Complex Numbers (Solutions)

1 HCI/2014/I/3

One of the roots of the equation $z^3 - 2z^2 + az + 1 + 3\mathrm{i} = 0$ is $z = \mathrm{i}$. Find the complex number a and the other roots. [5]

(a)	Let $\mathrm{P}(z) = z^3 - 2z^2 + az + 1 + 3\mathrm{i}$ $\Rightarrow \mathrm{P}(\mathrm{i}) = -1 - 2 - a + 1 + 3\mathrm{i} = 0$ $\Rightarrow a = -2 + 3\mathrm{i}$ Use long division or by comparing coefficient method, $\mathrm{P}(z) = z^3 - 2z^2 + (-2+3\mathrm{i})z + 1 + 3\mathrm{i}$ $= (z - \mathrm{i})\left[z^2 + (-2+\mathrm{i})z - 3 + \mathrm{i}\right]$ $z^2 + (-2+\mathrm{i})z - 3 + \mathrm{i} = 0 \Rightarrow z = \dfrac{-(-2+\mathrm{i}) \pm \sqrt{(-2+\mathrm{i})^2 - 4(-3+\mathrm{i})}}{2}$ $\Rightarrow z = \dfrac{(2-\mathrm{i}) \pm (4-\mathrm{i})}{2} \Rightarrow z = -1 \text{ or } 3 - \mathrm{i}$

2 MI/2014/I/10

(i) By using de Moivre's theorem, or otherwise, show that

$$(1-i)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right).$$ [2]

(ii) Using the result in **(i)** or otherwise, find the least positive integer n for which $(1-i)^n$ is real and negative. Solution by trial and error will not be accepted. [3]

(iii) For the equation $z^4 + az^3 + bz^2 + cz + d = 0$ where a, b, c and d are real, give a brief explanation and determine the possible number of complex roots the equation can have. [2]

(iv) Solve the equation $z^4 + 4 = 0$, expressing the solutions in the form $x + iy$ where x and y are real. [4]

i	$(1-i)^n = \left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)\right]^n$ $= 2^{\frac{n}{2}}\left(\cos\left(-\frac{n\pi}{4}\right)+i\sin\left(-\frac{n\pi}{4}\right)\right)$ $= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)$ *Alternatively,* $(1-i)^n = \left[\sqrt{2}e^{-i\frac{\pi}{4}}\right]^n$ $= \sqrt{2}e^{i\frac{n\pi}{4}}$ $= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)$ *Alternatively,* $\lvert 1-i\rvert^n = \sqrt{2}^n$ $= 2^{\frac{n}{2}}$ $\arg(1-i)^n = n\arg(1-i)$ $= -n\frac{\pi}{4}$ $\therefore (1-i)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)$
ii	For $(1-i)^n$ to be real and negative, $\arg\left[(1-i)^n\right] = -\frac{n\pi}{4} = \ldots,-3\pi,-\pi,\pi,3\pi,\ldots$ For least possible n, $-\frac{n\pi}{4} = -\pi$ $\therefore$ least $n = 4$
iii	Since the <u>coefficients of the equation are real</u>, <u>complex roots</u> will occur in <u>conjugate pairs</u> by Conjugate Roots Theorem. Furthermore, the order of the equation is 4, we would expect 4 roots ie. <u>0 pair of complex root with 4 real roots or 1 pair of complex conjugate roots with 2 real roots or 2 pairs of complex conjugate roots and no real root</u>.

