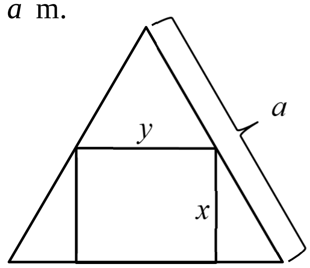
H2 Mathematics (9758)
Chapter 8 Applications of Differentiation
Discussion Solutions

<u>Level 1</u>

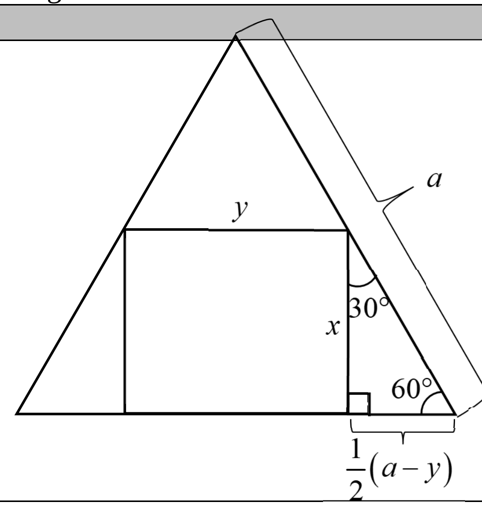
1 2012 ACJC JC1 Promo/9 (modified)

The diagram below shows a rectangle of height x m and width y m inscribed in an equilateral triangle of side a m.



(i) Show that $x=\frac{\sqrt{3}}{2}(a-y)$.

(ii) Hence find the maximum area of the rectangle.

Q1	**Solution**
(i)	$\tan 30° = \dfrac{\frac{1}{2}(a-y)}{x}$ $\Rightarrow \dfrac{1}{\sqrt{3}} = \dfrac{a-y}{2x}$ $\Rightarrow x = \dfrac{\sqrt{3}}{2}(a-y)$ (shown) 
(ii)	Let area of rectangle be $A\,\text{m}^2$ Then $A = xy = \dfrac{\sqrt{3}}{2}(a-y)y = \dfrac{\sqrt{3}}{2}ay - \dfrac{\sqrt{3}}{2}y^2$ $\Rightarrow \dfrac{\mathrm{d}A}{\mathrm{d}y} = \dfrac{\sqrt{3}}{2}a - \sqrt{3}y$ For maximum area, $\dfrac{\mathrm{d}A}{\mathrm{d}y} = 0 \Rightarrow \dfrac{\sqrt{3}}{2}a - \sqrt{3}y = 0 \Rightarrow y = \dfrac{a}{2}$ $\dfrac{\mathrm{d}^2A}{\mathrm{d}y^2} = -\sqrt{3} < 0$ Hence A is maximum when $y = \dfrac{a}{2}$. Maximum $A = \dfrac{\sqrt{3}}{2}a\left(\dfrac{a}{2}\right) - \dfrac{\sqrt{3}}{2}\left(\dfrac{a}{2}\right)^2 = \dfrac{\sqrt{3}}{8}a^2\ \text{m}^2$

2 **2018(9758)/I/7**

A curve C has equation $\dfrac{x^2-4y^2}{x^2+xy^2}=\dfrac{1}{2}$.

(i) Show that $\dfrac{dy}{dx}=\dfrac{2x-y^2}{2xy+16y}$. [3]

The points P and Q on C each have x-coordinate 1. The tangents to C at P and Q meet at the point N.

(ii) Find the exact coordinates of N. [6]

Q2	Suggested Solution
(i)	$\dfrac{x^2-4y^2}{x^2+xy^2}=\dfrac{1}{2}$ Cross multiply first $2x^2-8y^2=x^2+xy^2 \quad --(1)$ $x^2-8y^2=xy^2$ Differentiate w.r.t x, $2x-16y\dfrac{dy}{dx}=y^2+2xy\dfrac{dy}{dx}$ $\dfrac{dy}{dx}=\dfrac{2x-y^2}{2xy+16y} \quad \text{(Shown)}$
(ii)	Substitute $x=1$ into (1), $2-8y^2=1+y^2$ $9y^2=1$ $y=\pm\dfrac{1}{3}$ When $y=-\dfrac{1}{3}$, $\dfrac{dy}{dx}=\dfrac{2-\left(\frac{1}{3}\right)^2}{-2\left(\frac{1}{3}\right)-16\left(\frac{1}{3}\right)}=-\dfrac{17}{54}$ When $y=\dfrac{1}{3}$, $\dfrac{dy}{dx}=\dfrac{2-\left(\frac{1}{3}\right)^2}{2\left(\frac{1}{3}\right)+16\left(\frac{1}{3}\right)}=\dfrac{17}{54}$ Therefore, the equation of tangents at $x=1$ are $y-\left(-\dfrac{1}{3}\right)=-\dfrac{17}{54}(x-1) \quad --(2)$ and $y-\dfrac{1}{3}=\dfrac{17}{54}(x-1) \quad --(3)$. **Recall:** Equation of tangent is $(y-y_0)=m(x-x_0)$ Solving equation (2) and (3) simultaneously $(3)-(2)$,

	$-\frac{2}{3}=\frac{17}{27}(x-1)$ $x=-\frac{1}{17}$ $\Rightarrow y=0$ $\therefore$ coordinates of N is $\left(-\frac{1}{17},0\right)$.

3 Specimen Paper (9758)/I/1

A circular ink-blot is expanding such that the rate of change of its diameter D with respect to time t is 0.25cm/s. Find the rate of change of both the circumference and the area of the circle with respect to t when the radius of the circle is 1.5cm. Give your answers correct to 4 decimal places.

Q3	Suggested Solution
	Let the radius, circumference and area of the circle be r cm, C cm and A cm^2 respectively. Given: $\frac{\mathrm{d}D}{\mathrm{d}t}=0.25$ To find: $\frac{\mathrm{d}C}{\mathrm{d}t}$ and $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $r=1.5$ $C=2\pi r=\pi D$ Differentiate w.r.t t: **Strategy:** Use Implicit differentiation $\frac{\mathrm{d}C}{\mathrm{d}t}=\pi\frac{\mathrm{d}D}{\mathrm{d}t}$ $=0.25\pi$ $=0.7854$ (to 4 d.p.) **Look carefully at the requirement of the question** The rate of increase of circumference is 0.7854cm/s. $A=\pi r^2=\pi\left(\frac{D}{2}\right)^2=\pi\frac{D^2}{4}$ $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}D}\times\frac{\mathrm{d}D}{\mathrm{d}t}$ $=\pi\frac{D}{2}\times 0.25$ $=\frac{\pi D}{8}$ When $r=1.5,\ D=2r=2(1.5)=3$, $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\pi(3)}{8}$ $=1.1781$ (to 4 d.p.) **Look carefully at the requirement of the question** The rate of increase of area is 1.1781cm^2/s.

