1}{2}\ln(2)\right)-1$

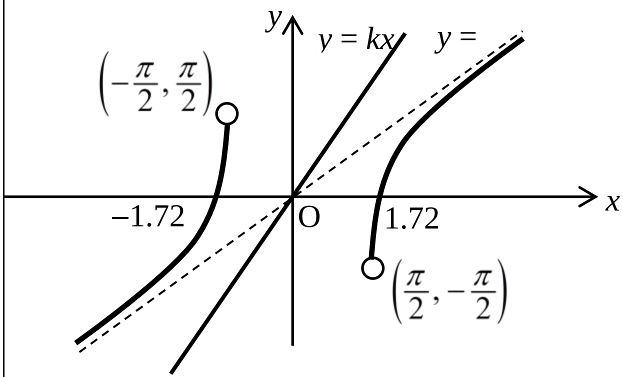
2	(i) $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{t^2}+\frac{1}{1+t^2}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{t^2}-\frac{1}{1+t^2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\Big/\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)=2t^2+1$ Since $y = ax$ is an oblique asymptote, $\therefore$ as $x\to\pm\infty$, the gradient of the curve approaches a. But $x\to\pm\infty \Rightarrow t\to 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=2t^2+1\to 1$, $\therefore$ a = 1 (proved)

(ii)



As $t \to -\infty$, $x \to 0 - \frac{\pi}{2}$ and $y \to 0 + \frac{\pi}{2}$.

As $t \to \infty$, $x \to 0 + \frac{\pi}{2}$ and $y \to 0 - \frac{\pi}{2}$.



From the diagram,
For $y = kx$ not to intersect the curve,
$k \geq 1$ or $k \leq -1$

(iii)

Eqn of normal at $t = 1$,

$$\frac{y-\left(1-\frac{\pi}{4}\right)}{x-\left(1+\frac{\pi}{4}\right)} = -\frac{1}{3}$$

$$3\left(y-1+\frac{\pi}{4}\right) = -\left(x-1-\frac{\pi}{4}\right)$$

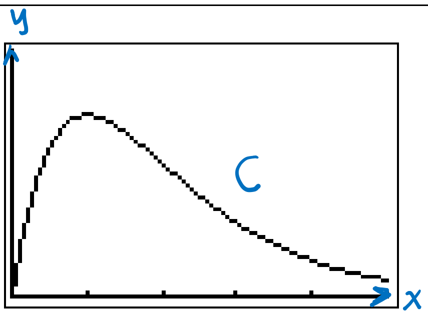
To find the points of intersection,

$$3\left(\frac{1}{t}-\tan^{-1}(t)-1+\frac{\pi}{4}\right) = -\left(\frac{1}{t}+\tan^{-1}(t)-1-\frac{\pi}{4}\right)$$

$$\frac{4}{t}-2\tan^{-1}(t)-4+\frac{\pi}{2} = 0$$

Using GC, $t = -8.41$

3(i)



(ii)

$y = xe^{-x}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = x(-\mathrm{e}^{-x}) + \mathrm{e}^{-x} = \mathrm{e}^{-x}(1-x)$$

At P, $x = a$, $y = a\mathrm{e}^{-a}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-a}(1-a)$

Equation of tangent to the curve at P is

$$y - a\mathrm{e}^{-a} = \mathrm{e}^{-a}(1-a)(x-a)$$

$$\Rightarrow \quad y = \mathrm{e}^{-a}(1-a)(x-a) + a\mathrm{e}^{-a}$$

At Q, $x = 0$, $y = h$

$$\Rightarrow \quad h = \mathrm{e}^{-a}(1-a)(0-a) + a\mathrm{e}^{-a}$$

$$= \mathrm{e}^{-a}(a-1)(a) + a\mathrm{e}^{-a} = a^2\mathrm{e}^{-a}$$

$$\Rightarrow \quad \frac{\mathrm{d}h}{\mathrm{d}a} = 2a\mathrm{e}^{-a} + a^2(-\mathrm{e}^{-a}) = a\mathrm{e}^{-a}(2-a)$$

For stationary values of h, $\frac{\mathrm{d}h}{\mathrm{d}a} = 0$

$\Rightarrow \quad a = 0$ (N.A. since $a > 0$) or $a = 2$

	2^-	2	2^+
$\frac{\mathrm{d}h}{\mathrm{d}a}$	>0	0	<0
Tangent	/	—	\

Maximum value of $h = 4\mathrm{e}^{-2}$

4 (i)

$x = a\left(1+\frac{1}{t}\right)$, $y = a\left(t - \frac{1}{t^2}\right)$, $a > 0$

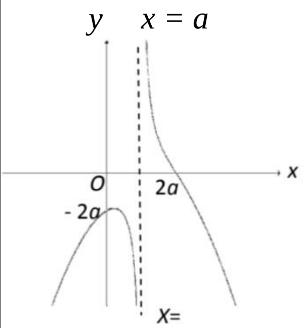
$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{a}{t^2}$ $\qquad$ $\frac{\mathrm{d}y}{\mathrm{d}t} = a\left(1+\frac{2}{t^3}\right)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x} = a\left(1+\frac{2}{t^3}\right)\left(-\frac{t^2}{a}\right) = -t^2\left(1+\frac{2}{t^3}\right)$$

(ii)

When $t = -\frac{1}{2}$, $x = -a$, $y = -\frac{9a}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{15}{4}$.

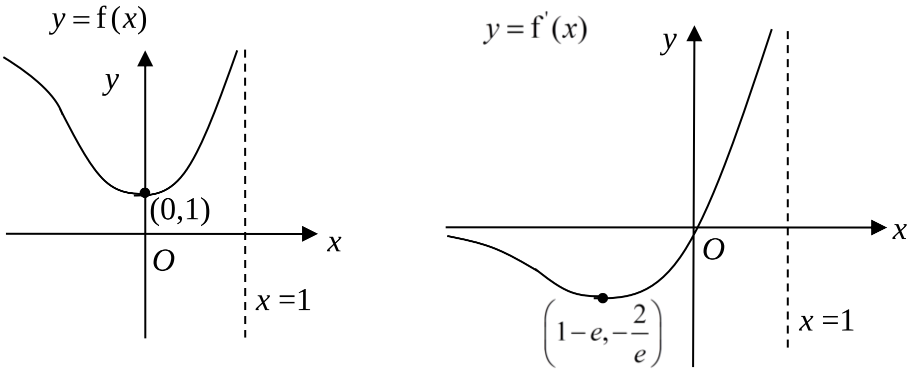
Eqn of tangent: $y + \frac{9a}{2} = \frac{15}{4}(x+a)$

	$4y+18a=15x+15a$ $4y=15x-3a$
(iii)	At Q, $4a\left(t-\frac{1}{t^2}\right)=15a\left(1+\frac{1}{t}\right)-3a$ $4t^3-4=15t^2+15t-3t^2$ $4t^3-12t^2-15t-4=0$ Using GC, $t=4$ or $t=-\frac{1}{2}$. Cdts of Q: $\left(\frac{5}{4}a,\frac{63}{16}a\right)$
(iv) (iv)	As $t\to\infty$, $x\to a$, $y\to\infty$. Asymptote: $x=a$ 
(v)	Required area = $\frac{1}{2}\left(\frac{5}{4}a-\frac{1}{5}a\right)\left(\frac{63}{16}a\right)+\int_4^1 a\left(t-\frac{1}{t^2}\right)\left(-\frac{a}{t^2}\right)\,\mathrm{d}t$ $=\frac{1323}{640}a^2+a^2\int_1^4 \frac{1}{t}-\frac{1}{t^4}\,\mathrm{d}t$ $=\frac{1323}{640}a^2+a^2\left[\ln\lvert t\rvert+\frac{1}{3t^3}\right]_1^4$ $=\frac{1323}{640}a^2+a^2\left(\ln 4-\frac{21}{64}\right)$ $=a^2\left(\frac{1113}{640}+\ln 4\right)$

5(i)	$x=a\sin 2t$ $\frac{\mathrm{d}x}{\mathrm{d}t}=2a\cos 2t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sin t}{2\cos 2t}$	$y=a\cos t$ $\frac{\mathrm{d}y}{\mathrm{d}t}=-a\sin t$

	When $t=\frac{\pi}{4}$, $2\cos 2t=0 \Rightarrow$ tangent at $t=\frac{\pi}{4}$ // y-axis When $t=\frac{\pi}{4}$, $x=a$. Thus, the tangent to the curve at $t=\frac{\pi}{4}$ is $x=a$.
(ii)	When $t=\frac{\pi}{3}$, $x=\frac{a\sqrt{3}}{2}$, $y=\frac{a}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\sqrt{3}}{2}$ Equation of normal: $y-\frac{a}{2}=-\frac{2}{\sqrt{3}}\left(x-\frac{a\sqrt{3}}{2}\right)$ At R: $-\frac{a}{2}=-\frac{2}{\sqrt{3}}\left(x-\frac{a\sqrt{3}}{2}\right)$ $x=\frac{a3\sqrt{3}}{4}$ Coordinates of R: $\left(\frac{a3\sqrt{3}}{4},0\right)$ At $x=a$, $y-\frac{a}{2}=-\frac{2}{\sqrt{3}}\left(a-\frac{a\sqrt{3}}{2}\right)$ $y=a\left(\frac{3}{2}-\frac{2}{\sqrt{3}}\right)$ Area enclosed = $\frac{1}{2}\times a\left(\frac{3}{2}-\frac{2}{\sqrt{3}}\right)\times\left(\frac{3\sqrt{3}a}{4}-a\right)$ $= a^2\left(\frac{43\sqrt{3}}{48}-\frac{3}{2}\right)$.

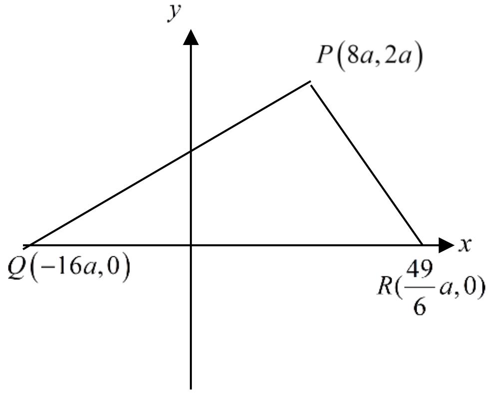
6(i)	$x=1-e^{-t} \Rightarrow \frac{dx}{dt}=e^{-t}$ $y=1+t^2 \Rightarrow \frac{dy}{dt}=2t$ $\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2t}{e^{-t}}=2te^{t}$ When $t=p$, $\frac{dy}{dx}=2pe^{p}$, point P is $(1-e^{-p},1+p^2)$,

	Equation of tangent at P is $y-(1+p^2)=2pe^p[x-(1-e^{-p})]$ $y=2pe^p x-2pe^p(1-e^{-p})+(1+p^2)$ Since tangent passes through (1, 0), $0=2pe^p(1)-2pe^p+2p+1+p^2$ $\Rightarrow p^2+2p+1=0$ $\Rightarrow p=-1$
(ii)	

7(i)	$y=xe^{-x}$ $\frac{dy}{dx}=e^{-x}-xe^{-x}=e^{-x}(1-x)$ Graph is decreasing: $\frac{dy}{dx}<0$ $e^{-x}(1-x)<0$ $x>1$
(ii)	$\frac{d^2y}{dx^2}=e^{-x}(-1)-e^{-x}(1-x)$ $=e^{-x}(x-2)$ Graph is concave downwards: $\frac{d^2y}{dx^2}<0$ $e^{-x}(x-2)<0$ $x<2$ Therefore, for graph to be decreasing and concave downwards: $1<x<2$.
(iii)	gradient at (x,y) = $e^{-x}(1-x)$ $e^{-x}(1-x)=\frac{y-h}{x-0}$ $xe^{-x}-h=xe^{-x}(1-x)$ $h=xe^{-x}-xe^{-x}(1-x)$ $=(x)^2e^{-x}$

	$\frac{\mathrm{d}h}{\mathrm{d}x} = 2x\mathrm{e}^{-x} - (x)^2\,\mathrm{e}^{-x}$ $= x\mathrm{e}^{-x}(2-x)$ At max/min point: $\frac{\mathrm{d}h}{\mathrm{d}x} = 0$ $x\mathrm{e}^{-x}(2-x) = 0$ $x = 2$ or $x = 0$ <table><tr><td>x</td><td>2-</td><td>2</td><td>2+</td></tr><tr><td>$\frac{\mathrm{d}h}{\mathrm{d}x}$</td><td>+</td><td>0</td><td>-</td></tr></table> <table><tr><td>x</td><td>0-</td><td>0</td><td>0-</td></tr><tr><td>$\frac{\mathrm{d}h}{\mathrm{d}x}$</td><td>-</td><td>0</td><td>+</td></tr></table> Greatest possible $h = 4\mathrm{e}^{-2}$

8	(i) $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{3a}{t^4}$ $\quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{a}{t^2}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{t^2}{3}$ At $t = \frac{1}{2}$, Gradient of tangent at $P = \frac{\left(\frac{1}{2}\right)^2}{3} = \frac{1}{12}$ Gradient of normal at $P = -12$ At P, $x = 8a$ and $y = 2a \to (8a, 2a)$ Equation of tangent: $y - 2a = \frac{1}{12}(x - 8a)$ $y = \frac{1}{12}x + \frac{4}{3}a$ Equation of normal: $y - 2a = -12(x - 8a)$ $y = -12x + 98a$ (ii) $\frac{a}{t} = \frac{1}{12}\left(\frac{a}{t^3}\right) + \frac{4}{3}a$ $12t^2 = 1 + 16t^3$ $16t^3 - 12t^2 + 1 = 0$ By G.C, $t = \frac{1}{2}$ (N.A.), $-\frac{1}{4}$

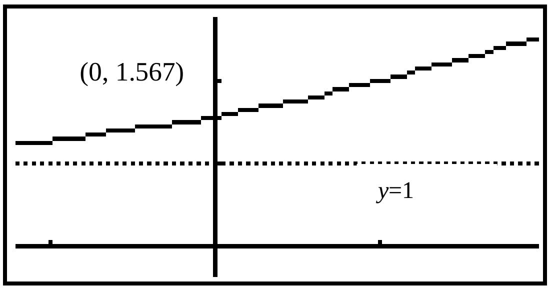
	When $t=-\frac{1}{4}, x=-64a, y=-4a$ Hence the tangent cuts the curve again at $\left(-64a,-4a\right)$ (iii) At Q: $y=0$ $\quad 0=\frac{1}{12}x+\frac{4}{3}a \rightarrow x=-16a \therefore Q\left(-16a,0\right)$ At R: $y=0$ $\quad 0=-12x+98a \quad \rightarrow \quad x=\frac{49}{6}a \therefore R(\frac{49}{6}a,0)$ Area of triangle PQR $=\frac{1}{2}\left(\frac{49}{6}a-\left(-16a\right)\right)\left(2a\right)=\frac{145}{6}a^2\text{units}^2$ 
9(i)	$\frac{d}{dx}\left(xy-2y^2+4x^2\right)=\frac{d}{dx}66$ $x\frac{dy}{dx}+y-4y\frac{dy}{dx}+8x=0$ $\frac{dy}{dx}\left(x-4y\right)=-8x-y$ $\frac{dy}{dx}=\frac{8x+y}{4y-x}$ For tangent parallel to y-axis, $4y-x=0$ $x=4y$ Substitute $x=4y$ into equation of curve, $\left(4y\right)y-2y^2+4\left(4y\right)^2=66$ $66y^2=66$ $y^2=1$ $y=\pm1.$ When $y=1,\ x=4$ When $y=-1,\ x=-4$ Coordinates are $\left(4,1\right),\ \left(-4,1\right)$

(ii)	Substitute $y=k$ into equation of the curve, $kx-2k^2+4x^2=66$ $4x^2+kx+\left(-2k^2-66\right)=0$ Considering the discriminant, $k^2-4(4)\left(-2k^2-66\right)$ $=33k^2+1056$ >0 for all real values of k The line $y=k$ cuts the curve for all real values of k.

10	(i) $4x^3+3x^2y=y^3-2$ Differentiating wr.t. x: $12x^2+6xy+3x^2\frac{\mathrm{d}y}{\mathrm{d}x}=3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$ $\left(3x^2-3y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-12x^2-6xy$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4x^2+2xy}{y^2-x^2}$ $=\frac{2x(2x+y)}{(y-x)(y+x)}$ Curve meets $y=-x$ when: $4x^3+3x^2(-x)=-x^3-2$ $2x^3=-2$ $\Rightarrow x=-1$ and $y=1$ Thus, coordinates of P is $(-1,1)$ (ii) At $(-1,1)$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined. Equation of tangent at P: $x=-1$ $OQPR$ is a square.