iv	
	$z^4+4=0$
	$z^4=-4=4e^{i(-\pi+2k\pi)},\ k=0,\pm1,-2$
	$z=\sqrt{2}e^{i\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)},\ k=0,\pm1,-2$
	when $k=0,\ z=\sqrt{2}e^{i\left(\frac{\pi}{4}\right)}=1+i$
	when $k=-1,\ z=\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)}=1-i$
	when $k=1,\ z=\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}=-1+i$
	when $k=-2,\ z=\sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)}=-1-i$
	Alternatively:
	$z^4+4=0$
	$\left(z^2-2i\right)\left(z^2+2i\right)=0$
	$z=\dfrac{\pm\sqrt{-4(-2i)}}{2}=\pm(1+i)$
	$z=\dfrac{\pm\sqrt{-4(2i)}}{2}=\pm(1-i)$
	Alternatively:
	Let $z=x+iy$.
	$(x+iy)^4+4=0$
	$x^4+4ix^3y-6x^2y^2-4ixy^3+y^4+4=0$
	$x^4-6x^2y^2+y^4+4=0$ and $4x^3y-4xy^3=0$
	$4xy\left(x^2-y^2\right)=0$
	$x=\pm y$
	when $x=y$, when $x=-y$,
	$y^4=1$ $y^4=1$
	$y=\pm1$ $y=\pm1$
	$x=\pm1$ $x=\mp1$
	$\therefore z=1+i,\ -1-i,\ 1-i,\ -1+i$

3 **ACJC/2014/II/2**

The complex number z satisfies the relations $\arg(z+3-3\mathrm{i}) = -\dfrac{\pi}{4}$ and $|z-3+3\mathrm{i}| \le b$, where b is a constant and $1 < b < 3$.

(i) Illustrate each of the above relations on a single Argand diagram. [2]

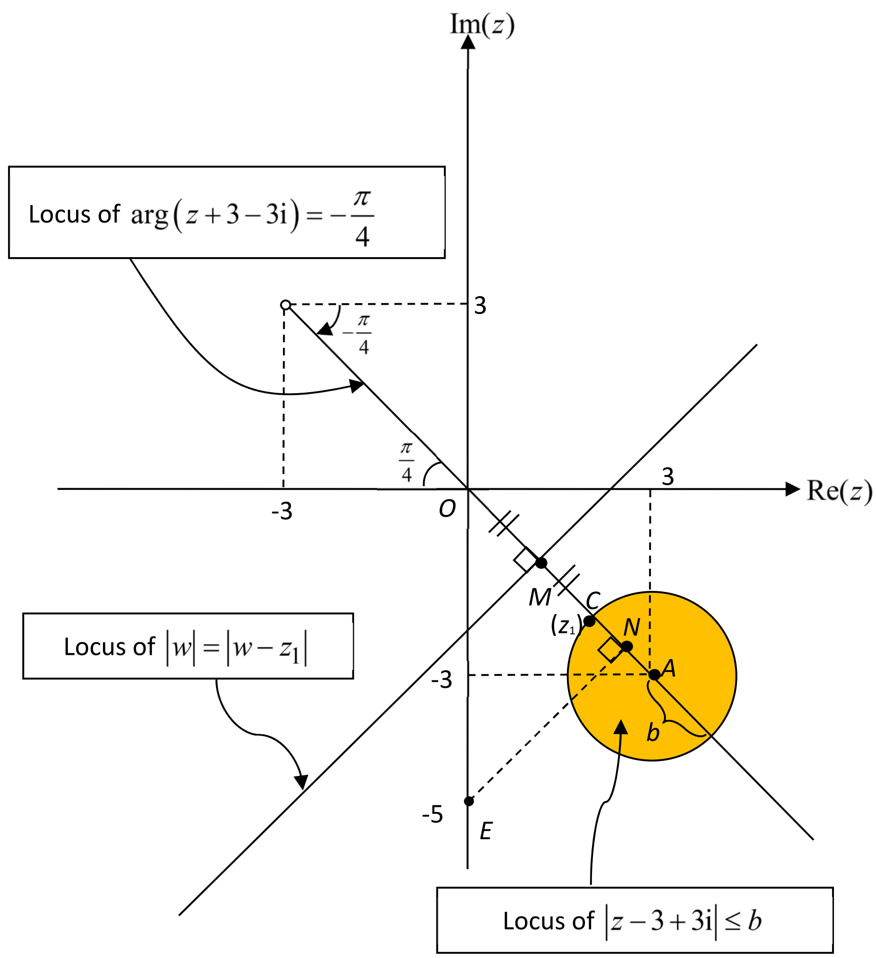
(ii) Find the exact least possible value of $|z+5\mathrm{i}|$. [1]

(iii) Given that the least possible value of $|z|$ is $\sqrt{18}-2$,

(a) find the value of b, [1]

(b) hence find an exact expression for z, in the form $x+\mathrm{i}y$. [2]

(c) State the cartesian equation of the locus of the point representing complex variable w such that $|w| = |w-z_1|$, where z_1 is the complex number found in part **(b)**. [1]

(i)	

(ii)

Let d be the least possible value of $|z+5\mathrm{i}|$.