Level 2

4 2009(9740)/II/1 (modified)

A curve C has parametric equations

$$x = t^2 + 4t, \quad y = t^3 + t^2.$$

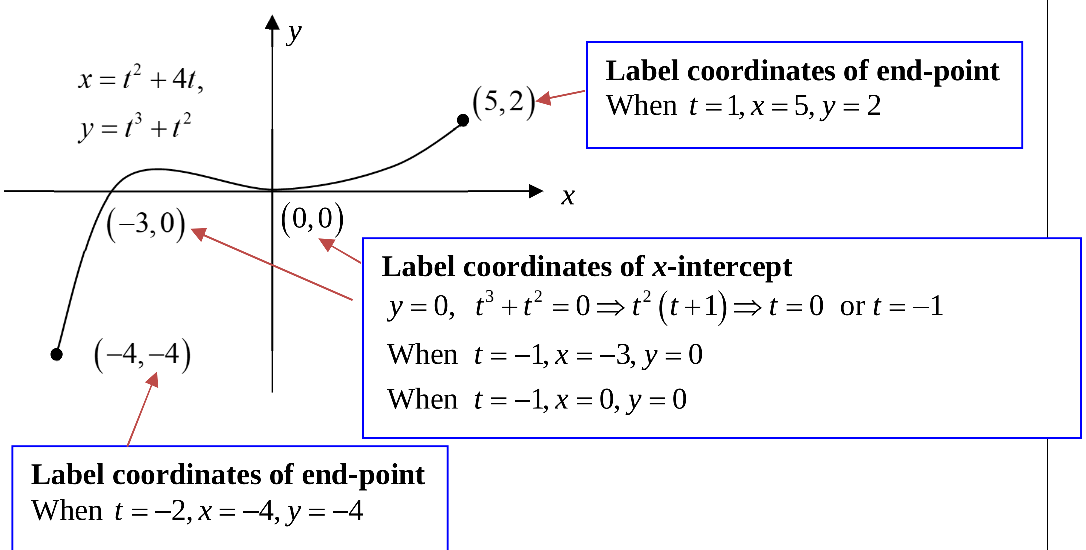
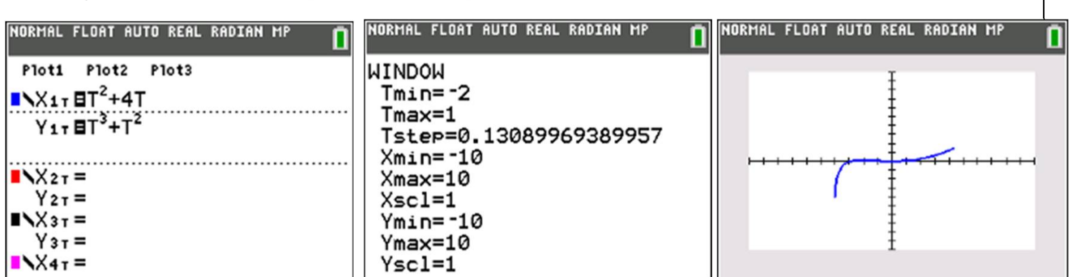
(i) Find the coordinates of the x-intercepts and sketch the curve for $-2 \le t \le 1$. You do not need to label the coordinates of the turning point(s). [3]

The tangent to the curve at the point P where $t = 2$ is denoted by l.

(ii) Find the Cartesian equation of l. [3]

(iii) The tangent l meets C again at the point Q. Use a non-calculator method to find the coordinates of Q. [4]

(iv) Determine the acute angle between the tangent l and the line $y = x + 3$. [2]

Q4	Suggested Solution
(i)	$y = 0, \quad t^3 + t^2 = 0 \Rightarrow t^2(t+1) \Rightarrow t = 0 \text{ or } t = -1$ When $t = -1, x = -3, y = 0$ When $t = -1, x = 0, y = 0$ Therefore the coordinates of the x-intercepts are $(-3, 0)$ and $(0, 0)$.  GC Keystrokes to sketch parametric equations: 

(ii)

$$\frac{dx}{dt} = 2t + 4 \qquad \frac{dy}{dt} = 3t^2 + 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 + 2t}{2t + 4}$$

Recall:
Equation of tangent is
$(y - y_0) = m(x - x_0)$

When $t = 2$, $x = 2^2 + 4(2) = 12$, $y = 2^3 + 2^2 = 12$

$\frac{dy}{dx} = 2,$

Equation of the tangent, l is $y - 12 = 2(x - 12)$

$$y = 2x - 12$$

(iii)

Curve C: . $x = t^2 + 4t$ and $y = t^3 + t^2$ --------(1)

Tangent line l: $y = 2x - 12$ --------- (2)

To find the point of intersection Q between tangent line l and curve C, we sub (1) into (2) and attempt to solve for the values of t:

Comments:
The tangent l meets C again at the point Q means point Q is the **point of intersection** between tangent line l and curve C.

Sub. $x = t^2 + 4t$ and $y = t^3 + t^2$ into l:

$$t^3 + t^2 = 2(t^2 + 4t) - 12$$

$$t^3 - t^2 - 8t + 12 = 0$$

$$(t-2)(t^2 + t - 6) = 0$$

$$(t-2)^2(t+3) = 0$$

$\therefore t = 2$ (rejected, this is point P) or $t = -3$

Note: "non-calculator method" means cannot use GC and get $t = -3$. You have to solve for the values of t algebraically, i.e. using factorisation. (You can use GC to guide/check your factorisation)
Note: Since the line is tangent to the curve at P (when t = 2), it will imply that t = 2 is a solution of the polynomial (hence t – 2 is a factor)

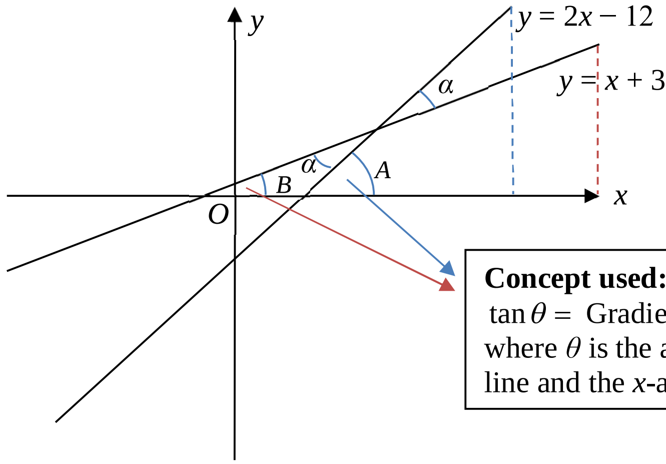
Always reject with a reason.

When $t = -3$,
$x = -3$, $y = -18$

Coordinates of Q: $(-3, -18)$

(iv) Let α be the acute angle between the two lines.