11	(i) $x=t+\ln t,\quad y=t+1,\quad t>0$ $\frac{\mathrm{d}x}{\mathrm{d}t}=1+\frac{1}{t},\quad \frac{\mathrm{d}y}{\mathrm{d}t}=1,$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+\frac{1}{t}}=\frac{t}{t+1}$ Since $t>0$, $t+1>0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{t}{t+1}>0$ for all $t>0$ Hence C does not have a stationary point

(ii)



When $x = 0$, $t + \ln t = 0 \Rightarrow t = 0.5671432904$ (by g.c.)

$y = 1 + 0.5671432904 = 1.5671432904$

When $t \to 0$, $x \to -\infty$, $y \to 0 + 1 = 1$

(iii) When $t = 1$, $x = 1 + \ln 1 = 1$ $y = 1 + 1 = 2$, $\dfrac{dy}{dx} = \dfrac{1}{2}$

Equation of normal : $y - 2 = -2(x - 1)$

$\Leftrightarrow y = -2x + 4$

(iv) Volume generated

$= \pi \int_{0.5671432904}^{1} (t+1)^2 \left(1 + \dfrac{1}{t}\right) \, dt + \dfrac{1}{3}\pi\left(2^2\right)(1)$

$= 14.10$ (2 decimal places)

12 (i)

$y^2 - xy = -1$

$2y\dfrac{dy}{dx} - \left(y + x\dfrac{dy}{dx}\right) = 0$

$(2y - x)\dfrac{dy}{dx} = y$

$\dfrac{dy}{dx} = \dfrac{y}{2y - x}$

Tangent parallel to y-axis implies $2y - x = 0$

$y = \dfrac{x}{2}$

$\left(\dfrac{x}{2}\right)^2 - x\left(\dfrac{x}{2}\right) = -1$

$-\dfrac{1}{4}x^2 = -1$

$x^2 = 4$

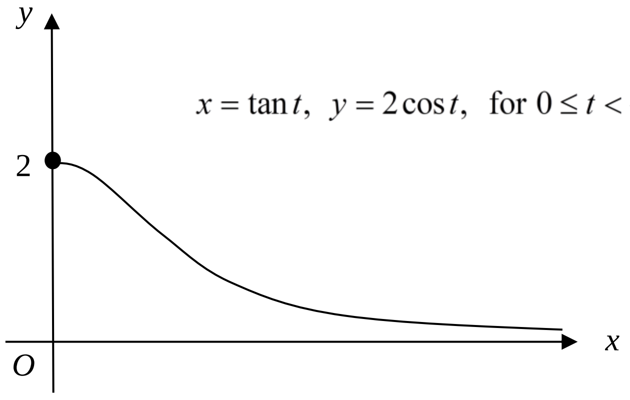
$x = \pm 2$

Hence **equation of tangents** are $x = 2$ and $x = -2$

	(ii) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{2y-x}$ Hence $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ implies $y=0$. But $y^2-xy=-1$. Thus $y=0 \Rightarrow \text{LHS}=0\neq\text{RHS}$ Hence there are no stationary points. (iii) $2^2-2x=-1$ $x=\frac{5}{2}$ $x=\frac{5}{2},\ y=2 \ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{2(2)-\frac{5}{2}}=\frac{4}{3}$ Hence gradient of normal is $-\frac{3}{4}$. $y-2=-\frac{3}{4}\left(x-\frac{5}{2}\right)$ Equation of normal: $y=-\frac{3}{4}x+\frac{31}{8}$ $x=0\Rightarrow y=\frac{31}{8}=3.875$ $y=0\Rightarrow x=\frac{31}{6}=5.1\dot{6}$ Hence area of region $=\frac{1}{2}\left(\frac{31}{6}\right)\left(\frac{31}{8}\right)$ $=\frac{961}{96}=10.0\ (3\text{ s.f.})$

13	(a) $4x+\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)-2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\frac{4x+y}{2y-x}=\frac{\mathrm{d}y}{\mathrm{d}x}$ When the tangent is parallel to the y – axis, $2y-x=0\Rightarrow x=2y$ Hence, $2(2y)^2+(2y)y-y^2=9$ $8y^2+2y^2-y^2=9$ Solving, $y=1$ or $y=-1$. Since $x = 2y$, $x = 2$ or $x = -2$. Hence, the coordinates are (2, 1) and (–2, –1).

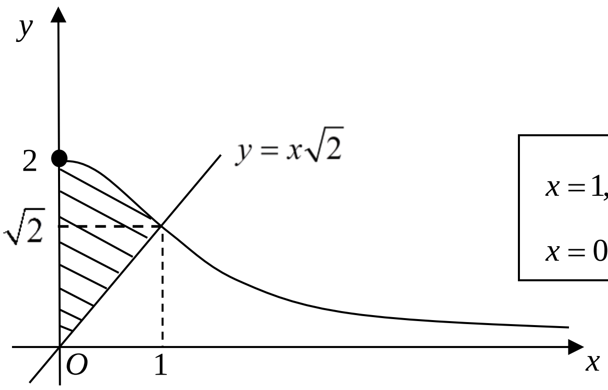
	(b) (i) $\frac{dx}{dt}=2t+1,\ \frac{dy}{dt}=-1$ $\Rightarrow \frac{dy}{dx}=\frac{-1}{2t+1}$ At the point P, $x=p^2+p,\ y=4-p, \frac{dy}{dx}=\frac{-1}{2p+1}$. Hence, equation of tangent at P: $y-(4-p)=\frac{-1}{2p+1}\left(x-(p^2+p)\right)$ $(2p+1)(4-p-y)=x-p^2-p$ (shown) (ii) For the tangent to meet the curve again, there must be another value of t such that: $(2p+1)(4-p-(4-t))=(t^2+t)-p^2-p$. $(2p+1)t-2p^2-p=t^2+t-p^2-p$ $t^2-2pt+p^2=0$ $(t-p)^2=0$ Since $t=p$ is the only solution, then every tangent to the curve C does not meet the curve again.

14(i)	$x=\tan t,\ \ y=2\cos t,\ \ \text{for } 0\le t<\frac{\pi}{2}$ $\frac{dx}{dt}=\sec^2 t,\ \ \frac{dy}{dt}=-2\sin t \Rightarrow \frac{dy}{dx}=\frac{-2\sin t}{\sec^2 t}=-2\sin t\cos^2 t$ As $t\to 0,\ \frac{dy}{dx}\to 0.$ The tangent becomes parallel to the x-axis/tangent is a horizontal line. $x=\tan 0=0,\ y=2\cos 0=2$ 
(ii)	At $P(\tan p,\ 2\cos p)$, gradient of normal $=-\frac{1}{\frac{dy}{dx}}=-\frac{1}{\left(-2\sin p\cos^2 p\right)}=\frac{1}{2\sin p\cos^2 p}$,

	Method 1 Since normal passes through origin, equation of normal : $y=\left(\dfrac{1}{2\sin p\cos^2 p}\right)x$ ..... (1) Since normal intersects curve also at P, substitute $x=\tan p, y=2\cos p$ into eqn (1)
	Equation of normal is $y=\dfrac{x}{2\sin\frac{\pi}{4}\cos^2\frac{\pi}{4}}$ $y=\dfrac{x}{2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^2}$ .... (2) $\therefore y=x\sqrt{2}$ (shown) Method 2 Equation of normal : $y-2\cos p=\dfrac{1}{2\sin p\cos^2 p}(x-\tan p)$ ..... (1) Since the normal passes through origin (0,0), substitute $x=0, y=0$ into eqn (1) $0-2\cos p=\dfrac{1}{2\sin p\cos^2 p}(0-\tan p)$ $-4\sin p\cos^3 p=\dfrac{-\sin p}{\cos p}$ $\sin p(4\cos^4 p-1)=0$ $\sin p=0$ or $\cos p=\pm\dfrac{1}{\sqrt{2}}$ $\therefore p=\dfrac{\pi}{4}\ \left(\because 0<p<\dfrac{\pi}{2}\right)$ Equation of normal which passes through origin is $y-2\cos\dfrac{\pi}{4}=\dfrac{1}{2\sin\frac{\pi}{4}\cos^2\frac{\pi}{4}}\left(x-\tan\dfrac{\pi}{4}\right)$ $y-2\left(\dfrac{1}{\sqrt{2}}\right)=\dfrac{1}{2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^2}(x-1)$ .... (2) $y-\sqrt{2}=\sqrt{2}(x-1)$ $\therefore y=x\sqrt{2}$ (shown)

(iii)	see below

(iii)



$$\boxed{\begin{array}{l} x=1, t=\dfrac{\pi}{4} \\ x=0, t=0 \end{array}}$$

When $p=\dfrac{\pi}{4}, x=\tan\dfrac{\pi}{4}=1, y=2\cos\dfrac{\pi}{4}=\sqrt{2}$

<u>Method 1 (with respect to *x*-axis)</u>

Required area

$$=\int_0^1 y\ \mathrm{d}x-\frac{1}{2}bh \quad \text{or} \quad \left(\int_0^1 x\sqrt{2}\ \mathrm{d}x\right)$$

$$=\int_0^{\frac{\pi}{4}} 2\cos t\ \left(\sec^2 t\right)\mathrm{d}t-\frac{1}{2}(1)\left(\sqrt{2}\right) \quad \text{or} \quad \left[\frac{x^2}{2}\sqrt{2}\right]_0^1$$

$$=2\int_0^{\frac{\pi}{4}}\sec t\ \mathrm{d}t-\frac{\sqrt{2}}{2}$$

$$=2\Big[\ln\left|\sec t+\tan t\right|\Big]_0^{\frac{\pi}{4}}-\frac{\sqrt{2}}{2}$$

$$=2\ln\left(\frac{1}{\cos\frac{\pi}{4}}+\tan\frac{\pi}{4}\right)-\frac{\sqrt{2}}{2}$$

$$=2\ln\left(\sqrt{2}+1\right)-\frac{\sqrt{2}}{2}\ \text{unit}^2$$

$$\boxed{\begin{array}{l} y=2, t=0 \\ y=\sqrt{2}, t=\dfrac{\pi}{4} \end{array}}$$

<u>Method 2 (with respect to *y*-axis)</u>

Required area

$$=\int_{\sqrt{2}}^2 x\ \mathrm{d}y+\frac{1}{2}bh \quad \left(\int_0^{\sqrt{2}}\frac{y}{\sqrt{2}}\mathrm{d}y\right)$$

$$=\int_{\frac{\pi}{4}}^0 \tan t\ \left(-2\sin t\right)\mathrm{d}t+\frac{1}{2}\left(\sqrt{2}\right)(1) \quad \text{or} \quad \frac{1}{\sqrt{2}}\left[\frac{y^2}{2}\right]_0^{\sqrt{2}}$$

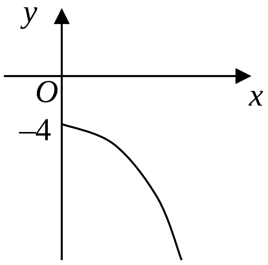
$$=2\int_0^{\frac{\pi}{4}}\frac{\sin^2 t}{\cos t}\ \mathrm{d}t+\frac{\sqrt{2}}{2}$$

$$=2\int_0^{\frac{\pi}{4}}\frac{1-\cos^2 t}{\cos t}\ \mathrm{d}t+\frac{\sqrt{2}}{2}$$

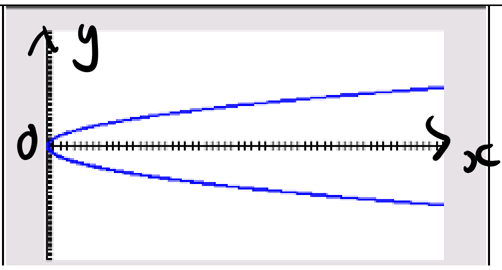
$$=2\int_0^{\frac{\pi}{4}}\left(\sec t-\cos t\right)\ \mathrm{d}t+\frac{\sqrt{2}}{2}$$

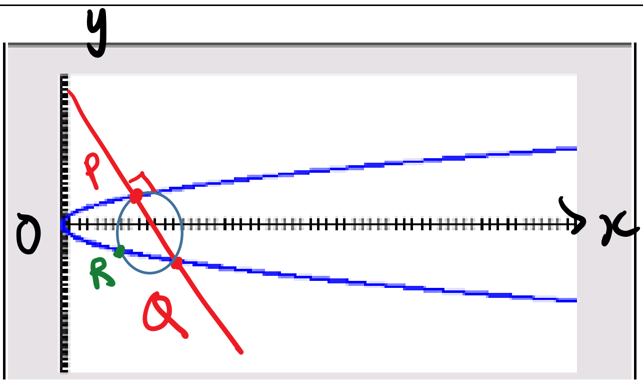
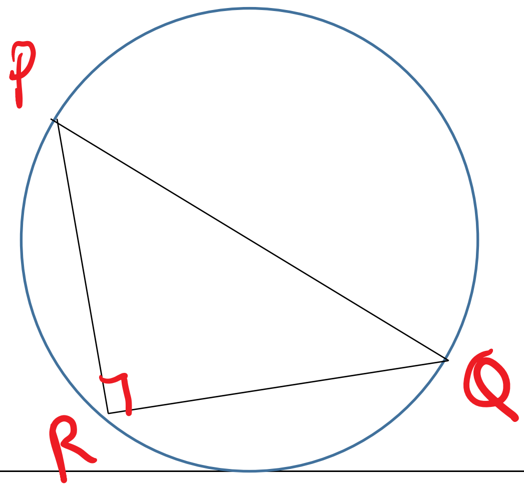
$$=2\Big[\ln\left|\sec t+\tan t\right|-\sin t\Big]_0^{\frac{\pi}{4}}+\frac{\sqrt{2}}{2}$$

	$= 2\ln\left(\frac{1}{\cos\frac{\pi}{4}} + \tan\frac{\pi}{4} - \sin\frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}$ $= 2\ln\left(\sqrt{2} + 1 - \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}$ $= 2\ln\left(\sqrt{2}+1\right) - \frac{\sqrt{2}}{2}$ unit^2 Note : Generally $\int \sec t \, \mathrm{d}t = \ln\left\lvert \sec t + \tan t \right\rvert$. But in this question where the limits are $0 \le t \le \frac{\pi}{4}$, $\int_0^{\frac{\pi}{4}} \sec t \, \mathrm{d}t = \ln\left(\sec t + \tan t\right)$ is acceptable.

15 (i)	$x = t^2, \quad y = t^3 - 4$ $\frac{\mathrm{d}x}{\mathrm{d}t} = 2t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3t}{2}$
	Tangent at P $\left(y - p^3 + 4\right) = \frac{3p}{2}\left(x - p^2\right)$ $2y = 3px - p^3 - 8$
(ii)	Since the tangent passes through the origin, subst. $x = 0$ and $y = 0$ into the equation of tangent in part (i). $-p^3 - 8 = 0$ $p = -2$ $x = 4, y = -12$ $P(4, -12)$
(iii)	$x = 0, y = -4, t = 0$ 

(iv)	$\text{Area} = \frac{1}{2}(4)(12) - \int_{-12}^{-4} x \, \mathrm{d}y$ $= 24 - \int_{-2}^{0} (t^2)(3t^2) \, \mathrm{d}t$ $= 24 - \left[\frac{3t^5}{5}\right]_{-2}^{0}$ $= \frac{24}{5}$

16 (i)	
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{1}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)} = \frac{2a}{2at} = \frac{1}{t}$ Gradient of normal = -*t*. Equation of normal at a point *P* is given by $\frac{y - 2ap}{x - ap^2} = -p$ $\Rightarrow y = 2ap - p(x - ap^2)$
(iii)	If the normal at point *P* meets *C* again at point *Q*, $2aq = 2ap - p(aq^2 - ap^2)$ $pq^2 + 2q - (2p + p^3) = 0$ $q = \frac{-2 \pm \sqrt{4 + 4p(2p + p^3)}}{2p}$ $= \frac{-2 \pm \sqrt{4 + 8p^2 + 4p^4}}{2p}$ $= \frac{-2 \pm \sqrt{(2p^2 + 2)^2}}{2p}$ $= \frac{-2 + (2p^2 + 2)}{2p}$ or $\frac{-2 - (2p^2 + 2)}{2p}$ $= p$ (rejected as it is the point *P*) or $-p - \frac{2}{p}$

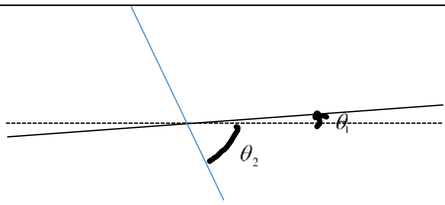
	Therefore Q will meet C again with $q=-p-\frac{2}{p}$. Coordinates of $P=(ap^2,2ap)$ Coordinates of $Q=\left(a(p+\frac{2}{p})^2,-2a\left(\frac{p^2+2}{p}\right)\right)$ $\lvert PQ\rvert^2$ $=(ap^2-a((p+\frac{2}{p})^2)^2+(2ap+2a(\frac{p^2+2}{p}))^2$ $=(ap^2-a(p^2+4+\frac{4}{p^2}))^2+(4ap+\frac{4a}{p})^2$ $=16a^2(1+\frac{1}{p^2})^2+16a^2(\frac{p^2+1}{p})^2$ $=16a^2(\frac{(p^2+1)^2}{p^4}+(\frac{p^2+1}{p})^2)$ $=\frac{16a^2}{p^4}((p^2+1)^2+p^2(p^2+1)^2)$ $=\frac{16a^2}{p^4}(p^2+1)^3$
(iv)	 

	Note that PR and QR are perpendicular to each other (angle PRQ is 90^0 – angle in a semi-circle). Gradient of PR = $\dfrac{2ap-2ar}{ap^2-ar^2}=\dfrac{2}{p+r}$ Gradient of QR = $\dfrac{-2a\left(\dfrac{p^2+2}{p}\right)-2ar}{a\left(\dfrac{p^2+2}{p}\right)^2-ar^2}=-\dfrac{2}{\left(\dfrac{p^2+2}{p}\right)-r}$ $-\dfrac{2}{\left(\dfrac{p^2+2}{p}\right)-r}\bullet\dfrac{2}{p+r}=-1$ $(p+r)(p+\dfrac{2}{p}-r)=4$ $p^2-r^2+\dfrac{2r}{p}=2$. (Shown).