Using ΔONE ,

$$\sin\frac{\pi}{4}=\frac{d}{5} \quad \Rightarrow d=\frac{5}{\sqrt{2}}=\frac{5\sqrt{2}}{2}$$

(iii) (a)

$$OA=\sqrt{3^2+3^2}=\sqrt{18}$$
$$b=OA-\left(\sqrt{18}-2\right)$$
$$b=2$$

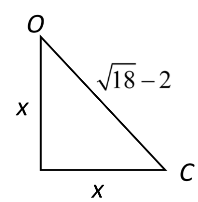
(iii) (b)

Method 1:

$$z=\left(\sqrt{18}-2\right)\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$$
$$=\left(3\sqrt{2}-2\right)\left[\frac{1}{\sqrt{2}}-\mathrm{i}\frac{1}{\sqrt{2}}\right]$$
$$=3-\frac{2}{\sqrt{2}}-\mathrm{i}\frac{\left(3\sqrt{2}-2\right)}{\sqrt{2}}$$
$$=3-\frac{2}{\sqrt{2}}-3\mathrm{i}+\mathrm{i}\frac{2}{\sqrt{2}}$$
$$=\left(3-\frac{2}{\sqrt{2}}\right)+\left(-3+\frac{2}{\sqrt{2}}\right)\mathrm{i}$$
$$=\left(3-\sqrt{2}\right)+\left(-3+\sqrt{2}\right)\mathrm{i}$$

Method 2:

$$x^2+x^2=\left(\sqrt{18}-2\right)^2$$
$$x^2=\frac{\left(\sqrt{18}-2\right)^2}{2}$$
$$x=\sqrt{\frac{\left(\sqrt{18}-2\right)^2}{2}}$$
$$x=\frac{\sqrt{18}-2}{\sqrt{2}}$$
$$x=3-\sqrt{2}$$
$$z=\left(3-\sqrt{2}\right)+\left(-3+\sqrt{2}\right)\mathrm{i}$$



(iii) (c)

The locus of $|w|=|w-z_1|$ is a perpendicular bisector of the line segment joining the points O and C.

Since gradient of line segment OC is -1,

Hence, gradient of perpendicular bisector = 1.

Let point M be the midpoint of OC.

	Hence, point M is $\left(\frac{\left(3-\sqrt{2}\right)}{2}, \frac{\left(-3+\sqrt{2}\right)}{2}\right)$. Equation of locus: $y-\frac{\left(-3+\sqrt{2}\right)}{2}=(1)\left(x-\frac{\left(3-\sqrt{2}\right)}{2}\right)$ $y=x-\frac{\left(3-\sqrt{2}\right)}{2}+\frac{\left(-3+\sqrt{2}\right)}{2}$ $y=x-3+\sqrt{2}$

4 **HCI/2014/II/3**

(a) **(i)** Find the fifth roots of –32, expressing the roots in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [2]

(ii) The roots representing z_1 and z_2 are such that $0<\arg(z_1)<\arg(z_2)<\pi$.