Strategy to solve this question:
Draw diagram



Concept used:
$\tan\theta =$ Gradient of line
where θ is the angle between the line and the x-axis

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{\text{rise}}{\text{run}} = \text{gradient of line}$$

$$\therefore \tan A = 2$$

$$A = \tan^{-1} 2$$

$$\tan B = \frac{\text{opp}}{\text{adj}} = \frac{\text{rise}}{\text{run}} = \text{gradient of line}$$

$$\therefore \tan B = 1$$

$$B = \tan^{-1} 1$$

$$\alpha = A - B \quad (\text{exterior angle of } \Delta = \text{sum of 2 interior opposite angles of } \Delta)$$

$$= \tan^{-1} 2 - \tan^{-1} 1$$

$$= 18.4°$$

<u>Alternatively, using MF26</u>

$$\tan\alpha = \tan(A - B)$$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{2-1}{1+(2)(1)} = \frac{1}{3}$$

$$\alpha = \tan^{-1}\frac{1}{3} = 18.4°$$

5 **2017(9758)/II/1 (modified)**

A curve C has parametric equations

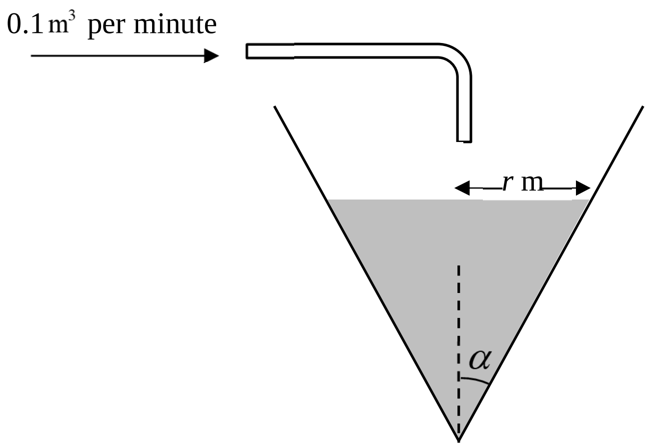
$$x=\frac{3}{t},\ y=2t.$$

(i) The tangent at the point $P\left(\frac{3}{p},2p\right)$ on C meets the x-axis at D and the y-axis at E. The point F is the midpoint of DE. Find a cartesian equation of the curve traced by F as p varies. [5]

(ii) Show that the area of triangle ODE is independent of p, where O is the origin.

Q5	Suggested Solution
(i)	$\frac{dx}{dt}=-\frac{3}{t^2}$, $\quad \frac{dy}{dt}=2$ $\frac{dy}{dx}=-\frac{2}{3}t^2$ Find $\frac{dy}{dx}$ in terms of the parameter t first. At point P, $t=p$ $\quad \therefore \left.\frac{dy}{dx}\right\|_{t=p}=-\frac{2}{3}p^2$ Equation of tangent at point P: $y-2p=-\frac{2}{3}p^2\left(x-\frac{3}{p}\right)$ When $x=0$, $y=4p.\Rightarrow E(0,4p)$ When $y=0$, $x=\frac{6}{p}.\Rightarrow D\left(\frac{6}{p},0\right)$ **Recall:** Equation of tangent is $(y-y_0)=m(x-x_0)$ Coordinates of F are $\left(\frac{\frac{6}{p}+0}{2},\frac{0+4p}{2}\right)\Rightarrow\left(\frac{3}{p},2p\right)$. Any point on the curved traced by F will have $x=\frac{3}{p}$, $\quad y=2p$ To obtain the cartestian equation, subst $p=\frac{3}{x}$ into $y=2p$ Cartesian equation traced by F : $y=\frac{6}{x}$
(ii)	Coordinates of D and E are $\left(\frac{6}{p},0\right)$ and $(0,4p)$ respectively. *OE* and *OD* are perpendicular to each other, thus the area of triangle *ODE* can be found using $\frac{1}{2}(\text{base})(\text{height})$ Area of triangle *ODE* = $\frac{1}{2}\left(\frac{6}{p}\right)4p=12$ which is independent of p. (shown)

6 2016(9740)/II/1



Water is poured at a rate of $0.1\,\text{m}^3$ per minute into a container in the form of an open cone. The semi-vertical angle of the cone is α, where $\tan\alpha = 0.5$. At time t minutes after the start, the radius of the water surface is r m (see diagram). Find the rate of increase of the depth of water when the volume of water in the container is $3\,\text{m}^3$. [7]

[The volume of a cone of base radius r and height h is given by $V = \dfrac{1}{3}\pi r^2 h$.]

Q6	**Suggested Solution**

Let the volume and depth of water be V m^3 and h m respectively.

Given: $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 0.1$

To find: $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ when $V = 3$

$$\frac{r}{h} = \tan\alpha$$

Obtain a relationship between r and h

$$r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{12}\pi h^3$$

Key idea is to express V in terms of a single variable.

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{4}\pi h^2$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$$

Apply chain rule

$$= \frac{4}{\pi h^2} \times 0.1$$

When $V = 3,\ 3 = \dfrac{1}{12}\pi h^3 \Rightarrow h = \left(\dfrac{36}{\pi}\right)^{\frac{1}{3}}$,

<u>Alternatively,</u>

$$V = \frac{1}{3}\pi h^3$$

Differentiate w.r.t. t,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi\left(3h^2\right)\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$0.1 = \frac{1}{12}\pi\left(\frac{36}{\pi}\right)^{\frac{2}{3}}\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.0251$$

	$\frac{dh}{dt} = \frac{4}{\pi\left[\left(\frac{36}{\pi}\right)^{\frac{1}{3}}\right]^2} \times 0.1$ $= 0.0251$ Answer the question rate of increase of depth of water $= 0.0251\,\text{m}$ per minute (3 s.f.)

7 **1981(9205)/I/6**

Two variables u and v are connected by the relation $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$, where f is a constant.

Given that u and v both vary with time, t, find an equation connecting $\frac{du}{dt}, \frac{dv}{dt}, u$ and v.

Given also that u is decreasing at a rate of 2 cm per second and that f = 10 cm, calculate the rate of increase of v when u = 50 cm.

Q7 **Suggested Solution**

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$$

Differentiate w.r.t. t,

$$\left(-\frac{1}{u^2}\right)\frac{du}{dt}+\left(-\frac{1}{v^2}\right)\frac{dv}{dt}=0$$

$$\Rightarrow \frac{1}{u^2}\frac{du}{dt}+\frac{1}{v^2}\frac{dv}{dt}=0 -----(1)$$

(This is an equation connecting $\frac{du}{dt}, \frac{dv}{dt}, u$ and v)

Look carefully at the question

1. u and v are **variables**
2. f is a **constant**

Since u and v both vary with time, ***t***, we differentiate with respect to ***t***.