DHS Prelim 9758/2018/02/Q5	
17 (a)(i)	$\left(x+y\right)^2=4\mathrm{e}^{xy}$ $2\left(x+y\right)\left(1+\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=4\mathrm{e}^{xy}\left(x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\right)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2y\mathrm{e}^{xy}-y-x}{y+x-2x\mathrm{e}^{xy}}$
(ii)	When x = 0, $\left(0+y\right)^2=4\mathrm{e}^{(0)y}$ $y=2\ (\because y>0)$ When at (0, 2), $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2(2)-2}{2}=1$ Equation of the tangent to the curve at (0, 2) is $y-2=1(x-0)$ $y=x+2$
(iii)	Substitute y = x + 2 into $\left(x+y\right)^2=4\mathrm{e}^{xy}$, $\left(x+x+2\right)^2=4\mathrm{e}^{x(x+2)}$ $\left(2x+2\right)^2=4\mathrm{e}^{x^2+2x}$ $\left(x+1\right)^2=\mathrm{e}^{x^2+2x}$ Using G.C., $x=-2$ or $x=0$ (reject $\because$ it's point A)

	$\therefore B(-2,0)$
(b)	Let r and h be the radius and the height of the cylinder respectively. Fixed vol. $p=\pi r^2 h \Rightarrow h=\dfrac{p}{\pi r^2}$ Surface area, S $=2\pi r^2+2\pi rh$ $=2\pi r^2+2\pi r\left(\dfrac{p}{\pi r^2}\right)$ $=2\pi r^2+\dfrac{2p}{r}$ $\dfrac{\mathrm{d}S}{\mathrm{d}r}=4\pi r-\dfrac{2p}{r^2}$ For min. S, $\dfrac{\mathrm{d}S}{\mathrm{d}r}=4\pi r-\dfrac{2p}{r^2}=0 \quad\Rightarrow\quad r=\left(\dfrac{p}{2\pi}\right)^{\frac{1}{3}}$ $\dfrac{\mathrm{d}^2S}{\mathrm{d}r^2}=4\pi+\dfrac{4p}{r^3}>0$ since r and p are positive. $\therefore$ S is minimum when $r=\left(\dfrac{p}{2\pi}\right)^{\frac{1}{3}}$ cm.

18(i)	$\dfrac{x^3-2y^2}{x^2+3xy}=1$ $x^3-2y^2=x^2+3xy$ Differentiating implicitly wrt x: $3x^2-4y\dfrac{\mathrm{d}y}{\mathrm{d}x}=2x+\left(3x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y(3)\right)$ $3x^2-2x-3y=(3x+4y)\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3x^2-2x-3y}{3x+4y}$ **Alternative (not advised)** Using quotient rule to differentiate: $\dfrac{\left(x^2+3xy\right)\left(3x^2-4y\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)-\left(x^3-2y^2\right)\left(2x+3x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y(3)\right)}{\left(x^2+3xy\right)^2}=0$ $\Rightarrow\left(x^2+3xy\right)\left(3x^2-4y\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)-\left(x^3-2y^2\right)\left(2x+3x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y(3)\right)=0$ (because $x^2+3xy\neq 0$)

	$\Rightarrow 3x^2\left(x^2+3xy\right)-\left(x^3-2y^2\right)\left(2x+3y\right)=4y\left(x^2+3xy\right)\frac{\mathrm{d}y}{\mathrm{d}x}+3x\left(x^3-2y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x^2\left(x^2+3xy\right)-\left(x^3-2y^2\right)\left(2x+3y\right)}{4y\left(x^2+3xy\right)+3x\left(x^3-2y^2\right)}=\frac{x^4+6x^3y+4xy^2+6y^3}{3x^4+4x^2y+6xy^2}$ Sub $x=1$, $\frac{1^3-2y^2}{1^2+3(1)y}=1$ $\Rightarrow 1-2y^2=1+3y$ $\Rightarrow 2y^2+3y=0$ $\Rightarrow y(2y+3)=0 \quad \therefore y=0$ or $y=-\frac{3}{2}$ Sub $x=1$ and $y=0$ into $\frac{\mathrm{d}y}{\mathrm{d}x}$: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3(1)^2-2(1)-3(0)}{3(1)+4(0)}=\frac{1}{3}$ Sub $x=1$ and $y=-\frac{3}{2}$ into $\frac{\mathrm{d}y}{\mathrm{d}x}$: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3(1)^2-2(1)-3\left(\frac{-3}{2}\right)}{3(1)+4\left(\frac{-3}{2}\right)}=-\frac{11}{6}$
(ii)	$\theta_1=\tan^{-1}\left(\frac{1}{3}\right)$ $\theta_2=\tan^{-1}\left(\frac{11}{6}\right)$  acute angle between tangents : $\theta_1+\theta_2=\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{11}{6}\right)=79.8^\circ$ (to 1d.p.) (or 1.39 rad)

19. ASRJC/2022/I/Q7

A curve C has parametric equations

$$x=\sin^3 t \ , \quad y=\cos^2 t, \ -\frac{\pi}{2}<t<0 .$$

The tangent at the point $P(\sin^3 p, \cos^2 p)$, $-\frac{\pi}{2}<p<0$, meets the x-axis and y-axis at Q and R respectively.

(i) By finding the equation of the tangent at the point P, show that the area of the triangle OQR is $-\frac{1}{12}\sin p\ (2+\cos^2 p)^2$. [6]

(ii) Find a cartesian equation of the locus of the mid-point of QR as p varies. You need not indicate its domain. [5]

	Solution	
	(i) $x=\sin^3 t \qquad y=\cos^2 t$ $\frac{dx}{dt}=3\sin^2 t\cos t \qquad \frac{dy}{dt}=-2\sin t\cos t$	
	$\frac{dy}{dx}=\frac{-2\sin t\cos t}{3\sin^2 t\cos t}=-\frac{2}{3\sin t}$	
	At the point P, $x=\sin^3 p$ $y=\cos^2 p$ $\frac{dy}{dx}=-\frac{2}{3\sin p}$	
	Equation of the tangent at the point P: $y-\cos^2 p=-\frac{2}{3\sin p}(x-\sin^3 p)$	
	When y = 0, $-\cos^2 p=-\frac{2}{3\sin p}(x-\sin^3 p)$ $x=\sin^3 p+\frac{3}{2}\sin p\cos^2 p$ $x=\frac{1}{2}\sin p(2\sin^2 p+3\cos^2 p)$ $x=\frac{1}{2}\sin p(2+\cos^2 p)$ $Q\left(\frac{1}{2}\sin p(2+\cos^2 p),0\right)$	

	When $x = 0$, $y-\cos^2 p = -\dfrac{2}{3\sin p}(0-\sin^3 p)$ $y=\dfrac{2}{3}\sin^2 p+\cos^2 p$ $y=\dfrac{1}{3}(2\sin^2 p+3\cos^2 p)=\dfrac{1}{3}(2+\cos^2 p)$ $R\left(0,\dfrac{1}{3}(2+\cos^2 p)\right)$	
	Area of the triange OQR $=\dfrac{1}{2}\times\left[0-\dfrac{1}{2}\sin p(2+\cos^2 p)\right]\times\dfrac{1}{3}(2+\cos^2 p)$ $=-\dfrac{1}{12}\sin p\left(2+\cos^2 p\right)^2$	
	(ii) Mid point of QR = $\left(\dfrac{\frac{1}{2}\sin p(2+\cos^2 p)+0}{2},\dfrac{0+\frac{1}{3}(2+\cos^2 p)}{2})\right)$ $=\left(\dfrac{1}{4}\sin p(2+\cos^2 p),\dfrac{1}{6}(2+\cos^2 p)\right)$	
	$x=\dfrac{1}{4}\sin p(2+\cos^2 p)$ ------------- (1) $y=\dfrac{1}{6}(2+\cos^2 p)$ ------------- (2)	
	$\dfrac{(1)}{(2)}$ gives	
	$\dfrac{x}{y}=\dfrac{\frac{1}{4}\sin p(2+\cos^2 p)}{\frac{1}{6}(2+\cos^2 p)}$ $\dfrac{x}{y}=\dfrac{3}{2}\sin p$ $\sin p=\dfrac{2x}{3y}$	
	$y=\dfrac{1}{6}(2+\cos^2 p)$ $y=\dfrac{1}{6}\left(2+(1-\sin^2 p)\right)$ $y=\dfrac{1}{6}\left(3-\dfrac{4x^2}{9y^2}\right)$	

	$y=\frac{1}{54y^2}\left(27y^2-4x^2\right)$	
	$54y^3=27y^2-4x^2$ Cartesian equation of the locus of the mid-point of QR is $54y^3 = 27y^2 - 4x^2$	

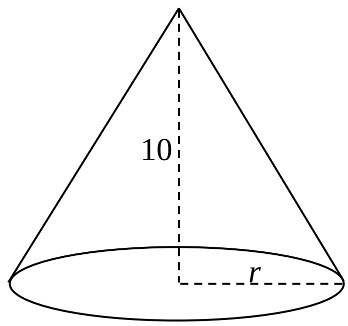
Differentiation (Rate of Change and Maxima Problems)

1(i)	At point P, $y=16\tan^{-1}t^3-4t+16$ and $y=4(5-\pi)$ $4\tan^{-1}t^3-t+4=5-\pi$ By observation, when $t=-1$, LHS = $4\left(-\frac{\pi}{4}\right)-(-1)+4=5-\pi=\text{RHS}$ OR by G.C, $t=-1$. $x=3(-1)^2-10(-1)-1=12$ $x=3t^2-10t-1$ $y=16\tan^{-1}t^3-4t+16$ $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{\frac{48t^2}{1+t^6}-4}{6t-10}$ When $t=-1$, $x=12$, $y=4(5-\pi)$, $\frac{dy}{dx}=\frac{24-4}{-6-10}=\frac{5}{-4}$ Equation of tangent: $y-4(5-\pi)=-\frac{5}{4}(x-12)$ $y+4\pi-20=-\frac{5}{4}x+15$ $5x+4y+16\pi-140=0$ (shown)
1(ii)	To find m, find coordinates of B. i.e solve the equations simultaneously. Eqn of curve: $x=3t^2-10t-1$, $y=16\tan^{-1}t^3-4t+16$ Eqn of tangent: $5x+4y+16\pi-140=0$ $5\left(3t^2-10t-1\right)+4\left(16\tan^{-1}t^3-4t+16\right)+16\pi-140=0$ Using G.C, $t=-0.568281$ or $t=-1$ (reject since at P) Hence, $m=16\tan^{-1}\left((-0.568281)^3\right)-4(-0.568281)+16$ = 15.369 (3 d.p.) (or 15.372 if 0.568 was used)
1(last part)	At $t=-1$, Equation of normal: $y-4(5-\pi)=\frac{4}{5}(x-12)$ $y+4\pi-20=\frac{4}{5}x-\frac{48}{5}$

	$y=\frac{4}{5}x+\frac{52}{5}-4\pi$ At point E, $y=0$. $\frac{4}{5}x = -\frac{52}{5}+4\pi$ $x = 5\pi-13$ Length of EP $=\sqrt{(5\pi-13-12)^2+(0+4\pi-20)^2}$ $=\sqrt{(5\pi-25)^2+(4\pi-20)^2}$ $=\sqrt{25(\pi-5)^2+16(\pi-5)^2}$ $=\sqrt{41}(5-\pi)$

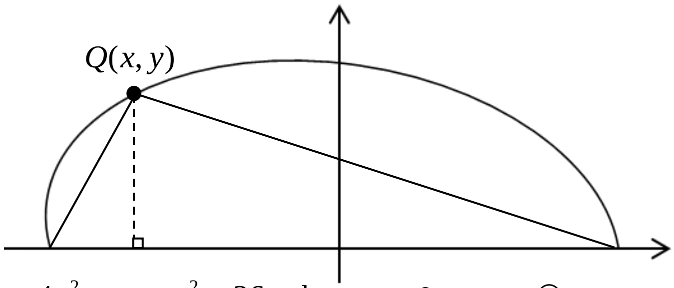
2	$AD=2\sqrt{r^2-x^2}$ $\rightarrow P=2x+2\left[2\sqrt{r^2-x^2}\right]$ $=2x+4\sqrt{r^2-x^2}$ $\frac{\mathrm{d}P}{\mathrm{d}x}=2-4x\left(r^2-x^2\right)^{-\frac{1}{2}}$ $\frac{\mathrm{d}P}{\mathrm{d}x}=0 \quad \rightarrow \quad 2-4x\left(r^2-x^2\right)^{-\frac{1}{2}}=0$ $\sqrt{r^2-x^2}=2x$ $\frac{AB}{BC}=\frac{1}{k}$ $\rightarrow \frac{x}{2\sqrt{r^2-x^2}}=\frac{1}{k}$ $\frac{1}{k}=\frac{1}{4} \quad \rightarrow k=4$

3	**(ii)** From (i), $\int \frac{-2}{1+x^2}\,\mathrm{d}x = \cos^{-1}\left(\frac{2x}{1+x^2}\right)+C''$ But $\int \frac{-2}{1+x^2}\,\mathrm{d}x = -2\tan^{-1}x+C'$ Hence, $\cos^{-1}\left(\frac{2x}{1+x^2}\right)=-2\tan^{-1}x+C$ When $x=0$, $\cos^{-1}(0)=\frac{\pi}{2}$, $-2\tan^{-1}0=0$ Therefore, $\cos^{-1}\left(\frac{2x}{1+x^2}\right)=-2\tan^{-1}x+\frac{\pi}{2}$.

	$\Rightarrow \cos^{-1}\left(\frac{2x}{1+x^2}\right)+2\tan^{-1}x=\frac{\pi}{2}$ $\Rightarrow A=2$ and $B=\frac{\pi}{2}$. Alternatively : Given $\cos^{-1}\left(\frac{2x}{1+x^2}\right)+A\tan^{-1}x=B$ ... (1) Therefore $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}\left(\frac{2x}{1+x^2}\right)+A\tan^{-1}x\right)=0$ $\Rightarrow \frac{-2}{1+x^2}+\frac{A}{1+x^2}=0 \Rightarrow \frac{A-2}{1+x^2}=0 \Rightarrow A-2=0 \Rightarrow A=2$ Next substitute A = 2 and x = 0 into (1) : $\cos^{-1}(0)+2\tan^{-1}0=B \Rightarrow B=\frac{\pi}{2}$.
	Let the volume of the cone by V, and area of the circular base be M. $V=\frac{1}{3}(M)(10)=\frac{10}{3}M \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t}=\frac{10}{3}\frac{\mathrm{d}M}{\mathrm{d}t}$ $\Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t}=\frac{10}{3}(2)=\frac{20}{3}$ cm^{3}s^{-1}.
	Let the radius of the circular base be r . $A=\pi r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}r}=2\pi r$ $V=\frac{1}{3}\pi r^2(10)=\frac{10}{3}\pi r^2 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r}=\frac{20}{3}\pi r$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}A}\times\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}A}\times\frac{\mathrm{d}A}{\mathrm{d}t}$ $=\left(\frac{20}{3}\pi r\right)\left(\frac{1}{2\pi r}\right)(2)=\frac{20}{3}$ cm^{3}s^{-1}. 