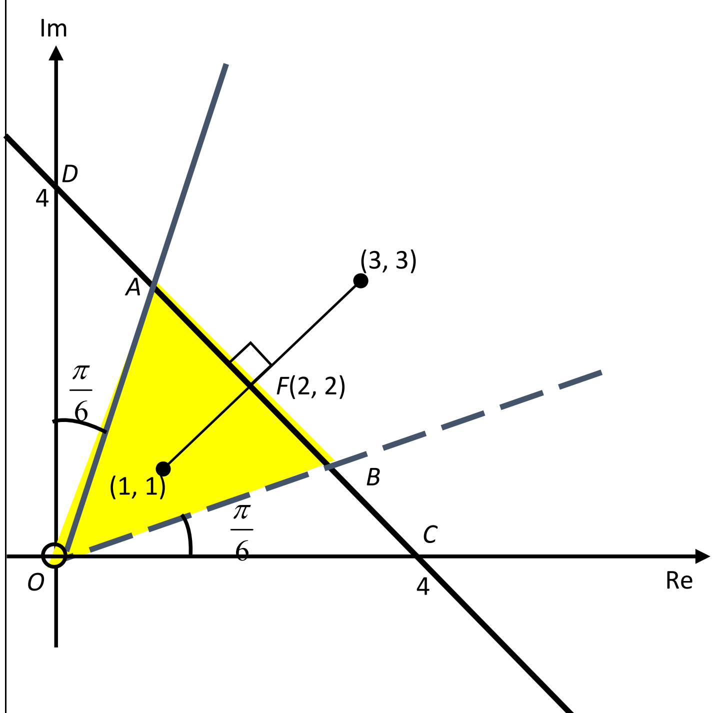
State the complex number w in the form $re^{i\theta}$ where $z_2 = wz_1$. [1]

(b) The complex number z satisfies $|z-3-3i|\ge|z-1-i|$ and $\frac{\pi}{6}<\arg(z)\le\frac{\pi}{3}$.

(i) On an Argand diagram, sketch the region in which the point representing z can lie. [3]

(ii) Find the area of the region in part **(b)(i)**. [3]

(iii) Find the range of values of $\arg(z-5+i)$. [2]

(a)(i)	$z^5=-32$ $z^5=32e^{i(\pi)}$ $z^5=2^5e^{i(\pi+2k\pi)}$ Therefore $z=2e^{i\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right)}$, $k=0,\pm1,\pm2$.
(a)(ii)	Since the two complex numbers are in the 1st and 2nd quadrants corresponding to $k=0$ and $k=1$, thus $w=e^{i\left(\frac{2\pi}{5}\right)}$.
(b)(i)	

(b) (ii)	<u>Method 1 (Using ½ x base x height)</u> $OF=\sqrt{2^2+2^2}=\sqrt{8}$ $\tan\angle FOA=\tan\left(\frac{\pi}{12}\right)=\frac{FA}{OF}$ $\Rightarrow FA=\sqrt{8}\tan\left(\frac{\pi}{12}\right)$ Area of triangle $=\frac{1}{2}\times AB\times OF$ $=\frac{1}{2}\left(\sqrt{8}\right)2\sqrt{8}\tan\left(\frac{\pi}{12}\right)=8\tan\left(\frac{\pi}{12}\right)=2.14$ (to 3 s.f.) <u>Method 2 (Using sum of areas of triangles)</u> Let $OA=OB=x$ Area $\Delta OAB+$ area $\Delta OBC+$ area $\Delta OAD=$ area ΔOCD $\Rightarrow \frac{1}{2}x^2\sin\frac{\pi}{6}+2\times\left[\frac{1}{2}(x)(4)\sin\frac{\pi}{6}\right]=\frac{1}{2}(4)^2$ $\Rightarrow x^2+8x-32=0$ $\Rightarrow x=-4\pm4\sqrt{3}$ $\Rightarrow x=-4+4\sqrt{3}$ since $x>0$ Thus, area of shaded region $=\frac{1}{2}x^2\sin\frac{\pi}{6}=4\left(\sqrt{3}-1\right)^2=16-8\sqrt{3}$
(b) (iii)	Note that the point (5, –1) lies on the perpendicular bisector. Therefore $\frac{3\pi}{4}\le\arg\left(z-5+\mathrm{i}\right)<\pi-\tan^{-1}\left(\frac{1}{5}\right)$. If correct to 3 s.f., answer is $2.36\le\arg\left(z-5+\mathrm{i}\right)<2.94$.