Given: $\frac{du}{dt}=-2,\ f=10$.

To find: $\frac{dv}{dt}$ when $u=50$

Since u is **decreasing** at a rate of 2 cm per second, $\frac{du}{dt}=-2$

When $u=50$, $f=10$,

$$\frac{1}{50}+\frac{1}{v}=\frac{1}{10}$$

$$\Rightarrow \frac{1}{v}=\frac{2}{25}$$

$$\Rightarrow v=12.5$$

Substitute $\frac{du}{dt}=-2, u=50, f=10, v=12.5$ into (1):

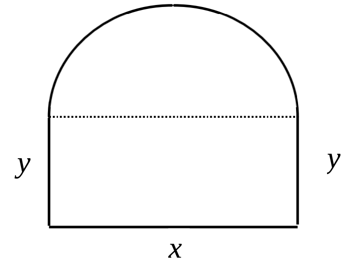
$$\frac{1}{u^2}\frac{du}{dt}+\frac{1}{v^2}\frac{dv}{dt}=0$$

$$\Rightarrow \frac{1}{50^2}(-2)+\frac{1}{12.5^2}\frac{dv}{dt}=0$$

$$\Rightarrow \frac{dv}{dt}=0.125$$

Therefore, v is increasing at a rate of 0.125 cm per second.

8 2008(9740)/I/7



A new flower-bed is being designed for a large garden. The flower-bed will occupy a rectangle x m by y m together with a semicircle of diameter x m, as shown in the diagram. A low wall will be built around the flower-bed. The time needed to build the wall will be 3 hours per metre for the straight parts and 9 hours per metre for the semicircular part. Given that a total time of 180 hours is taken to build the wall, find, using differentiation, the values of x and y which give a flower-bed of maximum area. [10]

Q8	**Suggested Solution**

Aim: Maximise area of the flower-bed

$$A=\underbrace{\frac{1}{2}\pi\left(\frac{x}{2}\right)^2}_{\text{Area of Semi-circle with radius }\left(\frac{x}{2}\right)}+\underbrace{xy}_{\text{Area of rectangle}}$$

A is in terms of **2 variables** x and y.

Steps to solve Maxima/Minima Problems

1. Draw a clear diagram and define all variables, where necessary.
2. Form equation(s) relating the variables.
3. Express quantity to be maximized/minimized in terms of a **single** variable, say x (if there are 2 variables, express one in terms of another).
4. Differentiate w.r.t. x and equate to 0 to find the stationary point(s).
5. Use 1st or 2nd derivative test to determine/prove nature of the stationary point.

Time taken to build wall,

$$\underbrace{3(x+2y)}_{\text{Length and breadth of rectangle}}+9\underbrace{\left(\pi\frac{x}{2}\right)}_{\text{Circumference of Semi-circle}}=180$$

$$\Rightarrow y=30-\left(\frac{3\pi}{4}+\frac{1}{2}\right)x$$

3 hours per metre for the straight parts and 9 hours per metre for the semicircular part

Area of flower bed is given by

$$A=\frac{1}{2}\pi\left(\frac{x}{2}\right)^2+xy$$

$$A=\frac{1}{8}\pi x^2+x\left[30-\left(\frac{3\pi}{4}+\frac{1}{2}\right)x\right]=30x-\left(\frac{3\pi}{4}-\frac{\pi}{8}+\frac{1}{2}\right)x^2$$

$$A=30x-\left(\frac{5}{8}\pi+\frac{1}{2}\right)x^2$$

Attempt to express A in a **single variable**, in this case, A in terms of x

At stationary point, $\frac{dA}{dx}=0$

$$\frac{dA}{dx}=30-2\left(\frac{5\pi}{8}+\frac{1}{2}\right)x=30-\left(\frac{5\pi}{4}+1\right)x=0$$

$30-\left(\frac{5\pi}{4}+1\right)x=0$

$x=6.0889\quad(5\text{ s.f})$

Note that: Question **does not require** the exact values of x and y, so we can leave our answer in 5 s.f. for intermediate working.

Using the first derivative test:

x	$(6.0889)^-$	6.0889	$(6.0889)^+$
$\frac{dA}{dx}$	$+$	0	$-$
Slope	/	—	\

Remember to prove A is maximum when $x=6.0889$

$\therefore A$ is maximum when $x=6.0889$.

Alternative method to prove that A is maximum: (using 2nd derivative test)

$\frac{d^2A}{dx^2}=-\left(\frac{5\pi}{4}+1\right)<0$

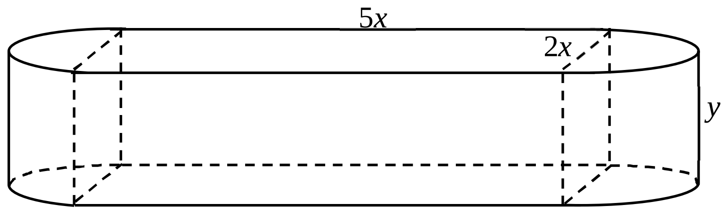
$\therefore A$ is maximum when $x=6.0889$

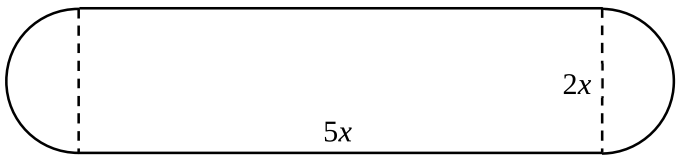
When $x=6.0889$, $y=30-\left(\frac{3\pi}{4}+\frac{1}{2}\right)(6.0889)=12.609$

The values of x and y which give a **flower-bed of maximum area is** $x=6.09$ and $y=12.6\ (3\text{ s.f})$

9 RI JC1 Promo 9758/2019/Q6

An **open** tin box of negligible thickness is to be made. The design of the box and its horizontal base are shown below.





The middle portion of the horizontal base of the box is a rectangle of length $5x$ cm and width $2x$ cm while the two ends are semicircles of radius x cm. The box has a depth of y cm and its volume is 800 cm^3.