4	(i) $\frac{dy}{dt}=\frac{-2t^2+2a}{(t^2+a)^2}$, $\frac{dx}{dt}=8t$ $\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}=\frac{a-t^2}{4t(t^2+a)^2}$ =0 t = $\pm\sqrt{a}$ $x=4a,\ y=\frac{\pm(2\sqrt{a})}{2a}=\pm\frac{\sqrt{a}}{a}$

	Turning points: $(4a,\dfrac{\sqrt{a}}{a})$, $(4a,-\dfrac{\sqrt{a}}{a})$

5	(i)  Given: $4x^2+xy+y^2=36$ where $y\geq 0$. ------- ① At $y=0$, $4x^2=36\Rightarrow x=-3,\ 3$ ie, base length of triangle PR = 6 $A=\dfrac{1}{2}(6)y=3y$ ------- ② (ii) Diff ① wrt x $8x+x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y+2y\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$------- ③ $\left(x+2y\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\left(8x+y\right)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{8x+y}{x+2y}$ (iii) Diff ② wrt x $\dfrac{\mathrm{d}A}{\mathrm{d}x}=3\dfrac{\mathrm{d}y}{\mathrm{d}x}$ For max A, $\dfrac{\mathrm{d}A}{\mathrm{d}x}=0\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ i.e. $8x+y=0$ or $y=-8x$ ----- ④ Substitute ④ into ①: $4x^2+x\left(-8x\right)+\left(-8x\right)^2=36$ $\therefore 60x^2=\dfrac{3}{5}\Rightarrow x=-\sqrt{\dfrac{3}{5}}$, $\sqrt{\dfrac{3}{5}}$ (reject since $y=-8x<0$)

<u>For 2nd derivative test, diff ③ wrt x</u>

$$8+x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}+2\left(y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)=0$$

At $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$$\therefore 8+\left(x+2y\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{8}{x+2y}=-\frac{8}{x+2(-8x)}=\frac{8}{15x}$$

At $x=-\sqrt{\frac{3}{5}}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}<0$ gives $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}<0$

ie. A is maximum at $x=-\sqrt{\frac{3}{5}}$

<u>Method 1</u>

At $x = 0$, ③: $0+0+y+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}$

Using chain rule on : $A=3y$

$$\Rightarrow\frac{\mathrm{d}A}{\mathrm{d}t}=3\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$=3\left(-\frac{1}{2}\right)(8)=-12\text{ units}^2/\text{s}$$

i.e., the area of the triangle decreases at a rate of $12\text{ units}^2/\text{s}$

<u>Method 2</u>

Diff ① wrt <u>t</u>

$$8x\frac{\mathrm{d}x}{\mathrm{d}t}+x\frac{\mathrm{d}y}{\mathrm{d}t}+y\frac{\mathrm{d}x}{\mathrm{d}t}+2y\frac{\mathrm{d}y}{\mathrm{d}t}=0$$

At $x=0$,

$$y\frac{\mathrm{d}x}{\mathrm{d}t}+2y\frac{\mathrm{d}y}{\mathrm{d}t}=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}$$

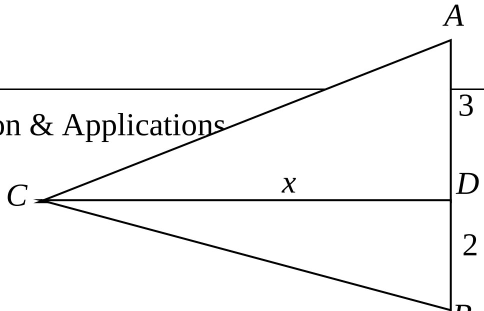
$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{2}(8)=-4$$

From $A=3y\Rightarrow\frac{\mathrm{d}A}{\mathrm{d}t}=3\frac{\mathrm{d}y}{\mathrm{d}t}$

$$\frac{\mathrm{d}A}{\mathrm{d}t}=3(-4)=-12\text{ units}^2/\text{s}$$

i.e., the area of the triangle decreases at a rate of $12\text{ units}^2/\text{s}$.

6



(i) Using triangles ACD and BCD,

$$\theta = \measuredangle ACB = \alpha + \beta = \tan^{-1}\frac{3}{x} + \tan^{-1}\frac{2}{x}$$

(ii) Given $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -10\left(\text{since } x \text{ is decreasing}\right)$, find $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$.

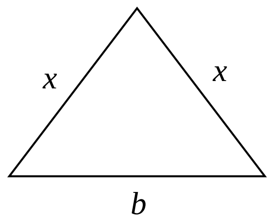
$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{1+\left(\frac{3}{x}\right)^2}\left(-\frac{3}{x^2}\right) + \frac{1}{1+\left(\frac{2}{x}\right)^2}\left(-\frac{2}{x^2}\right) = \frac{-3}{x^2+9} + \frac{-2}{x^2+4}$$

$$\begin{aligned}\therefore \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\mathrm{d}\theta}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \left(\frac{-3}{x^2+9} + \frac{-2}{x^2+4}\right)(-10)\end{aligned}$$

$$\therefore \text{ at } x = 10,\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = 10\left(\frac{3}{10^2+9} + \frac{2}{10^2+4}\right) = \frac{1325}{2834} \text{ rads}^{-1}$$

7 Let x be the length of each of the other 2 sides of the triangle.

$$\begin{aligned}\text{Area, } A &= \frac{1}{2} b \times \text{height} \\ &= \frac{1}{2} b \sqrt{x^2 - \left(\frac{b}{2}\right)^2}\end{aligned}$$

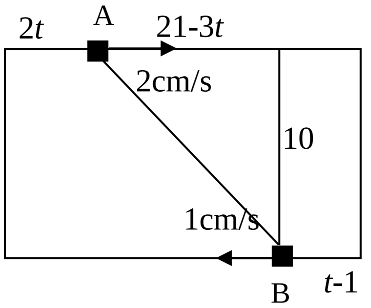


$$\Rightarrow \frac{dA}{dx} = \frac{1}{4} b \left(x^2 - \frac{b^2}{4}\right)^{-\frac{1}{2}} (2x)$$

$$\begin{aligned}\text{Now, } \frac{dA}{dt} &= \frac{dA}{dx} \times \frac{dx}{dt} \\ &= \frac{1}{4} b \left(x^2 - \frac{b^2}{4}\right)^{-\frac{1}{2}} (2x) \frac{dx}{dt}\end{aligned}$$

When $x = b$, $\quad \dfrac{dx}{dt} = -3$, $\quad \dfrac{dA}{dt} = \dfrac{1}{4} b \left(b^2 - \dfrac{b^2}{4}\right)^{-\frac{1}{2}} (2b)(-3)$

$$\Rightarrow \frac{dA}{dt} = \frac{-\frac{3}{2} b^2}{\sqrt{\frac{3}{4} b^2}} = -\sqrt{3} b \text{ cm}^2 / \text{s}$$

8(i)	$2y+2z=48$, $x+2z=18$ Expressing z and x in terms of y, $z=24-y$, $x=2y-30$ $V=xyz=(2y-30)y(24-y)=-2y^3+78y^2-720y$.
8(ii)	$\frac{dV}{dy}=0$ $-6y^2+156y-720=0$ $y^2-26y+120=0$ Using G.C, $y=6$ or $y=20$ y = 6 is not a feasible solution as x will be negative. $\frac{d^2V}{dy^2}=-12y+156$ When $y=20$, $\frac{d^2V}{dy^2}=-84<0$ Hence, when $y=20$, Maximum volume = $20(2\times 20-30)(24-20)=800$
8(iii)	 Let t be the time in seconds when robot A starts to move. m = 2t and n = t-1 Distance between A and B = l, $l^2=(21-3t)^2+10^2$ Differentiating wrt t, $2l\frac{\mathrm{d}l}{\mathrm{d}t}=2(21-3t)(-3)$ At n = 4, t = 5 $\frac{\mathrm{d}l}{\mathrm{d}t}=\frac{(6)(-3)}{\sqrt{6^2+10^2}}=-\frac{9}{\sqrt{34}}$ cm/s <u>Method 2:</u> $l^2=(20-m-n)^2+10^2$ Since m = 2n + 2, $l^2=(18-3n)^2+10^2$ Differentiating wrt n, $2l\frac{dl}{dn}=-6(18-3n)$ At n = 4, $l^2=10^2+6^2$. $\frac{\mathrm{d}l}{\mathrm{d}n}=\frac{-18}{\sqrt{10^2+6^2}}$ $\frac{\mathrm{d}l}{\mathrm{d}t}=\frac{\mathrm{d}l}{\mathrm{d}n}\frac{\mathrm{d}n}{\mathrm{d}t}=\frac{-18}{\sqrt{10^2+6^2}}(1)=\frac{-18}{\sqrt{10^2+6^2}}=-\frac{9}{\sqrt{34}}$ cm/s

9(i)

$$\frac{1}{3}\pi r^2 h = 10$$

$$h = \frac{30}{\pi r^2}$$

Let the slanted height of the cone be l cm.

$$\begin{aligned} A &= \pi r l \\ &= \pi r\sqrt{h^2 + r^2} \end{aligned}$$

$$\begin{aligned} A^2 &= \pi^2 r^2\left(h^2 + r^2\right) \\ &= \pi^2 r^4 + \pi^2 r^2\left(\frac{30}{\pi r^2}\right)^2 \\ &= \pi^2 r^4 + \frac{900}{r^2} \end{aligned}$$

(ii)

$$2A\frac{\mathrm{d}A}{\mathrm{d}r} = 4\pi^2 r^3 - \frac{1800}{r^3}$$

Since $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$,

$$4\pi^2 r^3 = \frac{1800}{r^3}$$

$$r^3\left(4\pi^2 - \frac{1800}{r^6}\right) = 0$$

$r^3 = 0$

$r = 0$ (Reject)

or

$$4\pi^2 - \frac{1800}{r^6} = 0$$

$$r^6 = \frac{450}{\pi^2}$$

$$r = \sqrt[6]{\frac{450}{\pi^2}} \approx 1.890$$

$$2\left(\frac{\mathrm{d}A}{\mathrm{d}r}\right)^2 + 2A\frac{\mathrm{d}^2A}{\mathrm{d}r^2} = 12\pi^2 r^2 + \frac{5400}{r^4}$$

At $r \approx 1.890$, $\frac{\mathrm{d}^2\mathrm{A}}{\mathrm{d}r^2} = 21.7655 > 0$ which indicates that the curved surface area of the cone is a minimum when $r \approx 1.890$.

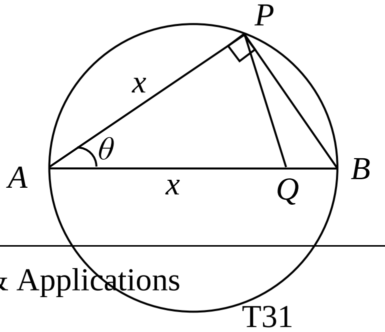
Alternatively,

r	1.890^-	1.890	1.890^+
Sign of $\frac{\mathrm{d}A}{\mathrm{d}r}$	-ve	0	+ve
Slope	\	-	/

Therefore the curved surface area of the cone is a minimum when $r \approx 1.890$.

10(i)	Base Area of Packaging $=6\left(\frac{1}{2}x^2\sin 60^0\right)=\frac{3\sqrt{3}x^2}{2}\,cm^2$.
(ii)	Volume $=\frac{3\sqrt{3}x^2}{2}h=972 \Rightarrow h=\frac{1944}{3\sqrt{3}x^2}$ Area of the material, $A=2\left(\frac{3\sqrt{3}x^2}{2}\right)+6xh=3\sqrt{3}x^2+\frac{6x(1944)}{3\sqrt{3}x^2}=3\sqrt{3}x^2+\frac{3888}{\sqrt{3}x}$ $\frac{dA}{dx}=6\sqrt{3}x-\frac{3888}{\sqrt{3}x^2}=0 \Rightarrow 18x^3-3888=0 \Rightarrow x^3=216 \Rightarrow x=6$ Check, $\frac{d^2A}{dx^2}=6\sqrt{3}+\frac{7776}{\sqrt{3}x^3}>0$ So area is minimum when x = 6cm Minimum Area $=3\sqrt{3}\left(6^2\right)+\frac{3888}{\sqrt{3}(6)}=561.1844... \approx 561.18 \text{ (2 d. p.)}$
(ii)	Minimum cost of packaging $=\frac{\$0.05}{100\,\text{cm}^2}\times\left(561.1844...\text{cm}^2\right)=\$0.28$ (nearest cent)

11	Let s be the length of the minor arc PQ and A be the area of the shaded segment. $s=2a\theta \;\Rightarrow\; \frac{\mathrm{d}s}{\mathrm{d}t}=2a\frac{\mathrm{d}\theta}{\mathrm{d}t}$ $A=\frac{1}{2}\left(4a^2\right)(\theta-\sin\theta)$ $\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t}=2a^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}-\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)$ $=2a^2\frac{\mathrm{d}\theta}{\mathrm{d}t}(1-\cos\theta)$ Given $\frac{\mathrm{d}A}{\mathrm{d}t}=\left(\frac{a}{2}\right)\frac{\mathrm{d}s}{\mathrm{d}t}$, $2a^2\frac{\mathrm{d}\theta}{\mathrm{d}t}(1-\cos\theta)=a^2\frac{\mathrm{d}\theta}{\mathrm{d}t}$ $\cos\theta=\frac{1}{2}$ $\theta=\frac{\pi}{3}$ or 60°

12	

	In ΔPAB, $\cos\theta = \dfrac{AP}{AB} = \dfrac{x}{\ell}$ $\therefore\ x = \ell\cos\theta$ $S = \dfrac{1}{2}x^2\sin\theta$ $= \dfrac{1}{2}(\ell\cos\theta)^2\sin\theta$ $= \dfrac{1}{2}\ell^2(1-\sin^2\theta)\sin\theta$ $= \dfrac{1}{2}\ell^2(\sin\theta - \sin^3\theta)$ (shown)
	$S = \dfrac{1}{2}\ell^2(\sin\theta-\sin^3\theta)$ $\dfrac{dS}{d\theta} = \dfrac{1}{2}\ell^2(\cos\theta - 3\sin^2\theta\cos\theta) = \dfrac{1}{2}\ell^2\cos\theta(1-3\sin^2\theta)$ $\dfrac{dS}{d\theta} = 0 \Rightarrow \dfrac{1}{2}\ell^2\cos\theta(1-3\sin^2\theta) = 0$ $\cos\theta = 0$ (rejected) or $\sin\theta = \dfrac{1}{\sqrt{3}}$ or $\sin\theta = -\dfrac{1}{\sqrt{3}}$ (rejected) since $0 < \theta < \dfrac{\pi}{2}$ $\dfrac{d^2S}{d\theta^2} = \dfrac{1}{2}\ell^2\left\{(\cos\theta)(-6\sin\theta\cos\theta) + (1-3\sin^2\theta)(-\sin\theta)\right\}$ When $\sin\theta = \dfrac{1}{\sqrt{3}}$, $\cos^2\theta = \dfrac{2}{3}$, $\dfrac{d^2S}{d\theta^2} = -\dfrac{2}{\sqrt{3}}\ell^2 < 0$ $\therefore$ S is max when $\sin\theta = \dfrac{1}{\sqrt{3}}$ max $S = \dfrac{1}{2}\ell^2\left(\dfrac{1}{\sqrt{3}} - \dfrac{1}{3\sqrt{3}}\right) = \dfrac{1}{2}\ell^2\left(\dfrac{2}{3}\right)\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{9}\ell^2$

13 (i)	Let l be the slant height of the cone. $l^2 = h^2 + r^2$ ----(1) Using similar triangles, $\dfrac{h-3}{l} = \dfrac{3}{r}$ $l = \dfrac{rh-3r}{3}$ ----(2) Equating (1) and (2),

$$\left(\frac{rh-3r}{3}\right)^2 = h^2 + r^2 \quad ----(*)$$

$$r^2h^2 - 6r^2h + 9r^2 = 9h^2 + 9r^2$$

$$r^2(h^2 - 6h) = 9h^2$$

$$\therefore r = \frac{3h}{\sqrt{h^2 - 6h}} \qquad (\text{Since } r > 0)$$

(ii) Volume of cone, $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\frac{3h}{\sqrt{h^2-6h}}\right)^2 h$$

$$= \frac{3\pi h^3}{h^2 - 6h}$$

$$= \frac{3\pi h^2}{h-6}$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{6\pi h(h-6) - 3\pi h^2}{(h-6)^2}$$

$$= \frac{3\pi h^2 - 36\pi h}{(h-6)^2}$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = 0 \quad \Rightarrow \quad 3\pi h^2 - 36\pi h = 0$$

$$h(h-12) = 0$$

$h = 12$ or $h = 0$ (reject $\because h > 0$)

h	12^-	12	12^+
Sign of $\frac{\mathrm{d}V}{\mathrm{d}h}$	– ve	0	+ ve
Tangent			

Thus, V is a minimum when $h = 12$

When $h = 12$,

$$r = \frac{3(12)}{\sqrt{(12)^2 - 6(12)}} = \frac{6}{\sqrt{2}} \qquad (\approx 4.2426)$$

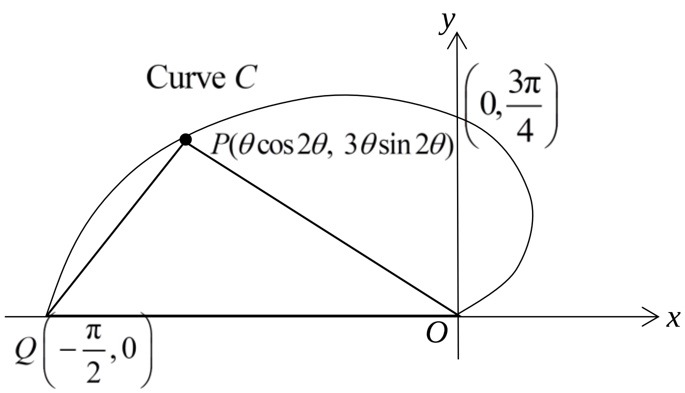
$$V = \frac{3\pi(12)^2}{12-6} = 72\pi \qquad (\approx 226.195)$$

(iii) Let R be the radius of the snowball

$$S = 4\pi R^2 \quad \Rightarrow \quad \frac{\mathrm{d}S}{\mathrm{d}t} = 8\pi R\frac{\mathrm{d}R}{\mathrm{d}t}$$

$$V = \frac{4}{3}\pi R^3 \quad \Rightarrow \quad \frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi R^2\frac{\mathrm{d}R}{\mathrm{d}t}$$

	When $R = 2.5$, $\dfrac{\mathrm{d}S}{\mathrm{d}t} = -0.75 \quad \Rightarrow \quad 8\pi(2.5)\dfrac{\mathrm{d}R}{\mathrm{d}t} = -0.75$ $\dfrac{\mathrm{d}R}{\mathrm{d}t} = -\dfrac{3}{80\pi} \quad or \quad -\dfrac{0.0375}{\pi} \quad or \quad -0.0119366$ $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 4\pi(2.5)^2\left(-\dfrac{3}{80\pi}\right) = -\dfrac{15}{16} \quad or \quad -0.9375$ At the instant when $R = 2.5$ m, the rate of decrease of volume is 0.9375 m^3 per minute.