5 **RI/2014/II/4**

Do not use a calculator in answering this question.

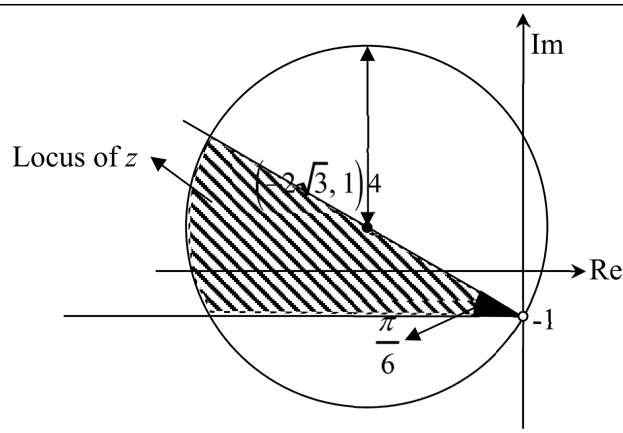
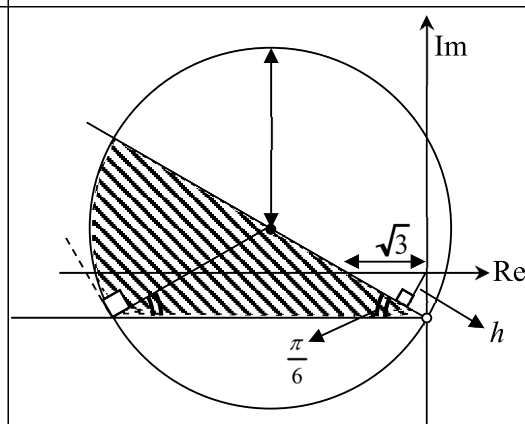
The complex number z satisfies both the relations $\left|z+2\sqrt{3}-\mathrm{i}\right|\le 4$ and $\frac{5}{6}\pi \le \arg\left(z+\mathrm{i}\right)\le \pi$.

(i) On an Argand diagram, shade the region in which the point representing z can lie. [4]

(ii) Find the least possible value of $|z|$. [2]

(iii) State the cartesian form of the complex number z when $|z+\mathrm{i}|$ is greatest. [1]

(iv) Find the range of values of $\arg\left(z+4\sqrt{3}+\mathrm{i}\right)$. [2]

(i)	
(ii)	 The least possible $\lvert z\rvert$ is given by the perpendicular distance from the origin to the half-line $\arg\left(z+\mathrm{i}\right)=\frac{5}{6}\pi$, denoted by h as shown in the diagram. $h=\sqrt{3}\sin\frac{\pi}{6}$ or $1.\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$
(iii)	$-4\sqrt{3}+3\mathrm{i}$
(iv)	$0\le\arg\left(z+4\sqrt{3}+\mathrm{i}\right)<\frac{1}{6}\pi+\frac{1}{2}\pi \quad\Rightarrow\quad 0\le\arg\left(z+4\sqrt{3}+\mathrm{i}\right)<\frac{2}{3}\pi$

6 RVHS/2014/II/2

(i) Solve the equation $z^6+64=0$, giving the roots in the form $re^{i\alpha}$, where $r>0$ and $-\pi<\alpha\le\pi$. [3]

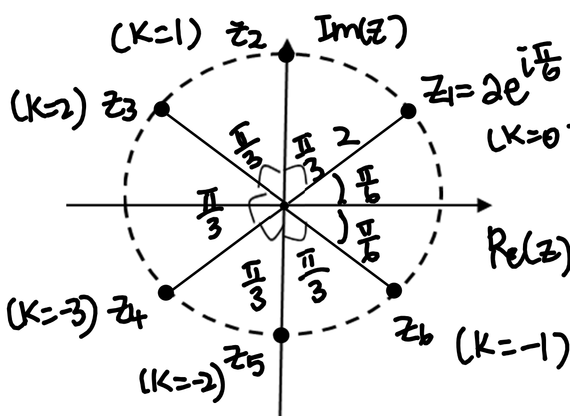
(ii) Show the roots on an Argand diagram. [2]