Show that the total external surface area, A in cm^2, of the box is given by

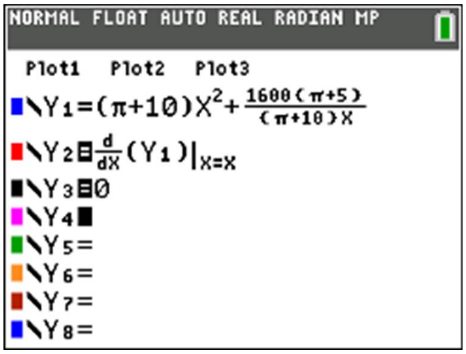
$$A=(\pi+10)x^2+\frac{1600(\pi+5)}{(\pi+10)x}.$$

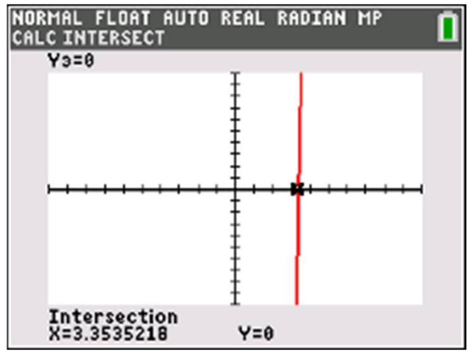
Use differentiation to find the value of x which minimizes A. [6]

Q9	**Solutions**
Since the volume of the box is 800 cm^3, $$(\pi x^2+10x^2)y=800 \Rightarrow y=\frac{800}{(\pi+10)x^2}$$ Let the external surface area be A cm^2. $$\begin{aligned}A&=10x^2+10xy+\pi x^2+2\pi xy\\&=(\pi+10)x^2+2(\pi+5)xy\\&=(\pi+10)x^2+2(\pi+5)x\left[\frac{800}{(\pi+10)x^2}\right]\\&=(\pi+10)x^2+\frac{1600(\pi+5)}{(\pi+10)x}\end{aligned}$$ $$\frac{\mathrm{d}A}{\mathrm{d}x}=2(\pi+10)x-\frac{1600(\pi+5)}{(\pi+10)x^2}=0$$	

Q9	**Solutions**

Using GC, $x = 3.3535 = 3.35$ (correct to 3 s.f.)





Using the first derivative test:

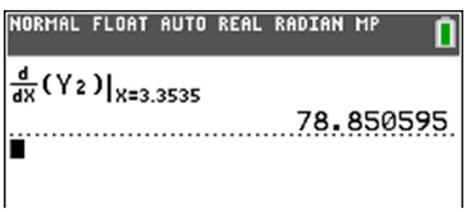
x	3.3535^-	3.3535	3.3535^+
$\frac{\mathrm{d}A}{\mathrm{d}x}$	negative	0	positive
Shape	\	—	/

Thus the external surface area is minimum at $x = 3.35$ (correct to 3 s.f.)

Alternatively, using the second derivative test.

$$\left.\frac{\mathrm{d}^2A}{\mathrm{d}x^2}\right|_{x=3.3535} = 78.851 > 0$$

Thus the external surface area is minimum at $x = 3.35$ (correct to 3 s.f.)



Level 3

10 2012(9740)/I/11(modified)

A curve C has parametric equations

$$x = \theta - \sin\theta, \qquad y = 1 - \cos\theta$$

where $0 \le \theta \le 2\pi$.

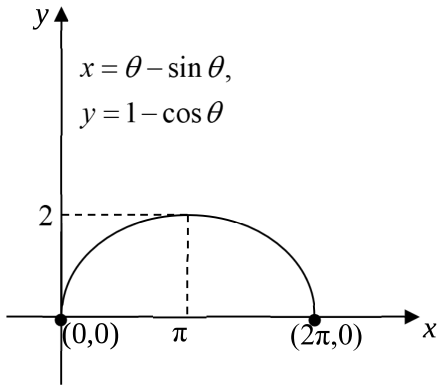
(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \cot\frac{1}{2}\theta$ and find the gradient of C at the point where $\theta = \pi$. What can be said about the tangents to C as $\theta \to 0$ and $\theta \to 2\pi$? [5]

(ii) Sketch C, showing clearly the features of the curve at the points where $\theta = 0$, π and 2π . [3]

(iii) A point P on C has parameter p, where $0 < p < \frac{1}{2}\pi$. Show that the normal to C at P crosses the x-axis at the point with coordinates $(p, 0)$. [5]

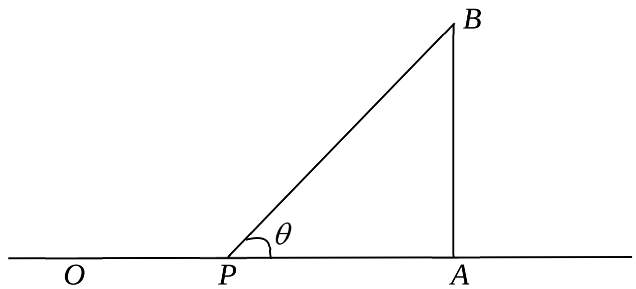
(iv) Given that θ is increasing at a rate of 2 radians per second, find the rate of change of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $\theta = \frac{\pi}{3}$.

Q10	Suggested Solution
(i)	$x = \theta - \sin\theta$ $y = 1 - \cos\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 1 - \cos\theta$ $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \sin\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\theta}{1-\cos\theta}$ $= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{1-\left(1-2\sin^2\frac{1}{2}\theta\right)}$ $= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta}$ $= \frac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \cot\frac{1}{2}\theta$ (shown) At $\theta = \pi$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \cot\frac{1}{2}\pi$ $= 0$ Gradient of C at the point where $\theta = \pi$ is 0. At $\theta = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined. At $\theta = 2\pi$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined. The tangents become parallel to the y-axis as $\theta \to 0$ and $\theta \to 2\pi$.

(ii)	 Students must show that the gradient of the curve is parallel to the y-axis at $\theta=0$ and 2π. Stationary maximum point at $\theta=\pi$.
(iii)	Gradient of normal at P = $-\tan\frac{1}{2}p$ Equation of normal at P is $y-(1-\cos p)=-\tan\frac{1}{2}p\left[x-(p-\sin p)\right]$ When y = 0, $\cos p-1=-\tan\frac{1}{2}p\left[x-(p-\sin p)\right]$ $x-p+\sin p=\dfrac{\left(1-2\sin^2\frac{1}{2}p\right)-1}{-\tan\frac{1}{2}p}$ $x-p+\sin p=2\sin\frac{1}{2}p\cos\frac{1}{2}p$ $x-p+\sin p=\sin p$ $x=p$ Therefore, the normal to C at P crosses the x-axis at $(p,0)$ (Shown) **<u>Alternative solution</u>** $\cos p-1=-\tan\frac{1}{2}p\left[x-(p-\sin p)\right]$ $\cos p-1=-\dfrac{1-\cos p}{\sin p}(x-p+\sin p)$ $\cos p-1=\dfrac{\cos p-1}{\sin p}(x-p+\sin p)$ $\sin p=x-p+\sin p$ $x=p$ Therefore, the normal to C at P crosses the x-axis at $(p,0)$ (Shown)