14(i)	 Point P has coordinates $(\theta\cos 2\theta,\ 3\theta\sin 2\theta)$. Let area of triangle OPQ be A. $A = \dfrac{1}{2}\left(\dfrac{\pi}{2}\right)(3\theta\sin 2\theta) = \dfrac{3\pi}{4}(\theta\sin 2\theta)$ $\dfrac{\mathrm{d}A}{\mathrm{d}\theta} = \dfrac{3\pi}{4}(2\theta\cos 2\theta + \sin 2\theta)$ $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \dfrac{\mathrm{d}A}{\mathrm{d}\theta} \times \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ $= \dfrac{3\pi}{4}(2\theta\cos 2\theta + \sin 2\theta)(0.01)$ When $\theta = \dfrac{\pi}{6}$, $\dfrac{\mathrm{d}A}{\mathrm{d}t}$ =0.0327 units^2/s (3 s.f.)

(ii)	When $\dfrac{\mathrm{d}A}{\mathrm{d}\theta}=0$, $\dfrac{3\pi}{4}\left(2\theta\cos 2\theta+\sin 2\theta\right)=0$ Since $\dfrac{3\pi}{4}\neq 0$, $2\theta\cos 2\theta+\sin 2\theta=0$ Using GC, $\theta=1.0144$ (5 s.f.) $=1.01$ (3 s.f.) $\dfrac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=\dfrac{3\pi}{4}\left(-4\theta\sin 2\theta+2\cos 2\theta+2\cos 2\theta\right)$ When $\theta=1.0144$, $\dfrac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=-12.7<0$ $\therefore$ $\theta=1.0144$ will result in maximum *A*. When $\theta=1.0144$, $A=\dfrac{3\pi}{4}\left(1.0144\right)\sin(2\times 1.0144)$ $=2.14\text{ units}^2\ \left(3\text{ s.f.}\right)$ When $\theta=1.0144$, $x=1.0144\cos\left[2\left(1.0144\right)\right]=-0.449$ $y=3\left(1.0144\right)\sin\left[2\left(1.0144\right)\right]=2.73$ $\therefore$ Location of the camera is at a point with coordinates $\left(-0.449,\ 2.73\right)$
(iii)	For triangle *OPQ* to be an isosceles triangle, $x=-\dfrac{\pi}{2}\div 2=-\dfrac{\pi}{4}$ $-\dfrac{\pi}{4}=\theta\cos 2\theta$ Using GC, $\theta=1.1581$ $y=3\left(1.1581\right)\sin\left(2\times 1.1581\right)=2.55$ $\therefore$ coordinates of $P\left(-0.785,\ 2.55\right)$

<table>
<tr><td>**15(i)**</td><td>

Let $\alpha = \angle TPB$ and $\beta = \angle CPB$

$\tan\theta = \tan(\alpha - \beta)$

$= \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$\tan\theta = \dfrac{\frac{b}{x} - \frac{a}{x}}{1+\left(\frac{b}{x}\right)\left(\frac{a}{x}\right)}$

$\tan\theta = \dfrac{(b-a)x}{x^2+ab}$

</td></tr>
<tr><td>**15(ii)**</td><td>

$\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan\theta) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{(b-a)x}{x^2+ab}\right)$

$\sec^2\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{(x^2+ab)(b-a)-(2x)(b-a)x}{(x^2+ab)^2}$

$\sec^2\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{(ab-x^2)(b-a)}{(x^2+ab)^2}$

OR <u>(Not recommended)</u>

$\theta = \tan^{-1}\left(\dfrac{(b-a)x}{x^2+ab}\right)$

$\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{1}{1+\left(\frac{(b-a)x}{x^2+ab}\right)^2} \times \dfrac{(x^2+ab)(b-a)-2x^2(b-a)}{(x^2+ab)^2}$

$= \vdots$

$\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0,\quad \dfrac{(ab-x^2)(b-a)}{(x^2+ab)^2}=0$

$x=\sqrt{ab}$ or $x=-\sqrt{ab}$ (rej $\because x>0$)

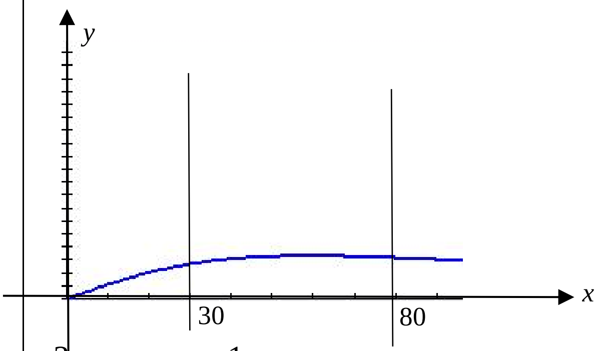
$\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{(ab-x^2)(b-a)}{(x^2+ab)^2\sec^2\theta}$

Since $(x^2+ab)^2>0$, $\sec^2\theta>0$ and $b>a \Rightarrow b-a>0$, it suffices to check $(ab-x^2)$

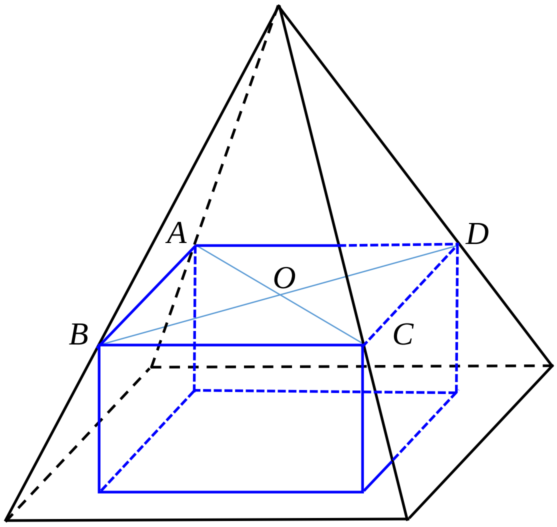
Using first derivative test:

x	$\left(\sqrt{ab}\right)^-$	$\sqrt{ab}$	$\left(\sqrt{ab}\right)^+$
$\dfrac{\mathrm{d}\theta}{\mathrm{d}x}$	+ve $\because ab-x^2>0$	0	-ve $\because ab-x^2<0$

</td></tr>
</table>

	$\therefore\ \theta$ is a maximum when $x=\sqrt{ab}$. When $x=\sqrt{ab}$, $\tan\theta=\dfrac{(b-a)\sqrt{ab}}{2ab}$
15(iii)	Given $a=50, b=70$, $\tan\theta=\dfrac{(b-a)x}{x^2+ab}=\dfrac{20x}{x^2+3500}$  $\dfrac{3}{22}<\tan\theta\le\dfrac{1}{\sqrt{35}}$
15(iv)	$\dfrac{\mathrm{d}x}{\mathrm{d}t}=-3$ When $a=50, b=70, x=10$, $\tan\theta=\dfrac{(b-a)x}{x^2+ab}$ $=\dfrac{1}{18}$ $\therefore \sec^2\theta=\dfrac{325}{324}$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}x}=\dfrac{(x^2-ab)(a-b)}{(x^2+ab)^2\sec^2\theta}=\dfrac{17}{3250}$ or 0.0052308 $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=\dfrac{\mathrm{d}\theta}{\mathrm{d}x}\times\dfrac{\mathrm{d}x}{\mathrm{d}t}$ $=-3\times\dfrac{17}{3250}$ $=-\dfrac{51}{3250}$ The angle θ is decreasing at a rate of $\dfrac{51}{3250}$ rad/s.

16. ASRJC/2022/2/Q4



The product engineer of a factory crafted the design of a rectangular box, using a right pyramid, that is shown on the diagram above (not drawn to scale). The rectangular box is contained in a right pyramid with a rectangular base such that the upper four corners of the box *A*, *B*, *C* and *D* touch the slant faces of the pyramid, and the bottom four corners lie on the base of the pyramid. *O* is the point of intersection of the two diagonals, *AC* and *BD*.

The height of the pyramid is $3\sqrt{2}$ units, the length of the diagonal of its rectangular base is $12\sqrt{2}$ units, the height of the box is *b* units, where $b < 3\sqrt{2}$, and the angle *AOB* is θ radians. It is given that the box is made of material with negligible thickness.

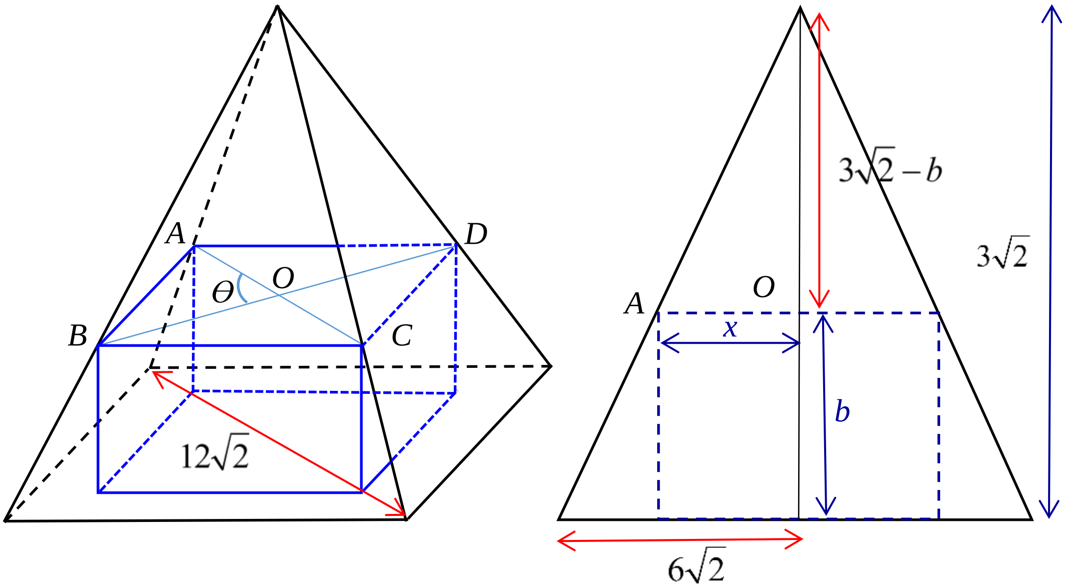
(i) By finding the length of *OA* in terms of *b*, show that the volume *V* of the rectangular box is given by $V = 8b\left(3\sqrt{2} - b\right)^2 \sin\theta$. [3]

For the rest of the question, it is given that $\theta = \dfrac{\pi}{3}$.

(ii) Find the exact value of *b* which maximises *V*. Hence find the cost of manufacturing one such box if the material used to make the box cost \$0.03 per unit^2. [6]

When the height of the box is at half the height of the pyramid, it is reducing at a rate of 2 units per second.

(iii) Determine whether the volume of the box is expanding or shrinking and find the rate at which this is happening. [3]

	Solution	
		
	(i) Let *OA* = *x* and *V* = volume of box By similar triangles, $\dfrac{x}{2(3\sqrt{2})}=\dfrac{3\sqrt{2}-b}{3\sqrt{2}} \Rightarrow x=2\left(3\sqrt{2}-b\right)$	
	$V = AB \times BC \times b$	
	$=\left(2x\sin\dfrac{\theta}{2}\right)\left(2x\cos\dfrac{\theta}{2}\right)b$	
	$=2b\left[2\left(3\sqrt{2}-b\right)\right]^2\sin\theta$	
	$V=8b\left(3\sqrt{2}-b\right)^2\sin\theta$ (shown)	
	(ii) $V=4\sqrt{3}b\left(3\sqrt{2}-b\right)^2$ $\dfrac{\mathrm{d}V}{\mathrm{d}b}=4\sqrt{3}\left[-2b\left(3\sqrt{2}-b\right)+\left(3\sqrt{2}-b\right)^2\right]$	
	$\dfrac{\mathrm{d}V}{\mathrm{d}b}=4\sqrt{3}\left(3\sqrt{2}-b\right)\left(3\sqrt{2}-3b\right)$	
	For stationary point, $4\sqrt{3}\left(3\sqrt{2}-b\right)\left(3\sqrt{2}-3b\right)=0$	
	$\Rightarrow b=\sqrt{2}$ or $b=3\sqrt{2}$ (rejected since $b<h$)	

	$\frac{d^2V}{db^2} = 4\sqrt{3}\left[-2b(-1)+\left(3\sqrt{2}-b\right)(-2)+2\left(3\sqrt{2}-b\right)(-1)\right]$ $= 4\sqrt{3}\left[6b-12\sqrt{2}\right]$ $= 24\sqrt{3}\left(b-2\sqrt{2}\right)$		
	$\left.\frac{d^2V}{db^2}\right\|_{b=\sqrt{2}} = -24\sqrt{6} < 0$		
	Thus V is maximised when $b=\sqrt{2}$.		
	$BC = 4\left(3\sqrt{2}-\sqrt{2}\right)\cos\frac{\pi}{6} = 4\sqrt{6}$ $AB = 4\left(3\sqrt{2}-\sqrt{2}\right)\sin\frac{\pi}{6} = 4\sqrt{2}$		
	Cost $= 0.03\times 2\left[4\sqrt{6}\left(4\sqrt{2}\right)+\sqrt{2}\left(4\sqrt{6}\right)+\sqrt{2}\left(4\sqrt{2}\right)\right]$ – to find surface area		
	$= \$4.64$		
	(iii) $\frac{dV}{dt} = \frac{dV}{db}\times\frac{db}{dt}$		
	When $b = \frac{3}{2}\sqrt{2}$,		
	$\left.\frac{dV}{dt}\right\|_{b=\frac{3}{2}\sqrt{2}} = 4\sqrt{3}\left(3\sqrt{2}-\frac{3}{2}\sqrt{2}\right)\left(3\sqrt{2}-\frac{9}{2}\sqrt{2}\right)\times(-2\text{ units/s})$		
	$= 36\sqrt{3}\ \text{units}^3/\text{s}$		
	Since $\left.\frac{dV}{dt}\right\|_{b=\frac{3}{2}\sqrt{2}} > 0$, the volume of the box is expanding.		