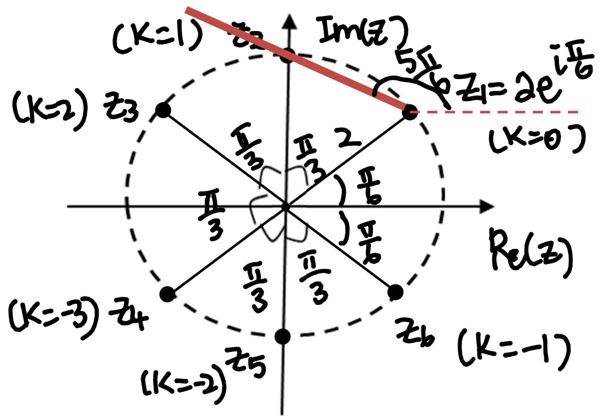
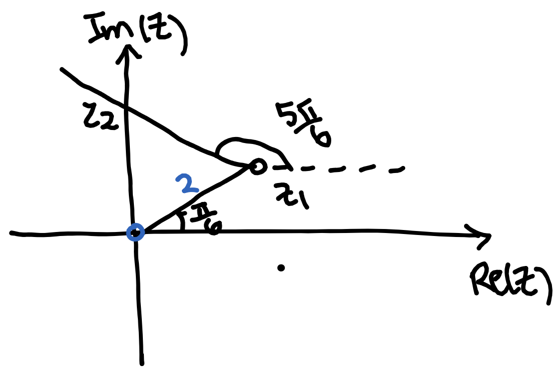
The roots denoted by z_1 and z_2 are such that $0<\arg(z_1)<\arg(z_2)\le\frac{\pi}{2}$. The complex numbers z_1 and z_2 are represented by the points Z_1 and Z_2 in the Argand diagram respectively.

(iii) Explain why the locus of all points z such that $|z|=|z-z_1|$ passes through the point Z_2. [1]

(iv) The complex number w satisfies the relation $\arg(w-z_1)=\arg(z_2-z_1)$. Sketch the locus of the points which represent w in the same Argand diagram. [2]

(v) Find the range of values of $\arg(w)$. [3]

i	$z^6=-64=64e^{i\pi}$ $z^6=64e^{i(\pi+2k\pi)}$ $z=2e^{i\left(\frac{\pi}{6}+\frac{k\pi}{3}\right)}, k=0,\pm1,\pm2,-3$
ii	 Note that the 6 points lie on the circumference of a circle with centre O and radius 2. The points are equally spaced out separated by an angle of $\frac{\pi}{3}$ radians. Z_1 and Z_6, Z_2 and Z_5 & Z_3 and Z_4 represent complex numbers which are conjugates to each other. Thus each conjugate pair lie on the same vertical line.
iii	Observe that OZ_1Z_2 is an equilateral triangle of side 2. Since $z=z_2$ satisfy the equation $\|z\|=\|z-z_1\|$ with $\|z_2\|=\|z_2-z_1\|=2$, locus of all points z such that $\|z\|=\|z-z_1\|$ passes through the point Z_2.

<table>
<tr><td>iv</td><td>
</td></tr>
<tr><td>v</td><td>

$\therefore \arg(z_1) < \arg(w) < \arg(z_2 - z_1) = \frac{5\pi}{6}$
$\therefore \frac{\pi}{6} < \arg(w) < \frac{5\pi}{6}$</td></tr>
</table>

7 CJC/2017/FM/Promo/7

It is given that the complex number $z = 1 + \cos\theta + \mathrm{i}\sin\theta$, where $-\pi < \theta \le \pi$.

