

Section A: Pure Mathematics [40 marks]

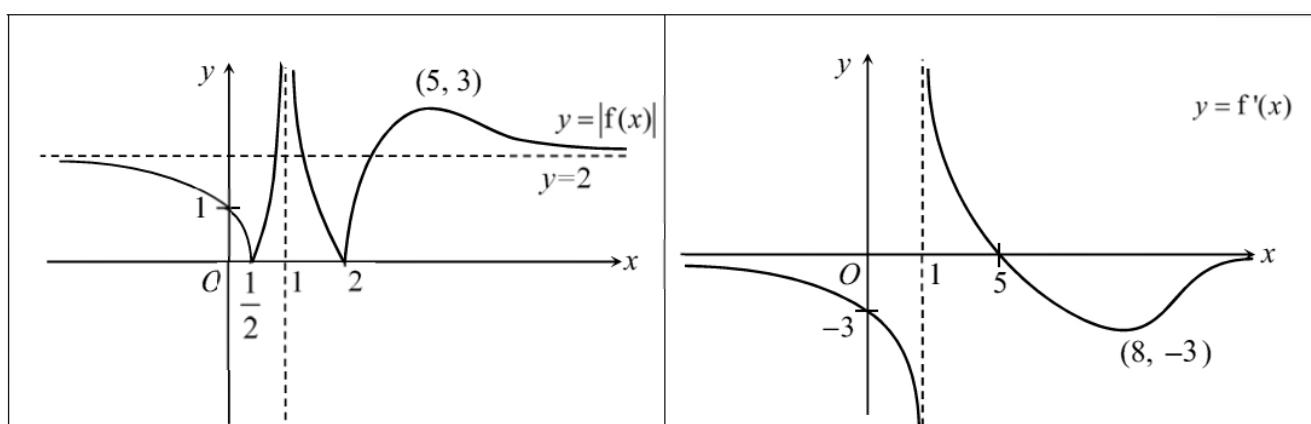
- 1 (i) Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ f\left(\frac{n+1}{n}\right) + f\left(\frac{n+2}{n}\right) + \dots + f\left(\frac{2n}{n}\right) \right\}$$

is $\int_1^2 f(x) \, dx$ [2]

- (ii) Hence evaluate $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{n}{n+r}$ exactly. [3]

- 2 The diagrams below show the graphs of $y = |f(x)|$ and $y = f'(x)$.



On separate diagrams, sketch the graphs of:

(i) $y = |f(2x)| + 1$ [2]

(ii) $y = \frac{1}{f'(x)}$ [3]

(iii) $y = f(x)$ [3]

showing clearly, in each case, the intersection(s) with the axes, the coordinates of the turning point(s) and the equation(s) of the asymptotes.

- 3 (a) A retirement savings account pays a compound interest of 0.2% per month on the amount of money in the account at the end of each month. A one-time principal amount of $\$P$ is deposited to open the account and a monthly pay-out of $\$x$ is withdrawn from the account at the beginning of each month, starting from the month that the account is opened.

- (i) Show that the amount in the account at the end of n months after the interest has been added is given by

$$P(1.002^n) - 501x(1.002^n - 1). \quad [4]$$

- (ii) Suppose a fixed monthly pay-out of $\$2,000$ is to be sustained for at least 25 years, find the minimum principal amount required correct to the nearest dollar. [2]

- (iii) If a principal amount of $\$600,000$ is placed in the account, find the number of years for which a monthly pay-out of $\$2,000$ per month can be sustained, leaving your answer correct to the nearest whole number. [2]

- (b) A different retirement savings account provides an increasing amount of monthly pay-out over a period of 25 years. The pay-out in the first month is $\$a$. The pay-out for each subsequent month is an increment of $\$c$ from the pay-out of the previous month.

The pay-out in the final month is $\$4,000$, and the total pay-out at the end of 25 complete years is $\$751,500$. Find the month in which the pay-out is $\$2,000$. [5]

4 The function f is defined by

$$f(x) = \begin{cases} 4-x & \text{for } 1 \leq x < 3, \\ (x-4)^2 & \text{for } 3 \leq x < 4, \end{cases}$$

and it is given that $f(x-3) = f(x)$ for all real values of x .

- (i) State a reason why f does not have an inverse. [1]
- (ii) Sketch the graph of $y = f(x)$ for $-1 < x < 6$. [3]
- (iii) Evaluate $f(2017)$. [2]

The function g has domain $[1, 4)$ and is defined by

$$g(x) = \begin{cases} 4-x & \text{for } 1 \leq x < 3, \\ (x-4)^2 & \text{for } 3 \leq x < 4. \end{cases}$$

- (iv) By sketching $y = g(x)$ and $y = g^{-1}(x)$ on the same diagram, state the values of x such that $g(x) = g^{-1}(x)$. [3]

The function h , is defined by

$$h(x) = \begin{cases} \sqrt{1-x} & \text{for } 0 \leq x < 1, \\ (x-1)^4 & \text{for } 1 \leq x \leq 3. \end{cases}$$

- (v) Explain why hg^{-1} doesn't exist. [1]
- (vi) Given that hg exist, define hg in similar form as function h . [2]
- (vii) Find the range of hg . [2]