Differentiation (Maclaurin Series)

1	$(1+x^2)\frac{dy}{dx}+xy=\sqrt{1+x^2}$ $(1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+x\frac{dy}{dx}+y=\frac{1}{2}\times\frac{2x}{\sqrt{1+x^2}}$ $(1+x^2)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{x}{\sqrt{1+x^2}}$ When $x=0$, $y=1$, $(1)\Rightarrow\frac{dy}{dx}=1$, $(2)\Rightarrow\frac{d^2y}{dx^2}=-1$, By Maclaurin's series, $y=f(x)=f(0)+f'(0)x+\frac{f''(0)}{(2!)}x^2+\cdots$ $y=1+x+\frac{-1}{(2!)}x^2+\cdots=1+x-\frac{1}{2}x^2+\cdots$ Method 1: $\int_0^{0.01}\frac{dy}{dx}\,dx=[y]_0^{0.01}$ $=\left[1+x-\frac{1}{2}x^2+\cdots\right]_0^{0.01}=1+0.01-\frac{1}{2}(0.01)^2-1\approx 0.00995(3s.f)\approx 0.010(3d.p.)$ Method 2: $\int_0^{0.01}\frac{dy}{dx}\,dx=\int_0^{0.01}1-x\,dx=\left[x-\frac{x^2}{2}\right]_0^{0.01}\approx 0.00995(3s.f)\approx 0.010(3d.p.)$ NOTE – Integration done by GC - accepted

2	$y=\cot\left(2x+\frac{\pi}{4}\right)$ $\Rightarrow\frac{dy}{dx}=-\operatorname{cosec}^2\left(2x+\frac{\pi}{4}\right).2=-2\left(1+y^2\right)$ $\frac{d^2y}{dx^2}=-4y\frac{dy}{dx}$ $\Rightarrow\frac{d^3y}{dx^3}=-4\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]$. Therefore, $k=-4$
	When $x=0$, $y=1$, $\frac{dy}{dx}=-4$, $\frac{d^2y}{dx^2}=16$, $\frac{d^3y}{dx^3}=-128$ Therefore, $\cot\left(2x+\frac{\pi}{4}\right)\approx 1-4x+8x^2-\frac{64}{3}x^3$

	Differentiating the expansion above, $-2\operatorname{cosec}^2\left(2x+\frac{\pi}{4}\right)\approx -4+16x-64x^2$ Therefore, $\operatorname{cosec}^2\left(2x+\frac{\pi}{4}\right)\approx 2-8x+32x^2$ Let $2x+\frac{\pi}{4}=\frac{13\pi}{50}$. Then $x=\frac{\pi}{200}$ Hence, $\operatorname{cosec}^2\left(\frac{13\pi}{50}\right)=2-\frac{\pi}{25}+\frac{\pi^2}{1250}$ where $a=2$, $b=-\frac{1}{25}$, $c=\frac{1}{1250}$

3	$y=\sin\left[\ln(1-3x)\right]$ $\frac{dy}{dx}=\cos\left[\ln(1-3x)\right]\left(\frac{-3}{1-3x}\right)$ $(1-3x)\frac{dy}{dx}=-3\cos\left[\ln(1-3x)\right]$ Diff. wrt x $(1-3x)\frac{d^2y}{dx^2}-3\frac{dy}{dx}=3\sin\left[\ln(1-3x)\right]\left(\frac{-3}{1-3x}\right)$ $(1-3x)^2\frac{d^2y}{dx^2}-3(1-3x)\frac{dy}{dx}=-9y$ $(1-3x)^2\frac{d^2y}{dx^2}-3(1-3x)\frac{dy}{dx}+9y=0$ (shown) Diff. wrt x $(1-3x)^2\frac{d^3y}{dx^3}-9(1-3x)\frac{d^2y}{dx^2}+18\frac{dy}{dx}=0$ When $x = 0$, $y=0$, $\frac{dy}{dx}=-3$, $\frac{d^2y}{dx^2}=-9$, $\frac{d^3y}{dx^3}=-27$ $\therefore\ y=-3x-(9)\frac{x^2}{2}-\frac{27}{3!}x^3\ldots$ $\approx -3x-\frac{9}{2}x^2-\frac{9}{2}x^3$ From $\frac{dy}{dx}=\cos\left[\ln(1-3x)\right]\left(\frac{-3}{1-3x}\right)$, $(1-3x)\frac{dy}{dx}=-3\cos\left[\ln(1-3x)\right]$ $\cos\left[\ln(1-3x)\right]=-\frac{1}{3}(1-3x)\frac{dy}{dx}$ $\cos\left[\ln(1-3x)\right]\approx-\frac{1}{3}(1-3x)\left(-3-9x-\frac{27}{2}x^2\right)$ $\approx 1-\frac{9}{2}x^2$

4	$BC = BD + DC$ $= \dfrac{h}{\tan\frac{\pi}{6}} + \dfrac{h}{\tan\left(\frac{\pi}{4}+x\right)}$ $= \dfrac{h}{\frac{1}{\sqrt{3}}} + \dfrac{h}{\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}}$ $= h\sqrt{3} + \dfrac{h(1-\tan x)}{1+\tan x}$ $\approx h\sqrt{3} + \dfrac{h(1-x)}{1+x}$ $= h\sqrt{3} + h(1-x)(1+x)^{-1}$ $= h\sqrt{3} + h(1-x)[1+(-1)x+\dfrac{(-1)(-2)}{2!})x^2+...]$ $= h\sqrt{3} + h(1-x)[1-x+x^2+...]$ $= h\sqrt{3} + h(1-2x+2x^2+...]$ $\approx h(1+\sqrt{3}-2x+2x^2)$

5	$\tan y\left(\dfrac{dy}{dx}\right) = \ln\left(\dfrac{e^x+1}{2}\right)$ ……. (1) Differentiate (1) wrt x: $\sec^2 y\left(\dfrac{dy}{dx}\right)^2 + \tan y\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{e^x}{e^x+1}$ ……. (2) Differentiate (2) wrt x: $2\sec^2 y\tan y\left(\dfrac{dy}{dx}\right)^3 + 2\sec^2 y\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right) + \sec^2 y\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right) + \tan y\left(\dfrac{d^3y}{dx^3}\right) = \dfrac{e^x}{e^x+1} - \dfrac{e^{2x}}{\left(e^x+1\right)^2}$. $2\sec^2 y\tan y\left(\dfrac{dy}{dx}\right)^3 + 3\sec^2 y\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right) + \tan y\left(\dfrac{d^3y}{dx^3}\right) = \dfrac{e^x}{e^x+1} - \dfrac{e^{2x}}{\left(e^x+1\right)^2}$ ……. (3) At $x = 0$: $f(0) = \dfrac{\pi}{4}$ $f'(0) = 0$ $f''(0) = \dfrac{1}{2}$ Hence, Maclaurin's series: $y = \dfrac{\pi}{4} + \dfrac{x^2}{4} + \dfrac{x^3}{24} + ...$ $f'''(0) = \dfrac{1}{4}$ Maclaurin's series of $(\cot y)\ln\left(\dfrac{e^x+1}{2}\right) = \dfrac{d}{dx}\left(\dfrac{\pi}{4}+\dfrac{x^2}{4}+\dfrac{x^3}{24}\right) = \dfrac{x}{2}+\dfrac{x^2}{8}$

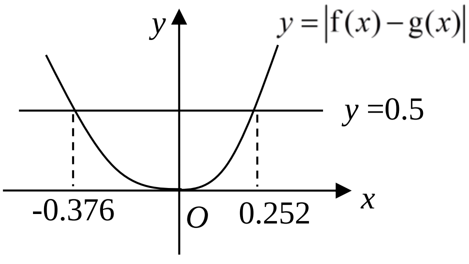
	$\tan y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\ln\left(\frac{e^x+1}{2}\right)$ ……. (1) Differentiate (1) wrt x: $\sec^2 y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\tan y\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=\frac{e^x}{e^x+1}$ ……. (2) Differentiate (2) wrt x: $2\sec^2 y\tan y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+2\sec^2 y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\sec^2 y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\tan y\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)=\frac{e^x}{e^x+1}-\frac{e^{2x}}{\left(e^x+1\right)^2}$. $2\sec^2 y\tan y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+3\sec^2 y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\tan y\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)=\frac{e^x}{e^x+1}-\frac{e^{2x}}{\left(e^x+1\right)^2}$ ……. (3) At $x = 0$: $\mathrm{f}(0)=\frac{\pi}{4}$ $\mathrm{f}'(0)=0$ $\mathrm{f}''(0)=\frac{1}{2}$ Hence, Maclaurin's series: $y=\frac{\pi}{4}+\frac{x^2}{4}+\frac{x^3}{24}+\ldots$ $\mathrm{f}'''(0)=\frac{1}{4}$ Maclaurin's series of $(\cot y)\ln\left(\frac{e^x+1}{2}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\pi}{4}+\frac{x^2}{4}+\frac{x^3}{24}\right)=\frac{x}{2}+\frac{x^2}{8}$

6	$y=\ln(1+2x)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{1+2x}$ $\Rightarrow(1+2x)\frac{\mathrm{d}y}{\mathrm{d}x}=2$ Differentiate w.r.t x : $(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}=0$ **(i)** Differentiate w.r.t x, we have $(1+2x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+4\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ $(1+2x)\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+6\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0$ $(1+2x)\frac{\mathrm{d}^5y}{\mathrm{d}x^5}+8\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=0$ When $x=0$, $y=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=2$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-4;\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=16;\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=-96;\frac{\mathrm{d}^5y}{\mathrm{d}x^5}=768$

	$y=0+2x+\frac{-4}{2!}x^2+\frac{16}{3!}x^3+\frac{-96}{4!}x^4+\frac{768}{5!}x^5......$ $=2x-2x^2+\frac{8}{3}x^3-4x^4+\frac{32}{5}x^5+......$ **(ii)** $\ln(1-2x)=\ln(1+(-2x))=-2x-2x^2-\frac{8}{3}x^3-4x^4-\frac{32}{5}x^5$ $\ln\left(\frac{1+2x}{1-2x}\right)=\ln(1+2x)-\ln(1-2x)$ $=2\left(2x+\frac{8}{3}x^3+\frac{32}{5}x^5+\cdots\right)$ **(iii)** When $x=\frac{1}{4}$, $\ln\left(\frac{6}{2}\right)=2\left[\frac{1}{2}+\frac{1}{3(2^3)}+\frac{1}{5(2^5)}+\cdots\right]$ $\therefore \sum_{r=0}^{\infty}\frac{1}{(2r+1)2^{2r+1}}=\frac{1}{2}\ln 3$

7	$\ln y=\sin^{-1}2x$ $\left(\frac{1}{y}\right)\frac{dy}{dx}=\frac{2}{\sqrt{1-4x^2}}\quad\Rightarrow\frac{dy}{dx}\sqrt{1-4x^2}=2y$ (shown) ------- (1) $\frac{d^2y}{dx^2}\sqrt{1-4x^2}+\frac{dy}{dx}\left(\frac{-4x}{\sqrt{1-4x^2}}\right)=2\frac{dy}{dx}\;\Rightarrow\;\frac{d^2y}{dx^2}(1-4x^2)-4x\left(\frac{dy}{dx}\right)=4y$ ---- (2) $\frac{d^3y}{dx^3}(1-4x^2)+\frac{d^2y}{dx^2}(-8x)-4\left(\frac{dy}{dx}\right)-4x\left(\frac{d^2y}{dx^2}\right)=4\frac{dy}{dx}$ $\frac{d^3y}{dx^3}(1-4x^2)+\frac{d^2y}{dx^2}(-12x)-8\left(\frac{dy}{dx}\right)=0$ ---------------- (3) When $x=0, y=1,\ \frac{dy}{dx}=2,\ \frac{d^2y}{dx^2}=4,\ \frac{d^3y}{dx^3}=16$ $\therefore y=1+2x+\frac{4x^2}{2!}+\frac{16x^3}{3!}+...\quad\therefore y\approx 1+2x+2x^2+\frac{8x^3}{3}$ Let $e^{\frac{\pi}{3}}=e^{\sin^{-1}2x}$ $\frac{\pi}{3}=\sin^{-1}2x\quad\Rightarrow\quad x=\frac{\sqrt{3}}{4}$ $e^{\frac{\pi}{3}}\approx 1+2\left(\frac{\sqrt{3}}{4}\right)+2\left(\frac{\sqrt{3}}{4}\right)^2+\frac{8}{3}\left(\frac{\sqrt{3}}{4}\right)^3=\frac{1}{8}(11+5\sqrt{3})$ $\therefore a=11,\ b=5$

8	$y=\frac{1}{(2+\sin 2x)} \Rightarrow \frac{dy}{dx}=\frac{-2\cos 2x}{(2+\sin 2x)^2}=-2y^2\cos 2x$ Differentiating wrt *x*: $\frac{d^2y}{dx^2}=-4y\frac{dy}{dx}\cos 2x-2y^2(-2\sin 2x)$ $\frac{d^2y}{dx^2}=2(-2y\cos 2x)\frac{dy}{dx}+4y^2\sin 2x$ $\frac{d^2y}{dx^2}=2\frac{1}{y}\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)+4y^2\sin 2x$ $\frac{d^2y}{dx^2}=\frac{2}{y}\left(\frac{dy}{dx}\right)^2+4y^2\sin 2x\ (shown)$ (i) $\frac{d^3y}{dx^3}=-\frac{2}{y^2}\left(\frac{dy}{dx}\right)^3+\frac{4}{y}\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+8y\left(\frac{dy}{dx}\right)\sin 2x+8y^2\cos 2x$ At $x=0$, $y=\frac{1}{2}$, $\frac{dy}{dx}=-\frac{1}{2}$, $\frac{d^2y}{dx^2}=1$, $\frac{d^3y}{dx^3}=-1$ By Maclaurin's theorem, $y=\frac{1}{2}-\frac{1}{2}x+\frac{1}{2!}x^2+\frac{(-1)}{3!}x^3+\ldots$ $=\frac{1}{2}-\frac{1}{2}x+\frac{1}{2}x^2-\frac{1}{6}x^3$ (up to term in x^3)
	(ii) $\frac{1}{(2+\sin 2x)}=(2+\sin 2x)^{-1}=\frac{1}{2}\left(1+\frac{1}{2}\left(2x-\frac{8x^3}{6}+\ldots\right)\right)^{-1}$ $\approx \frac{1}{2}\left(1-\frac{1}{2}\left(2x-\frac{8x^3}{6}\right)+\left(\frac{1}{2}(2x+\ldots)\right)^2-\left(\frac{1}{2}(2x+\ldots)\right)^3+\ldots\right)$ $=\frac{1}{2}\left(1-\frac{1}{2}\left(2x-\frac{8x^3}{6}\right)+\frac{1}{4}(2x+..)^2-\frac{1}{8}(2x+\ldots)^3+\ldots\right)$ $=\frac{1}{2}\left(1-x+\frac{2x^3}{3}+x^2-x^3+\ldots\right)$ $=\frac{1}{2}-\frac{1}{2}x+\frac{1}{2}x^2-\frac{1}{6}x^3+..$ which is the same as the series obtained by using Maclaurin's theorem.