(i) By considering appropriate trigonometric identities, or otherwise, show that the argument of z is $\frac{\theta}{2}$ and find the modulus of z in terms of θ. [3]

(ii) Hence, find the real and imaginary parts of $(1 + \cos\theta + \mathrm{i}\sin\theta)^n$, where $n \in \mathbb{Z}^+$. [3]

(iii) By considering the binomial expansion of $\left[1 + (\cos\theta + \mathrm{i}\sin\theta)\right]^n$, show that

$$1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos(2\theta) + \cdots + \binom{n}{n}\cos(n\theta) = \left[2\cos\left(\frac{\theta}{2}\right)\right]^n \cos\left(\frac{n\theta}{2}\right),$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. [3]

(i)	<u>Method 1:</u> $z = 1 + \cos\theta + \mathrm{i}\sin\theta$ $= 2\cos^2\frac{\theta}{2} + \mathrm{i}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ $= 2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2} + \mathrm{i}\sin\frac{\theta}{2}\right]$ Therefore $\lvert z\rvert = 2\cos\frac{\theta}{2}$ and $\arg(z) = \frac{\theta}{2}$.
	<u>Method 2</u> $z = 1 + \cos\theta + \mathrm{i}\sin\theta$ $= 1 + e^{\mathrm{i}\theta}$ $= \mathrm{e}^0 + \mathrm{e}^{\mathrm{i}\theta}$ $= \mathrm{e}^{\mathrm{i}\left(\frac{\theta}{2} - \frac{\theta}{2}\right)} + \mathrm{e}^{\mathrm{i}\left(\frac{\theta}{2} + \frac{\theta}{2}\right)}$ $= \mathrm{e}^{\mathrm{i}\left(\frac{\theta}{2}\right)}(\mathrm{e}^{\mathrm{i}\left(-\frac{\theta}{2}\right)} + \mathrm{e}^{\mathrm{i}\left(\frac{\theta}{2}\right)})$ $= \mathrm{e}^{\mathrm{i}\left(\frac{\theta}{2}\right)}\left(2\cos\frac{\theta}{2}\right)$ Therefore $\lvert z\rvert = 2\cos\frac{\theta}{2}$ and $\arg(z) = \frac{\theta}{2}$.

(ii)	<u>Method 1</u> $\left(1+\cos\theta+\mathrm{i}\sin\theta\right)^n=\left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)\right]^n$ $=\left(2\cos\frac{\theta}{2}\right)^n\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)^n$ $=\left(2\cos\frac{\theta}{2}\right)^n\left(\cos\frac{n\theta}{2}+\mathrm{i}\sin\frac{n\theta}{2}\right)$, by DMT $=\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}+\mathrm{i}\left(2\cos\frac{\theta}{2}\right)^n\sin\frac{n\theta}{2}$ $\mathrm{Re}\left(1+\cos\theta+\mathrm{i}\sin\theta\right)^n=\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$ $\mathrm{Im}\left(1+\cos\theta+\mathrm{i}\sin\theta\right)^n=\left(2\cos\frac{\theta}{2}\right)^n\sin\frac{n\theta}{2}$ <u>Method 2</u> $\left(1+\cos\theta+\mathrm{i}\sin\theta\right)^n=\left[2\cos\frac{\theta}{2}e^{\mathrm{i}\frac{\theta}{2}}\right]^n$ $=\left(2\cos\frac{\theta}{2}\right)^n e^{\mathrm{i}\frac{n\theta}{2}}$ $=\left(2\cos\frac{\theta}{2}\right)^n\left(\cos\frac{n\theta}{2}+\mathrm{i}\sin\frac{n\theta}{2}\right)$ $=\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}+\mathrm{i}\left(2\cos\frac{\theta}{2}\right)^n\sin\frac{n\theta}{2}$ $\mathrm{Re}\left(1+\cos\theta+\mathrm{i}\sin\theta\right)^n=\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$ $\mathrm{Im}\left(1+\cos\theta+\mathrm{i}\sin\theta\right)^n=\left(2\cos\frac{\theta}{2}\right)^n\sin\frac{n\theta}{2}$
(iii)	$\left[1+\left(\cos\theta+\mathrm{i}\sin\theta\right)\right]^n$ $=1+\binom{n}{1}\left(\cos\theta+\mathrm{i}\sin\theta\right)+\binom{n}{2}\left(\cos\theta+\mathrm{i}\sin\theta\right)^2+\cdots+\left(\cos\theta+\mathrm{i}\sin\theta\right)^n$ $=1+\binom{n}{1}\left(\cos\theta+\mathrm{i}\sin\theta\right)+\binom{n}{2}\left(\cos(2\theta)+\mathrm{i}\sin(2\theta)\right)+\cdots+\left(\cos(n\theta)+\mathrm{i}\sin(n\theta)\right)$, by DMT $=\left(1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos(2\theta)+\cdots+\cos(n\theta)\right)+\mathrm{i}\left(\binom{n}{1}\sin\theta+\binom{n}{2}\sin(2\theta)+\cdots+\sin(n\theta)\right)$ From (ii) $\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}+\mathrm{i}\left(2\cos\frac{\theta}{2}\right)^n\sin\frac{n\theta}{2}$ Comparing real and imaginary, $1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos(2\theta)+\cdots+\cos(n\theta)=\left[2\cos\left(\frac{\theta}{2}\right)\right]^n\cos\left(\frac{n\theta}{2}\right)$