(iv)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\cot\dfrac{1}{2}\theta$ Let m be $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. $\dfrac{\mathrm{d}(m)}{\mathrm{d}t}=\dfrac{\mathrm{d}(m)}{\mathrm{d}\theta}\cdot\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ [so we need $\dfrac{\mathrm{d}(m)}{\mathrm{d}\theta}$] $\dfrac{\mathrm{d}(m)}{\mathrm{d}\theta}=-\dfrac{1}{2}\operatorname{cosec}^2\dfrac{\theta}{2}$ $\therefore$ when $\theta=\dfrac{\pi}{3}$, $\dfrac{\mathrm{d}(m)}{\mathrm{d}t}=\dfrac{\mathrm{d}(m)}{\mathrm{d}\theta}\cdot\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=-\operatorname{cosec}^2\dfrac{\pi}{6}=-(2)^2=-4$ units/s

11 1987(9202)/I/17

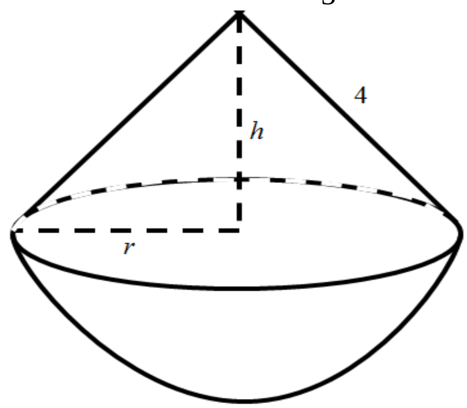


In the diagram, *O* and *A* are fixed points 1000m apart on horizontal ground. The point *B* is vertically above *A*, and represents a balloon which is ascending at a steady rate of 2ms^{-1}. The balloon is being observed from a moving point *P* on the line *OA*. At time $t=0$, the balloon is at *A* and the observer is at *O*. The observation point *P* moves towards *A* with steady speed 6ms^{-1}. At time *t*, the angle *APB* is θ radians. Show that $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=\dfrac{500}{t^2+(500-3t)^2}$.

Q11	Suggested Solution
	At time *t*, $PA = OA - OP = 1000-6t$ and $AB = 2t$. From ΔAPB $\tan\theta=\dfrac{2t}{1000-6t}=\dfrac{t}{500-3t}$. $\sec^2\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=\dfrac{500}{(500-3t)^2}$ $\left(1+\tan^2\theta\right)\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=\dfrac{500}{(500-3t)^2}$ $\left(1+\dfrac{t^2}{(500-3t)^2}\right)\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=\dfrac{500}{(500-3t)^2}$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=\dfrac{500}{t^2+(500-3t)^2}$ <u>**Alternative solution**</u> $\theta=\tan^{-1}\dfrac{2t}{1000-6t}$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=\dfrac{1}{1+\left(\dfrac{2t}{1000-6t}\right)^2}\left(\dfrac{2000}{(1000-6t)^2}\right)$ $=\dfrac{2000}{(1000-6t)^2+4t^2}$ $=\dfrac{500}{t^2+(500-3t)^2}$

12 N2014/I/11

[It is given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



A toy manufacturer makes a toy which consists of a hemisphere of radius r cm joined to a circular cone of base radius r cm and height h cm (see diagram). The manufacturer determines that the length of the slant edge of the cone must be 4 cm and that the total volume of the toy, V cm^3, should be as large as possible.

(i) Find a formula for V in terms of r. Given that $r = r_1$ is the value of r which gives the maximum value of V, show that r_1 satisfies the equation $45r^4 - 768r^2 + 1024 = 0$. [6]

(ii) Find the two solutions to the equation in part **(i)** for which $r > 0$, giving your answers correct to 3 decimal places. [2]

(iii) Show that one of the solutions found in part **(ii)** does not give a stationary value of V. Hence write down the value of r_1 and find the corresponding value of h. [3]

(iv) Sketch the graph showing the volume of the toy as the radius of the hemisphere varies. [3]

Q12	Suggested Solution

(i)

Vol of cone

Vol of hemisphere

V is in terms of 2 variables.

$$V=\frac{1}{3}\pi r^2h+\frac{1}{2}\left(\frac{4}{3}\pi r^3\right)=\frac{1}{3}\pi r^2h+\frac{2}{3}\pi r^3 \text{ --- (1)}$$

By Pythagoras Theorem, $h^2+r^2=4^2 \Rightarrow h=\sqrt{16-r^2}$ --- (2)

Solving (1) and (2), $V=\frac{1}{3}\pi r^2\sqrt{16-r^2}+\frac{2}{3}\pi r^3$

Attempt to express *V* in a **single variable**, in this case, *V* in terms of ***r***

$$\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{1}{3}\pi\left(2r\sqrt{16-r^2}+r^2\left(\frac{1}{2}\left(16-r^2\right)^{-\frac{1}{2}}(-2r)\right)\right)+\frac{2}{3}\left(3\pi r^2\right)$$

$$=\frac{1}{3}\pi\left(2r\sqrt{16-r^2}-\frac{r^3}{\sqrt{16-r^2}}\right)+2\pi r^2$$

For maximum *V*, $\frac{\mathrm{d}V}{\mathrm{d}r}=0$.

$$\therefore \frac{\mathrm{d}V}{\mathrm{d}r}=\frac{1}{3}\pi\left(2r\sqrt{16-r^2}-\frac{r^3}{\sqrt{16-r^2}}\right)+2\pi r^2=0$$

$$2r\sqrt{16-r^2}-\frac{r^3}{\sqrt{16-r^2}}+6r^2=0$$

$$2r\left(16-r^2\right)-r^3+6r^2\sqrt{16-r^2}=0$$

$$2\left(16-r^2\right)-r^2+6r\sqrt{16-r^2}=0 \quad , r\neq 0$$

Need to get rid of the square root sign and obtain a polynomial equation in *r*.

$$32-2r^2-r^2=-6r\sqrt{16-r^2}$$

$$32-3r^2=-6r\sqrt{16-r^2}$$

$$\left(32-3r^2\right)^2=\left(-6r\sqrt{16-r^2}\right)^2$$

Can only square both sides to get rid of the square root sign at this step.

$$1024-192r^2+9r^4=36r^2\left(16-r^2\right)$$

$$1024-192r^2+9r^4=576r^2-36r^4$$

$$45r^4-768r^2+1024=0 \quad \text{(shown)}$$

(ii)

Using GC,

$r=3.9508=3.951$ cm (3 d.p)

or $r=-3.9508=-3.951$ (rejected $\because r>0$)

or $r=1.2074=1.207$ cm (3 d.p)

or $r=-1.2074=-1.207$ (rejected $\because r>0$)

Read question carefully. 3 decimal places is required.