Section B: Probability and Statistics [60 marks]

- 5** Two fair 4-sided dice each has its faces labelled '1', '2', '3' and '4'. The two dice are thrown and the absolute difference in score on their bottom faces is denoted by X .
- (i) Find $P(X = x)$ for all possible values of x . [2]
- (ii) Find $E(X)$ and $\text{Var}(X)$. [2]
- 6** (a) Two events A and B occur with probabilities a and b respectively.
Let $c = P(A \cap B)$.
- (i) Find the minimum value of c , in terms of a and b , if $a + b > 1$. [2]
- (ii) Find the minimum value of c if $a = 0.6$ and $b = 0.2$. [1]
- (b) For independent events A and B , prove that the events A' and B' are also independent. [3]
- 7** When Mr. Lee sends a text message to any of his students over the weekend, he gets a reply, on average, about 6 out of 10 times. On a particular weekend, Mr. Lee sends a text message to 25 students.
- (i) State, in the context of this question, two assumptions needed to model the number of students that reply by a binomial distribution. [2]
- (ii) Explain why one of the assumptions stated in part (i) may not hold in this context. [1]
- Assume now that these assumptions do in fact hold. Stating the distribution clearly,
- (iii) find the probability that at least half of the students reply. [3]
- It is given instead that the probability of a randomly chosen student replying is p . Find the least value of p such that there is at least a 90% chance of more than 20 students replying to the text message. [3]

- 8 A teacher, Mr. Ku, suspects that the average time a student spends on his or her mobile phone per day is μ_0 minutes. He selected a random sample of 97 students in the school who own mobile phones and recorded the amount of time each student spent on his or her phone in a randomly selected day. The results are displayed in the table below.

Time spent per day (to nearest minute)	60	65	72	90	110	180
Number of people	11	20	32	18	10	6

- (i) Calculate unbiased estimates of the population mean and variance of the time a student spends on his or her mobile phone per day. [2]

The null hypothesis that the average time a student spends on his or her mobile phone per day is μ_0 minutes is tested, at 5% level of significance, against the alternative hypothesis that the average time a student spends on his or her mobile phone per day differs from μ_0 minutes.

- (ii) Determine the range of values of μ_0 for which the null hypothesis is rejected. [5]
- (iii) Explain, in the context of this question, the meaning of ‘at 5% level of significance’. [1]
- (iv) If the null hypothesis in (ii) is rejected at 5% significance level, can we reject the null hypothesis at 1% level of significance? Explain your answer. [1]

- 9 The table below gives the world record time, in seconds, for the 100 metre race for the various years.

Year, x	1993	1994	1996	1999	2002	2005	2007	2008	2009
Time, t	9.87	9.85	9.84	9.79	9.78	9.77	9.74	9.69	9.58

- (i) Draw the scatter diagram for these values, labelling the axes clearly. [2]

It is thought that the world record time t can be modelled by one of the following formulae

$$t = a + bY \quad \text{or} \quad t = c + dY^3$$

where a, b, c, d are constants and $Y = (x - 2000)$.

- (ii) Using the value of the product moment correlation coefficient, explain which of $t = a + bY$ and $t = c + dY^3$ is the better model. [3]
- (iii) Using the better model found in part (ii), find the equation of a suitable regression line. Use it to estimate the world record time in the year 2018. Comment on the reliability of this estimate. [3]
- (iv) Deduce, with justification, the product moment correlation coefficient between t and x . [1]

- 10** In this question, you should state clearly the distribution of any random variables that you define.

The volume, S , in ml, of perfume in a randomly chosen small bottle has mean 20 and variance σ^2 .

- (i) If $\sigma = 15$, explain why S may not be appropriately modelled by a normal distribution. [2]

It is now assumed that S follows a normal distribution

- (ii) Given that 6.68% of the small bottles contains more than 23 ml of perfume, find the value of σ . [2]

For the rest of the question, the volume of perfume, S , in ml, in a randomly chosen small bottle follows the distribution $N(20, 4)$ and the volume of perfume, L , in ml, in a randomly chosen large bottle follows the distribution $N(100, 25)$.

- (iii) Calculate the probability that 6 randomly chosen small bottles and 9 randomly chosen large bottles contain a total volume of at least 1 litre of perfume. [3]
- (iv) Calculate the probability that the volume of perfume in a randomly chosen large bottle differs from 6 times the volume of perfume in a randomly chosen small bottle by more than 25ml. [3]
- (v) State, in this context, an assumption needed for your calculations in parts (iii) and (iv). [1]

- 11** A bag contains 5 cards with the letter 'A', 3 cards with the letter 'B' and 2 cards with the letter 'C'.

A A A A A B B B C C

The 10 cards are arranged at random in a row to form a letter sequence. For example, AABBCAAACB is a possible letter sequence.

- (i) Find the number of possible letter sequences. [1]
- (ii) Find the number of possible letter sequences if no two 'B's are next to each other **and** no two 'C's are next to each other. [4]
- (iii) Find the probability that the first two letters are identical given that the second letter is **not** an 'A'. [4]

The 10 cards are now arranged at random in a circle.

- (iv) Find the probability that no two 'A's are next to each other. [3]