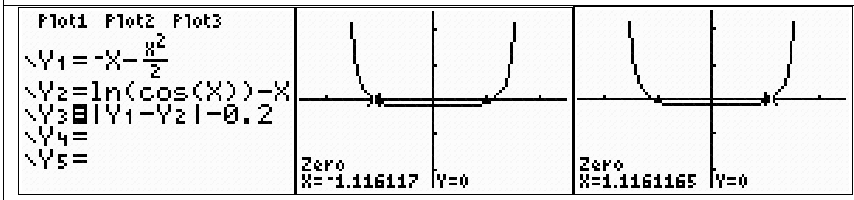
9	$\tan^{-1} y = 2\tan^{-1} x + \frac{\pi}{4}$ Differentiate with respect to *x*, $\frac{1}{1+y^2}\frac{dy}{dx} = \frac{2}{1+x^2}$ $\Rightarrow \left(1+x^2\right)\frac{dy}{dx} = 2\left(1+y^2\right)$ Differentiate with respect to *x*, $\Rightarrow \left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 4y\frac{dy}{dx}$ $\Rightarrow \left(1+x^2\right)\frac{d^2y}{dx^2} + (2x-4y)\frac{dy}{dx} = 0$ Differentiate with respect to *x*, $\Rightarrow \left(1+x^2\right)\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} + (2x-4y)\frac{d^2y}{dx^2} + (2-4\frac{dy}{dx})\frac{dy}{dx} = 0$ $\Rightarrow \left(1+x^2\right)\frac{d^3y}{dx^3} + (4x-4y)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4\left(\frac{dy}{dx}\right)^2 = 0$ When $x = 0$, $y = \tan\frac{\pi}{4} = 1$ $\frac{1}{1+1}\frac{dy}{dx} = \frac{2}{1+0} \Rightarrow \frac{dy}{dx} = 4$ $\left(1+0\right)\frac{d^2y}{dx^2} + (0-4)(4) = 0 \Rightarrow \frac{d^2y}{dx^2} = 16$ $\left(1+0\right)\frac{d^3y}{dx^3} + (0-4)(4) + 2(4) - 4(16) = 0 \Rightarrow \frac{d^3y}{dx^3} = 120$ $y = \tan\left[2\tan^{-1} x + \frac{\pi}{4}\right]$ $= 1 + 4x + \frac{16}{2!}x^2 + \frac{120}{3!}x^3$ $= 1 + 4x + 8x^2 + 20x^3$ Sketch $y = \left\|f(x) - g(x)\right\|$ and $y = 0.5$  For $\left\|f(x) - g(x)\right\| < 0.5$, $-0.376 < x < 0.252$

2023 Differentiation & Applications

10	$\mathrm{e}^{(1+5x)^{\frac{1}{3}}}=\mathrm{e}^{1+\frac{1}{3}(5x)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(5x)^2+\cdots}$ $=\mathrm{e}^{1+\frac{5}{3}x-\frac{25}{9}x^2+\cdots}$ $=\mathrm{e}\times\mathrm{e}^{\frac{5}{3}x-\frac{25}{9}x^2+\cdots}$ $=\mathrm{e}\left(1+\left(\frac{5}{3}x-\frac{25}{9}x^2\right)+\frac{\left(\frac{5}{3}x\right)^2}{2!}+\cdots\right)$ $=\mathrm{e}\left(1+\frac{5}{3}x-\frac{25}{18}x^2+\cdots\right)$ i.e. $a=1,\ b=\frac{5}{3},\ c=-\frac{25}{18}$ The expansion is valid for $\lvert 5x\rvert<1$ i.e. $-\frac{1}{5}<x<\frac{1}{5}$. **Method 1** Let $\mathrm{f}(x)=\cos(\alpha x-\beta)$ $\mathrm{f}'(x)=-\alpha\sin(\alpha x-\beta)$ $\mathrm{f}''(x)=-\alpha^2\cos(\alpha x-\beta)$ When $x=0$, $\mathrm{f}(0)=\cos\beta,\ \ \mathrm{f}''(0)=-\alpha^2\cos\beta$ Comparing coefficients, $\cos\beta=1\Rightarrow\beta=0$ & $\frac{\mathrm{f}''(0)}{2!}=-\frac{25}{18}$ $-\frac{\alpha^2}{2}=-\frac{25}{18}\Rightarrow\alpha=\frac{5}{3}$ **Method 2** Let $\mathrm{f}(x)=\cos(\alpha x-\beta)$ $=\cos\alpha x\cos\beta+\sin\alpha x\sin\beta$ $=\left(1-\frac{\alpha^2x^2}{2}+\cdots\right)\cos\beta+\left(\alpha x-\cdots\right)\sin\beta$ $=\cos\beta+\alpha\left(\sin\beta\right)x-\frac{\alpha^2\left(\cos\beta\right)}{2}x^2+\cdots$ Comparing coefficients, $\cos\beta=1\Rightarrow\beta=0$ & $-\frac{\alpha^2}{2}=-\frac{25}{18}\Rightarrow\alpha=\frac{5}{3}$

11	(a)(i) $\dfrac{AC}{AB}=\tan\left(\dfrac{\pi}{3}-x\right)$ $=\dfrac{\tan\frac{\pi}{3}-\tan x}{1+\tan\frac{\pi}{3}\tan x}$ $\dfrac{AB}{AC}=\dfrac{1+\sqrt{3}\tan x}{\sqrt{3}-\tan x}$ (ii) $\dfrac{AB}{AC}\approx\dfrac{1+\sqrt{3}x}{\sqrt{3}-x}$ when x is small $=\left(1+\sqrt{3}x\right)\left(\sqrt{3}-x\right)^{-1}$ $=\left(1+\sqrt{3}x\right)\dfrac{1}{\sqrt{3}}\left(1-\dfrac{x}{\sqrt{3}}\right)^{-1}$ $=\dfrac{1}{\sqrt{3}}\left(1+\sqrt{3}x\right)\left(1+\dfrac{x}{\sqrt{3}}\right)$ $=\dfrac{1}{\sqrt{3}}\left(1+\dfrac{x}{\sqrt{3}}+\sqrt{3}x+...\right)$ $=\dfrac{1}{\sqrt{3}}+\dfrac{4}{3}x+...$ Hence, $a=\dfrac{1}{\sqrt{3}}$, $b=\dfrac{4}{3}$ (b)(i) $\left(1+x^2\right)\dfrac{dy}{dx}+xy=\sqrt{1+x^2}$ $\left(1+x^2\right)\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+x\dfrac{dy}{dx}+y=\dfrac{x}{\sqrt{1+x^2}}$ $\left(1+x^2\right)\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=\dfrac{x}{\sqrt{1+x^2}}$ When $x = 0$, $y = 1$, $\dfrac{dy}{dx}=1$, $\dfrac{d^2y}{dx^2}=-1$ $\therefore y=1+x-\dfrac{x^2}{2}+...$ (ii) $e^y\approx e^{1+x-\frac{x^2}{2}}=e\left(e^{x-\frac{x^2}{2}}\right)$ $\approx e\left[1+\left(x-\dfrac{x^2}{2}\right)+\dfrac{1}{2}\left(x-\dfrac{x^2}{2}\right)^2\right]$ $\approx e\left[1+\left(x-\dfrac{x^2}{2}\right)+\dfrac{1}{2}\left(x^2\right)\right]$ $\approx e\left(1+x\right)$

12	$\frac{dy}{dx} = \frac{6-2y}{\cos 2x}$ $\cos 2x\frac{dy}{dx} = 6-2y$ Differentiate w.r.t. x $\cos 2x\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(-2\sin 2x\right)=-2\frac{dy}{dx}$ $\cos 2x\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(2-2\sin 2x\right)=0$ Differentiate w.r.t. x again $\cos 2x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}\left(-2\sin 2x\right)+\frac{dy}{dx}\left(-4\cos 2x\right)+\left(2-2\sin 2x\right)\frac{d^2y}{dx^2}=0$ When $x=0,\ y=1,\ \frac{dy}{dx}=4,\ \frac{d^2y}{dx^2}=-8,\ \frac{d^3y}{dx^3}=32$. Using Maclaurin's theorem: $f\left(x\right)=1+4x+\frac{-8}{2!}x^2+\frac{32}{3!}x^3+.......$ $\quad=1+4x-4x^2+\frac{16}{3}x^3+......$ $\frac{1-\sin x}{\cos x}=\left(1-x+\frac{x^3}{6}\right)\left(1-\frac{x^2}{2}\right)^{-1}$ $=\left(1-x+\frac{x^3}{6}\right)\left(1+\frac{x^2}{2}+.....\right)$ $=1-x+\frac{x^2}{2}-\frac{x^3}{3}+....$ $\frac{1-\sin x}{\cos x}=\sec x-\tan x=1-x+\frac{x^2}{2}-\frac{x^3}{3}+....$ $\therefore \tan 2x-\sec 2x=-\left(1-2x+\frac{4x^2}{2}-\frac{8x^3}{3}\right)$ Substitute power series into $f(x)=a\left(\tan 2x-\sec 2x\right)+b$ and compare coefficients of constant term and term in x $b\text{-}a=1\ ,\ \ 2a=4\ \Rightarrow\ a=2\ \textit{and}\ b=3$

13(a) (i)	$e^{y+x} = \cos x$ $y = \ln(\cos x) - x$ Differentiate wrt x $\frac{dy}{dx} = -\tan x - 1$
	Differentiate wrt x $\frac{d^2y}{dx^2} = -\sec^2 x$
	$x = 0$ $y = 0,\ \frac{dy}{dx} = -1,\ \frac{d^2y}{dx^2} = -1,$ $y = -x - \frac{x^2}{2} + ...$
(a)(ii)	$\lvert h(x) - y \rvert < 0.2$ $\lvert h(x) - y \rvert - 0.2 < 0$
	 From GC, $-1.12 < x < 1.12$ (to 3 s.f)
(b)	$\left(a - \frac{x}{3}\right)^n = a^n\left(1 - \frac{x}{3a}\right)^n$ $= a^n\left(1 - \frac{xn}{3a} + \frac{n(n-1)}{2}\left(\frac{x}{3a}\right)^2 + ..\right)$ $= a^n\left(1 - \frac{n}{3a}x + \frac{n(n-1)}{18a^2}x^2 + ..\right)$ $-\frac{n}{3a} = \frac{4n(n-1)}{18a^2}$ $n = 1 - \frac{3a}{2}$ $a^n = \frac{1}{4}$ Sub $n = 1 - \frac{3a}{2}$ $a^{1-\frac{3a}{2}} = \frac{1}{4}$

	From GC, $a = 2$ or 0.16086 (rejected) when $a = 2$, $n = -2$

14	$\dfrac{\sqrt{9+ax}}{1+bx^2} = (9+ax)^{\frac{1}{2}}(1+bx^2)^{-1} \quad = 3\left(1+\dfrac{a}{18}x+\dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\dfrac{a}{9}x\right)^2+\ldots\right)(1-bx^2+\ldots)$ $= 3\left(1+\dfrac{a}{18}x-\dfrac{a^2}{648}x^2+\ldots\right)(1-bx^2+\ldots) \qquad \approx 3+\dfrac{a}{6}x+\left(-\dfrac{a^2}{216}-3b\right)x^2$ By comparing coefficients, a = 6 and b = -2 The valid range for expansion of $\dfrac{\sqrt{9+ax}}{1+bx^2}$ is $\lvert x\rvert < \dfrac{1}{\sqrt{2}}$.

15	$\dfrac{2}{(1+3x)^n} = 2(1+3x)^{-n}$ Consider $(1+3x)^{-n} = 1+(-n)(3x)+\dfrac{-n(-n-1)}{2!}(3x)^2+\ldots$ $= 1-3nx+\dfrac{n(n+1)}{2}(9x^2)+\ldots$ $\dfrac{2}{(1+3x)^n} = 2\left(1-3nx+\dfrac{n(n+1)}{2}(9x^2)+\ldots\right)$ $= \ldots+2\left(\dfrac{n(n+1)}{2}9x^2\right)+\ldots$ Given: coefficient of $x^2 = 108$ $\Rightarrow \quad 9n(n+1) = 108$ $n^2+n-12 = 0$ $(n+4)(n-3) = 0$ $n = -4$ (rejected since $n \in \mathbb{Z}^+$) or $n = 3$ Thus, value of $\underline{\underline{n = 3}}$

16	(i) Let $\mathrm{f}(x) = \dfrac{x-4}{(x+1)(3x+2)} = \dfrac{A}{x+1}+\dfrac{B}{3x+2}$ $\Rightarrow x-4 = A(3x+2)+B(x+1)$ Using cover up rule, For $x = -\dfrac{2}{3}$, $-\dfrac{2}{3}-4 = B\left(\dfrac{1}{3}\right) \Rightarrow B = -14$

	For $x=-1$, $-1-4=A(-1)\Rightarrow A=5$ $\therefore \mathrm{f}(x)=\dfrac{5}{x+1}-\dfrac{14}{3x+2}$ $\mathrm{f}(x)=\dfrac{5}{x+1}-\dfrac{14}{3x+2}=5(1+x)^{-1}-14(3x+2)^{-1}$ $=5(1+x)^{-1}-14\left[2\left(1+\dfrac{3x}{2}\right)\right]^{-1}$ $=5(1+x)^{-1}-\dfrac{14}{2}\left(1+\dfrac{3x}{2}\right)^{-1}$ $=5(1-x+x^2-x^3+\ldots)$ $-7\left(1+(-1)\left(\dfrac{3x}{2}\right)+\dfrac{(-1)(-2)}{2!}\left(\dfrac{3x}{2}\right)^2+\dfrac{(-1)(-2)(-3)}{3!}\left(\dfrac{3x}{2}\right)^3+\ldots\right)$ $=5(1-x+x^2-x^3+\ldots)-7\left(1-\dfrac{3x}{2}+\dfrac{9x^2}{4}-\dfrac{27x^3}{8}+\ldots\right)$ $=-2+\dfrac{11}{2}x-\dfrac{43}{4}x^2+\dfrac{149}{8}x^3+\ldots$ (ii) Expansion of $(1+x)^{-1}$ is valid for $-1<x<1$ Expansion of $\left(1+\dfrac{3x}{2}\right)^{-1}$ is valid for $-\dfrac{2}{3}<x<\dfrac{2}{3}$ Therefore, the range of values of *x* for the expansion of f(*x*) to be valid is $-\dfrac{2}{3}<x<\dfrac{2}{3}$. (iii) Coefficient of $x^n=(-1)^n 5+(-1)^{n+1}7\left(\dfrac{3}{2}\right)^n=(-1)^n\left[5-7\left(\dfrac{3}{2}\right)^n\right]$

17	$\left(\dfrac{1+2x^2}{4-x}\right)^{\frac{1}{2}}$ $=\left(1+2x^2\right)^{1/2}\cdot 4^{-1/2}\left(1-\dfrac{x}{4}\right)^{-1/2}$ $=\dfrac{1}{2}\left(1+\dfrac{1}{2}\left(2x^2\right)+\ldots\right)$ $\left(1-\dfrac{1}{2}\left(-\dfrac{x}{4}\right)+\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\dfrac{x}{4}\right)^2+\ldots\right)$ $=\dfrac{1}{2}\left(1+x^2+\ldots\right)\left(1+\dfrac{1}{8}x+\dfrac{3}{128}x^2+\ldots\right)$ $=\dfrac{1}{2}\left(1+\dfrac{1}{8}x+\dfrac{3}{128}x^2+x^2\right)+\ldots$ $=\dfrac{1}{2}\left(1+\dfrac{1}{8}x+\dfrac{131}{128}x^2\right)+\ldots$
(i)	$\left\|2x^2\right\|<1$ and $\left\|\dfrac{x}{4}\right\|<1$ $-\dfrac{1}{\sqrt{2}}<x<\dfrac{1}{\sqrt{2}}$ and $-4<x<4$ Taking intersection, the set of values is $\left\{x\in\mathbb{R}:-\dfrac{1}{\sqrt{2}}<x<\dfrac{1}{\sqrt{2}}\right\}$.
(ii)	$\sqrt{\dfrac{1+2\left(\frac{1}{4}\right)^2}{4-\frac{1}{4}}}\approx\dfrac{1}{2}\left(1+\dfrac{1}{8}\left(\dfrac{1}{4}\right)+\dfrac{131}{128}\left(\dfrac{1}{4}\right)^2\right)$ $\sqrt{\dfrac{9}{30}}\approx\dfrac{2243}{4096}$ $\sqrt{30}\approx 3\left(\dfrac{4096}{2243}\right)=\dfrac{12288}{2243}$ Alternatively, $\sqrt{\dfrac{9}{30}}\approx\dfrac{2243}{4096}$ $\sqrt{\dfrac{30}{100}}\approx\dfrac{2243}{4096}$ $\sqrt{30}\approx\dfrac{11215}{2048}$

18	(i)

	$\dfrac{1}{\sqrt{4+x^2}}=\dfrac{1}{\sqrt{4}}\left(1+\dfrac{x^2}{4}\right)^{-\frac{1}{2}}$ $=\dfrac{1}{2}\left(1+\left(-\dfrac{1}{2}\right)\left(\dfrac{x^2}{4}\right)+\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\dfrac{x^2}{4}\right)^2+...\right)$ $=\dfrac{1}{2}\left(1-\dfrac{x^2}{8}+\dfrac{3x^4}{128}-...\right)$ $\approx\dfrac{1}{2}-\dfrac{1}{16}x^2+\dfrac{3}{256}x^4$ (ii) $\dfrac{x+1}{\sqrt{4+x^2}}=(x+1)\left(\dfrac{1}{\sqrt{4+x^2}}\right)$ $=(x+1)\left(\dfrac{1}{2}-\dfrac{1}{16}x^2+\dfrac{3}{256}x^4-...\right)$ $\approx\dfrac{1}{2}+\dfrac{1}{2}x-\dfrac{1}{16}x^2-\dfrac{1}{16}x^3$ (iii) $\left\lvert\dfrac{x^2}{4}\right\rvert<1$ $x^2<4$ $-2<x<2$

19	$PR^2=3^2+\left(\sqrt{2}\right)^2-2(3)\left(\sqrt{2}\right)\cos\left(\theta+\dfrac{\pi}{4}\right)$ $=11-6\sqrt{2}\left(\cos\theta\cos\dfrac{\pi}{4}-\sin\theta\sin\dfrac{\pi}{4}\right)$ $=11-6\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\cos\theta-\dfrac{1}{\sqrt{2}}\sin\theta\right)$ $=11-6\cos\theta+6\sin\theta$ $\approx11-6\left(1-\dfrac{\theta^2}{2}\right)+6\theta$ $=5+6\theta+3\theta^2$ $PR=\left(5+6\theta+3\theta^2\right)^{\frac{1}{2}}$ (shown) $PR=\left(5+6\theta+3\theta^2\right)^{\frac{1}{2}}$

$$=\left[5\left(1+\frac{6}{5}\theta+\frac{3}{5}\theta^2\right)\right]^{\frac{1}{2}}$$

$$=5^{\frac{1}{2}}\left[1+\frac{1}{2}\left(\frac{6}{5}\theta+\frac{3}{5}\theta^2\right)-\frac{1}{8}\left(\frac{6}{5}\theta+\frac{3}{5}\theta^2\right)^2+\ldots\right]$$

$$\approx\sqrt{5}+\frac{3\sqrt{5}}{5}\theta+\frac{3\sqrt{5}}{25}\theta^2$$

$$\therefore a=\sqrt{5},\quad b=\frac{3\sqrt{5}}{5},\quad c=\frac{3\sqrt{5}}{25}$$