$\therefore$ the two solutions are $r=3.951$ cm or 1.207 cm. (3 d.p)

(iii)

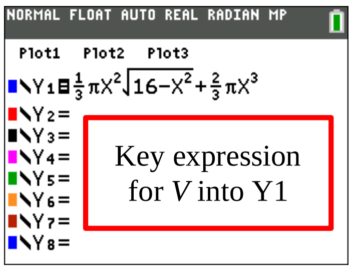
Does not give a stationary value of $V \Rightarrow \dfrac{\mathrm{d}V}{\mathrm{d}r} \neq 0$

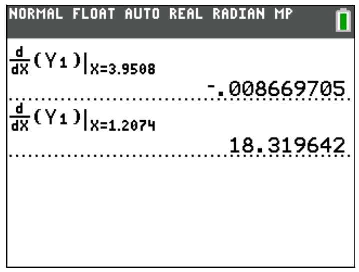
For $r = 3.9508$, Use at least 5 s.f for intermediate working

$\dfrac{\mathrm{d}V}{\mathrm{d}r} = -0.00867\text{(using GC) or } -0.003\text{(sub into dV/dr expression)} \approx 0$

For $r = 1.2074$, Use at least 5 s.f for intermediate working

$\dfrac{\mathrm{d}V}{\mathrm{d}r} = 18.3 \neq 0$





$\therefore$ only $r = 3.951$ cm gives a stationary value of $V \Rightarrow r_1 = 3.951$ cm

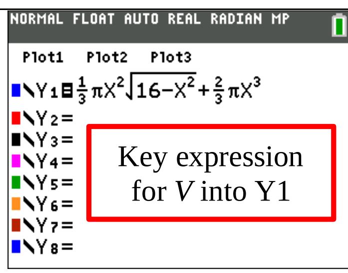
For $r_1 = 3.951$ cm,

$h = \sqrt{16 - 3.9508^2}$

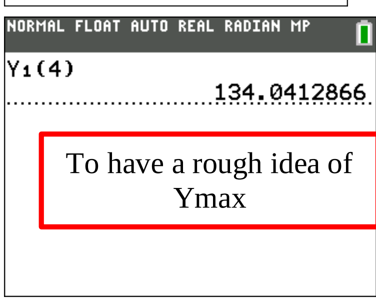
$= 0.62544$ Use at least 5 s.f for intermediate working

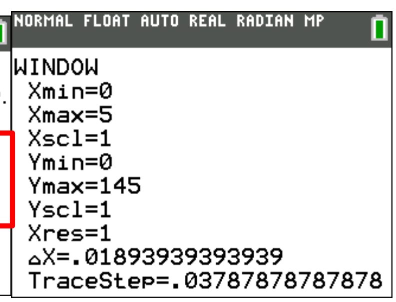
≈ 0.625 cm (3 s.f)

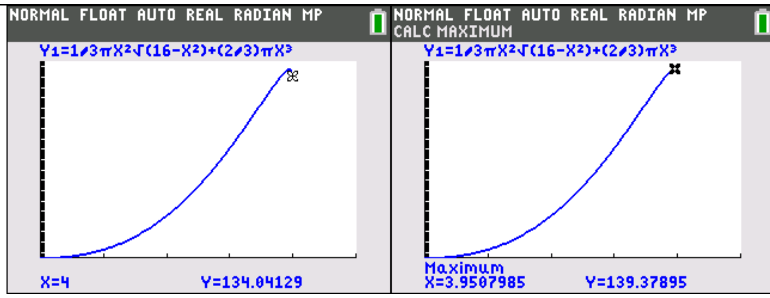
(iv)

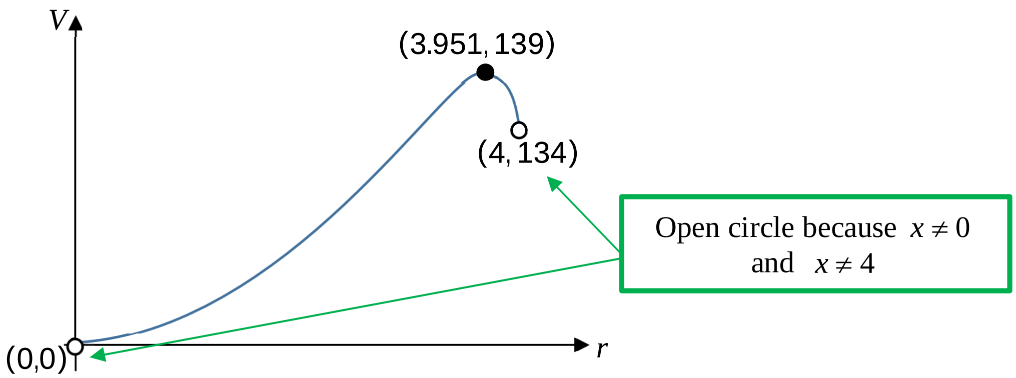


Notice that $x < 4$ in order for the toy to have a cone.






NORMAL FLOAT AUTO REAL RADIAN MP
Y1=1/3πX²√(16-X²)+(2/3)πX³
X=4
Y=134.04129
NORMAL FLOAT AUTO REAL RADIAN MP
CALC MAXIMUM
Y1=1/3πX²√(16-X²)+(2/3)πX³
Maximum
X=3.9507985
Y=139.37895


V
(3.951, 139)
(4, 134)
Open circle because x ≠ 0 and x ≠ 4
(0,0)
r

13 2013(9740)/II/2

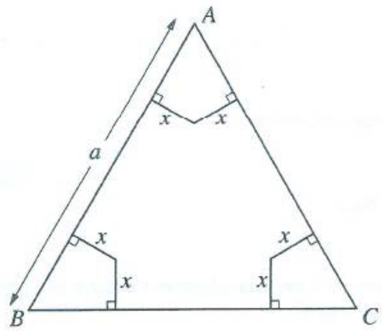


Fig. 1

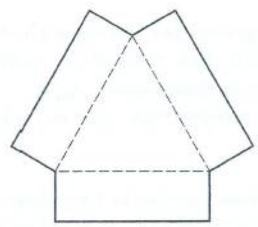


Fig. 2

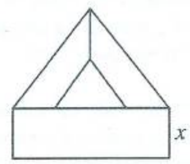


Fig. 3

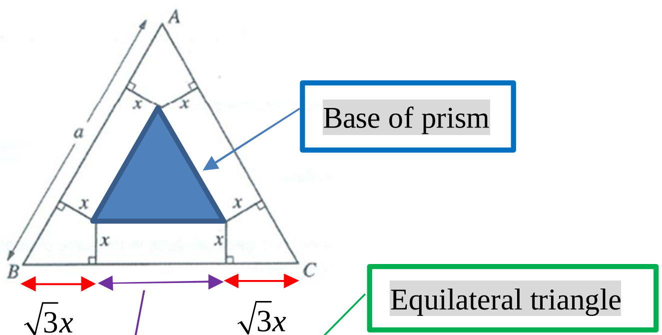
Fig. 1 shows a piece of card, ABC, in the form of an equilateral triangle of side a. A kite shape is cut from each corner, to give the shape shown in Fig. 2. The remaining card shown in Fig. 2 is folded along the dotted lines, to form the open triangular prism of height x shown in Fig. 3.

(i) Show that the volume V of the prism is given by $V = \frac{1}{4}x\sqrt{3}\left(a - 2x\sqrt{3}\right)^2$. [3]

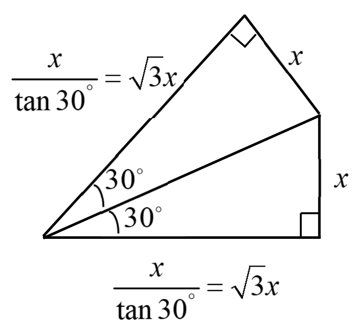
(ii) Use differentiation to find, in terms of a, the maximum value of V, proving that it is a maximum. [6]

Solution:

(i)



$$
\begin{aligned}
V &= \text{Cross Sectional Area} \times \text{Height} \\
&= \frac{1}{2}\left(a - 2\sqrt{3}x\right)^2\left(\sin 60^\circ\right)x \\
&= \frac{x}{2}\left(a - 2\sqrt{3}x\right)^2\frac{\sqrt{3}}{2} \\
&= \frac{x\sqrt{3}}{4}\left(a - 2\sqrt{3}x\right)^2 \text{ (shown)}
\end{aligned}
$$



(ii)

$$V = \frac{1}{4}x\sqrt{3}\left(a - 2x\sqrt{3}\right)^2 = \frac{\sqrt{3}}{4}\left[x\left(a - 2x\sqrt{3}\right)^2\right]$$

$$\frac{dV}{dx} = \frac{\sqrt{3}}{4}\left[\left(a-2\sqrt{3}x\right)^2 + x.2\left(a-2\sqrt{3}x\right)\left(-2\sqrt{3}\right)\right]$$

$$= \frac{\sqrt{3}}{4}\left(a-2\sqrt{3}x\right)\left[\left(a-2\sqrt{3}x\right)-4\sqrt{3}x\right]$$

$$= \frac{\sqrt{3}}{4}\left(a-2\sqrt{3}x\right)\left(a-6\sqrt{3}x\right) - - - - - - (1)$$

When $\frac{dV}{dx} = 0$,

Note: $a-2\sqrt{3}x \neq 0$ as it is the length of the base of the prism.

$$\frac{\sqrt{3}}{4}\left(a-2\sqrt{3}x\right)\left(a-6\sqrt{3}x\right) = 0$$

$a = 2\sqrt{3}x$ or $a = 6\sqrt{3}x$.

$\therefore x = \frac{a}{2\sqrt{3}} = \frac{a\sqrt{3}}{6}$ or $x = \frac{a}{6\sqrt{3}} = \frac{a\sqrt{3}}{18}$

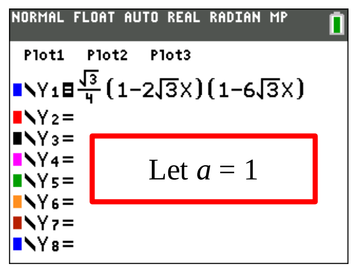
(reject $\because x < \frac{a}{2\sqrt{3}}$ for prism to have base)

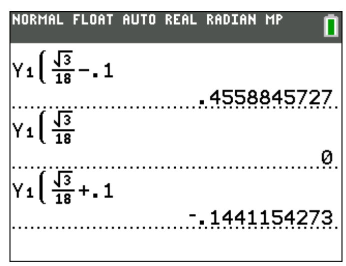
Either 1st Derivative Test

$$\frac{dV}{dx} = \frac{\sqrt{3}}{4}(a-2\sqrt{3}x)\left[a-6\sqrt{3}x\right]$$

x	$\left(\frac{a\sqrt{3}}{18}\right)^-$	$\frac{a\sqrt{3}}{18}$	$\left(\frac{a\sqrt{3}}{18}\right)^+$
Sign of $\frac{dV}{dx}$	+	0	–
Slope of tangent	/	—	\

$\therefore V$ is maximum at $x = \frac{a\sqrt{3}}{18}$.





Or 2nd Derivative Test (not recommended)

From (1),

$$\frac{d^2V}{dx^2} = \frac{\sqrt{3}}{4}\left[-2\sqrt{3}(a-6\sqrt{3}x)+(a-2\sqrt{3}x)(-6\sqrt{3})\right]$$

$$= -\frac{3}{2}\left[(a-6\sqrt{3}x)+3(a-2\sqrt{3}x)\right]$$

$$= -\frac{3}{2}\left[a-6\sqrt{3}x+3a-6\sqrt{3}x\right]$$

$$= -\frac{3}{2}\left[4a-12\sqrt{3}x\right]$$

$$= -6(a-3\sqrt{3}x)$$

When $x = \dfrac{a\sqrt{3}}{18}$,

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -6\left(a - 3\sqrt{3}\cdot a\frac{\sqrt{3}}{18}\right) = -3a < 0$$

$\Rightarrow V$ is maximum at $x = \dfrac{a\sqrt{3}}{18}$.

$$\begin{aligned}\text{Max } V &= \frac{\sqrt{3}}{4}\left(\frac{a\sqrt{3}}{18}\right)\left[a - 2\sqrt{3}\left(\frac{a\sqrt{3}}{18}\right)\right]^2 \\ &= \frac{a}{24}\left(\frac{2a}{3}\right)^2 \\ &= \frac{a^3}{54}\end{aligned}$$

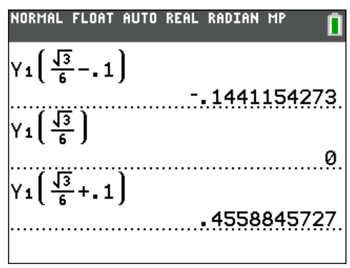
Remember to find the value of *V*.

<u>Note</u>: In the event it is not obvious that $x = \dfrac{a}{2\sqrt{3}} = \dfrac{a\sqrt{3}}{6}$ has to be rejected, one can proceed to show that this *x* value gives a minimum value of *V* and go on to reject this *x* value.

Use
<u>Either</u> 1st Derivative Test

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\sqrt{3}}{4}(a - 2\sqrt{3}x)\left[a - 6\sqrt{3}x\right]$$

x	$\left(\dfrac{a\sqrt{3}}{6}\right)^-$	$\dfrac{a\sqrt{3}}{6}$	$\left(\dfrac{a\sqrt{3}}{6}\right)^+$
Sign of $\dfrac{\mathrm{d}V}{\mathrm{d}x}$	$-$	0	$+$
Slope of tangent			



$\therefore V$ is minimum at $x = \dfrac{a\sqrt{3}}{6}$.

<u>Or</u> 2nd Derivative Test

When $x = \dfrac{a\sqrt{3}}{6}$,

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -6\left(a - 3\sqrt{3}.a\frac{\sqrt{3}}{6}\right) = 3a > 0$$

$\Rightarrow V$ is minimum at $x = \dfrac{a\sqrt{3}}{6}$.