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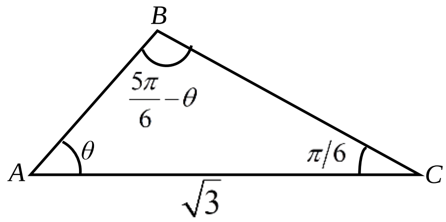
20 By sine rule,



$$\frac{AB}{\sin\frac{\pi}{6}}=\frac{\sqrt{3}}{\sin(\frac{5\pi}{6}-\theta)}$$

$$AB=\frac{\frac{1}{2}\sqrt{3}}{\sin\frac{5\pi}{6}\cos\theta-\cos\frac{5\pi}{6}\sin\theta}$$

$$=\frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta}$$

$$\approx\frac{2\sqrt{3}}{2\left(1-\frac{1}{2}\theta^2\right)+2\sqrt{3}\left(\theta\right)}\quad\text{since }\theta\text{ is small}$$

$$=\frac{2\sqrt{3}}{2+2\sqrt{3}\theta-\theta^2}\quad\text{(shown)}$$

Applying binomial expansion,

$$AB\approx\sqrt{3}\left[1+\left(\sqrt{3}\theta-\frac{1}{2}\theta^2\right)\right]^{-1}$$

$$\approx\sqrt{3}\left[1-\left(\sqrt{3}\theta-\frac{1}{2}\theta^2\right)+\left(\sqrt{3}\theta-\frac{1}{2}\theta^2\right)^2\right]$$

$$\approx\sqrt{3}\left[1-\sqrt{3}\theta+\frac{1}{2}\theta^2+3\theta^2\right]$$

$$=\sqrt{3}-3\theta+\frac{7\sqrt{3}}{2}\theta^2\qquad\left(a=\sqrt{3},\quad b=-3,\quad c=\frac{7\sqrt{3}}{2}\right)$$

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21 $x^3+1=\left(x+1\right)\left(x^2-x+1\right)$

	$\dfrac{x^2+2}{(x+1)(x^2-x+1)}=\dfrac{1}{x+1}+\dfrac{1}{x^2-x+1}$ $\Rightarrow (1-x+x^2)^{-1}=\dfrac{x^2+2}{x^3+1}-\dfrac{1}{x+1}$ $=(x^2+2)(1+x^3)^{-1}-(1+x)^{-1}$ $=(x^2+2)\left(1+(-1)(x^3)+\dfrac{(-1)(-2)}{2!}(x^3)^2+\dfrac{(-1)(-2)(-3)}{3!}(x^3)^3+...\right)$ $-\left(1+(-1)(x)+\dfrac{(-1)(-2)}{2!}(x^2)+\dfrac{(-1)(-2)(-3)}{3!}(x^3)+...\right)$ $=(x^2+2)(1-x^3+x^6-x^9+...+(-1)x^{3n}+...)$ $-(1-x+x^2-x^3+...+(-1)^n x^n+...)$ Hence Coefficient of x^{3n} is $2(-1)^n-(-1)^{3n}=(-1)^n$ **OR** $2(-1)^n+(-1)^{n-1}$ **OR** $2(-1)^n-(-1)^n$ **OR** { -1 if *n* is odd; 1 if *n* is even}

JJC Prelim 9758/2018/02/Q1	
22	$\mathrm{f}(x) = \mathrm{e}^{\sin x}$ $= \mathrm{e}^{x-\frac{x^3}{3!}+\ldots}$ $=1+\left(x-\frac{x^3}{3!}\right)+\frac{\left(x-\frac{x^3}{3!}\right)^2}{2!}+\frac{\left(x-\frac{x^3}{3!}\right)^3}{3!}+\ldots..$ $=1+x-\frac{x^3}{6}+\frac{x^2}{2}+\frac{x^3}{6}+\ldots$ $=1+x+\frac{x^2}{2}+\ldots.$ $\therefore\ a=1,\ b=1,\ c=\frac{1}{2}$ and $d=0$.
	$\frac{1}{\left(\mathrm{e}^{\sin x}\right)^2}\approx\left(1+x+\frac{x^2}{2}\right)^{-2}$
	$=1+(-2)\left(x+\frac{x^2}{2}\right)+\frac{(-2)(-3)}{2!}\left(x+\frac{x^2}{2}\right)^2+\ldots..$ $\approx 1-2x+2x^2$
(i)	$4\frac{\mathrm{d}y}{\mathrm{d}x}=(y+1)^2$ Differentiating with respect to x, $4\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2(y+1)\frac{\mathrm{d}y}{\mathrm{d}x}$ $4\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=2(y+1)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$
	Sub $x=0$, $y=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{3}{2}$
	Using Maclaurin's formula, $\mathrm{g}(x)=1+x+\frac{x^2}{2!}+\left(\frac{3}{2}\right)\frac{x^3}{3!}+\ldots..$
	$\mathrm{g}(x)=1+x+\frac{x^2}{2}+\frac{x^3}{4}+\ldots..$
	$\mathrm{g}(x)-\mathrm{f}(x)=\left(1+x+\frac{x^2}{2}+\frac{x^3}{4}+\ldots..\right)-\left(1+x+\frac{1}{2}x^2+\ldots..\right)$ $\approx\frac{x^3}{4}$
(ii)	As $x\to 0$, $\mathrm{g}(x)-\mathrm{f}(x)\approx\frac{1}{4}x^3\to 0$. Therefore, $\mathrm{f}(x)$ is a good approximation to $\mathrm{g}(x)$ for values of x close to zero.

23(i)	$y=\sqrt{\mathrm{e}^{\cos x}}$ --- (1) $y^2=\mathrm{e}^{\cos x}$ Differentiate with respect to *x*, $2y\dfrac{\mathrm{d}y}{\mathrm{d}x}=(-\sin x)\mathrm{e}^{\cos x}$ $y\left(2\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\sin x\right)=0$ $y=0$ (rejected $y>0$) or $2\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\sin x=0$ $2\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\sin x=0$ (Shown) --- (1)
(ii)	Differentiate with respect to x, $2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(\sin x)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)+(\cos x)y=0$ $x=0$, $y=\mathrm{e}^{\frac{1}{2}}$ from (1) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ from (2) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\dfrac{\sqrt{\mathrm{e}}}{2}$ $y=\mathrm{e}^{\frac{1}{2}}-\dfrac{\mathrm{e}^{\frac{1}{2}}}{4}x^2+\ldots$ $\mathrm{e}^{\sin^2(\frac{x}{2})}=\mathrm{e}^{\frac{1-\cos x}{2}}=\mathrm{e}^{\frac{1}{2}}\left(\mathrm{e}^{\cos x}\right)^{-\frac{1}{2}}$ $\approx \mathrm{e}^{\frac{1}{2}}\left(\mathrm{e}^{\frac{1}{2}}-\dfrac{\mathrm{e}^{\frac{1}{2}}}{4}x^2\right)^{-1}$ $=\left(1-\dfrac{x^2}{4}\right)^{-1}$ $=1+\dfrac{x^2}{4}+\ldots$
(iii)	$\int_0^{\sqrt{2}}\mathrm{e}^{\sin^2\left(\frac{x}{2}\right)}\mathrm{d}x\approx\int_0^{\sqrt{2}}\left(1+\dfrac{x^2}{4}\right)\mathrm{d}x$ =1.649915 =1.650 *(understand that the Maclaurin's series is a good estimation of the original function.)*

24(a)	$\left(b-\frac{x}{2}\right)^n=b^n\left(1-\frac{x}{2b}\right)^n$ $=b^n\left(1-\frac{xn}{2b}+\frac{n(n-1)}{2!}\left(\frac{x}{2b}\right)^2+..\right)$ $=b^n\left(1-\frac{n}{2b}x+\frac{n(n-1)}{8b^2}x^2+..\right)$ Since the coefficient of x is four times the coefficient of x^2, $-\frac{n}{2b}=\frac{4n(n-1)}{8b^2}$ $-1=\frac{n-1}{b}$ $-b=n-1$ $n=1-b$ Since the constant term in the expansion is $\frac{1}{2}$, $b^n=\frac{1}{2}$ Sub $n=1-b$ $b^{1-b}=\frac{1}{2}$ Using GC, $b=0.346$ (rejected because b is an integer) or $b=2$ $\therefore\ b=2$ and $n=-1$
(bi)	Let $\mathrm{f}(x)=\ln\left(2x^2\right)$. As $\mathrm{f}(0)$ is undefined, it is not possible to obtain a Maclaurin series for $\ln\left(2x^2\right)$.
(bii)	$\mathrm{f}(x)=\ln\left(2x^2\right)$ $\mathrm{f}'(x)=\frac{4x}{2x^2}=\frac{2}{x}$ $\mathrm{f}''(x)=-\frac{2}{x^2}$ When $x=2$, $\mathrm{f}(2)=\ln 8$, $\mathrm{f}'(2)=1$, $\mathrm{f}''(2)=-\frac{1}{2}$ $\therefore \ln\left(2x^2\right)=\ln 8+1(x-2)+\frac{\left(\frac{-1}{2}\right)}{2!}(x-2)^2+...$ $=\ln 8+(x-2)-\frac{1}{4}(x-2)^2+...$

25. ACJC/2022/I/Q7

It is given that $y = \ln(2+\sin 2x)$.

(i) Show that $e^y \dfrac{d^2y}{dx^2} + e^y\left(\dfrac{dy}{dx}\right)^2 = -4\sin 2x$. [2]

(ii) By further differentiation of the above results, find the Maclaurin series for *y*, up to and including the term in x^3. [3]

(iii) Verify that the series for $\ln(2+\sin 2x)$ is the same as the result obtained in part **(ii)**, if the standard series from the List of Formulae (MF26) are used. [3]

(iv) Hence deduce the series expansion for $\dfrac{\ln(2+\sin 2x)}{\sqrt{1-x}}$, up to and including the term in x^2. [2]

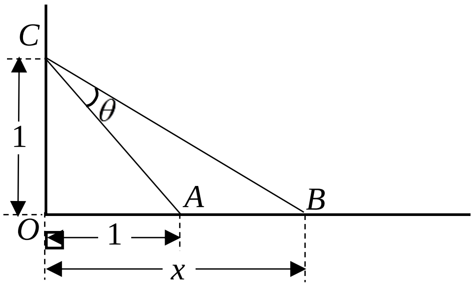
ACJC Prelim 9758/2022/01/Q7

(i)	$y = \ln(2+\sin 2x)$ $e^y \dfrac{dy}{dx} = 2\cos 2x$ $e^y \dfrac{d^2y}{dx^2} + e^y\left(\dfrac{dy}{dx}\right)^2 = -4\sin 2x$ (shown)
(ii)	$e^y \dfrac{d^3y}{dx^3} + e^y \dfrac{d^2y}{dx^2}\dfrac{dy}{dx} + 2e^y \dfrac{d^2y}{dx^2}\dfrac{dy}{dx} + e^y\left(\dfrac{dy}{dx}\right)^3 = -2\cos 2x$ $e^y \dfrac{d^3y}{dx^3} + 3e^y \dfrac{d^2y}{dx^2}\dfrac{dy}{dx} + e^y\left(\dfrac{dy}{dx}\right)^3 = -2\cos 2x$ When $x=0, y=\ln 2, \dfrac{dy}{dx}=1, \dfrac{d^2y}{dx^2}=-1, \dfrac{d^3y}{dx^3}=-2$ $\therefore y = \ln 2 + x + \dfrac{(-1)}{2!}x^2 + \dfrac{(-2)}{3!}x^3 + \ldots$ $= \ln 2 + x - \dfrac{1}{2}x^2 - \dfrac{1}{3}x^3 + \ldots$

(iii)	$y=\ln\left(2+\sin 2x\right)$ $\approx \ln\left(2+2x-\dfrac{(2x)^3}{6}\right)$ $=\ln\left(2\left(1+x-\dfrac{2}{3}x^3\right)\right)$ $=\ln 2+\ln\left(1+x-\dfrac{2}{3}x^3\right)$ $=\ln 2+\left(x-\dfrac{2}{3}x^3\right)-\dfrac{\left(x-\frac{2}{3}x^3\right)^2}{2}+\dfrac{\left(x-\frac{2}{3}x^3\right)^3}{3}+\ldots$ $\approx \ln 2+x-\dfrac{2}{3}x^3-\dfrac{1}{2}x^2+\dfrac{1}{3}x^3$ $=\ln 2+x-\dfrac{1}{2}x^2-\dfrac{1}{3}x^3+\ldots$ (verified)
(iv)	$\dfrac{\ln(2+\sin 2x)}{\sqrt{1-x}}$ $\approx \dfrac{\ln 2+x-\frac{1}{2}x^2-\frac{1}{3}x^3}{\sqrt{1-x}}$ $=\left(\ln 2+x-\dfrac{1}{2}x^2-\dfrac{1}{3}x^3\right)(1-x)^{-\frac{1}{2}}$ $\approx\left(\ln 2+x-\dfrac{1}{2}x^2\right)\left(1+\dfrac{1}{2}x+\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2\right)$ $=\left(\ln 2+x-\dfrac{1}{2}x^2\right)\left(1+\dfrac{1}{2}x+\dfrac{3}{8}x^2\right)$ $\approx \ln 2+x-\dfrac{1}{2}x^2+\left(\dfrac{1}{2}\ln 2\right)x+\dfrac{1}{2}x^2+\left(\dfrac{3}{8}\ln 2\right)x^2$ $=\ln 2+\left(1+\dfrac{1}{2}\ln 2\right)x+\left(\dfrac{3}{8}\ln 2\right)x^2$

26. ASRJC/2022/I/Q4

(i) Show that the first two non-zero terms of the Maclaurin series for $\tan\theta$ is given by $\theta+\frac{1}{3}\theta^3$. You may use the standard results given in the List of Formulae (MF26). [2]



In the right-angle triangle OBC shown above, point A lies on OB such that $OA=1$, $OB=x$, where $x > 1$ and $OC=1$. It is given that angle COB is $\frac{\pi}{2}$ radians and that angle ACB is θ radians (see diagram).

(ii) Show that $AB=\dfrac{2\tan\theta}{1-\tan\theta}$. [2]

(iii) Given that θ is a sufficently small angle, show that

$$AB\approx a\theta+b\theta^2+c\theta^3$$

for exact real constants a, b and c to be determined. [3]

ASRJC Prelim 9758/2022/01/Q4

	Solutions	
	(i) $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ $=\sin\theta(\cos\theta)^{-1}$ $\approx\left(\theta-\dfrac{\theta^3}{3!}\right)\left(1-\dfrac{\theta^2}{2!}\right)^{-1}$	
	$\approx\left(\theta-\dfrac{\theta^3}{3!}\right)\left(1+\dfrac{\theta^2}{2!}\right)$	
	$\approx\theta+\dfrac{\theta^3}{2!}-\dfrac{\theta^3}{3!}$ $=\theta+\dfrac{1}{3}\theta^3$	
	(ii) $\tan\left(\dfrac{\pi}{4}+\theta\right)=\dfrac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}$	

	$x \quad = \dfrac{1+\tan\theta}{1-\tan\theta}$	
	$AB \quad = \dfrac{1+\tan\theta}{1-\tan\theta} - 1$	
	$AB \quad = \dfrac{2\tan\theta}{1-\tan\theta}$	
	(iii) $AB = 2\tan\theta\left(1-\tan\theta\right)^{-1}$	
	$\approx 2\left(\theta+\dfrac{\theta^3}{3}\right)\left(1-\left(\theta+\dfrac{\theta^3}{3}\right)\right)^{-1}$	
	$\approx 2\left(\theta+\dfrac{\theta^3}{3}\right)\left(1+\left(\theta+\dfrac{\theta^3}{3}\right)+\left(\theta+\dfrac{\theta^3}{3}\right)^2\right)$	
	$\approx \left(2\theta+\dfrac{2\theta^3}{3}\right)\left(1+\theta+\theta^2\right)$ $\approx 2\theta+2\theta^2+\dfrac{8\theta^3}{3}$	
	$a=2,\ b=2,\ c=\dfrac{8}{3}$	