8 RI/2017/FM/Promo/6

(a) On the same Argand diagram, sketch the loci of points given by each of the following equations:

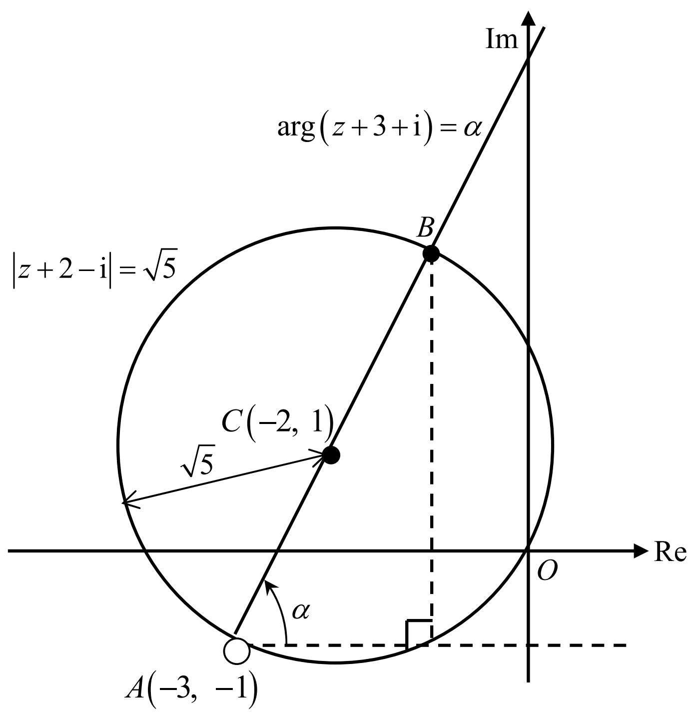
$L_1: \; |z+2-\mathrm{i}| = \sqrt{5}$,

$L_2: \; \arg(z+3+\mathrm{i}) = \alpha$, where $\alpha = \tan^{-1} 2$.

Find, in the form $x+\mathrm{i}y$, the complex number which represents the point in the Argand diagram which is on both L_1 and L_2, giving the exact values of x and y.

[5]

Solution



Method 1:

$x-$coordinate of $B = -3 + AB\cos\alpha = -3 + 2\sqrt{5}\left(\frac{1}{\sqrt{5}}\right) = -1$

$y-$coordinate of $B = -1 + AB\sin\alpha = -1 + 2\sqrt{5}\left(\frac{2}{\sqrt{5}}\right) = 3$

Therefore the required complex number is $-1+3\mathrm{i}$.

Method 2:

Cartesian equation of L_1: $(x+2)^2 + (y-1)^2 = 5$.

Cartesian equation of L_2: $y = 2x+5, x > -3$.

Solving, we have $x = -1$ and $y = 3$.

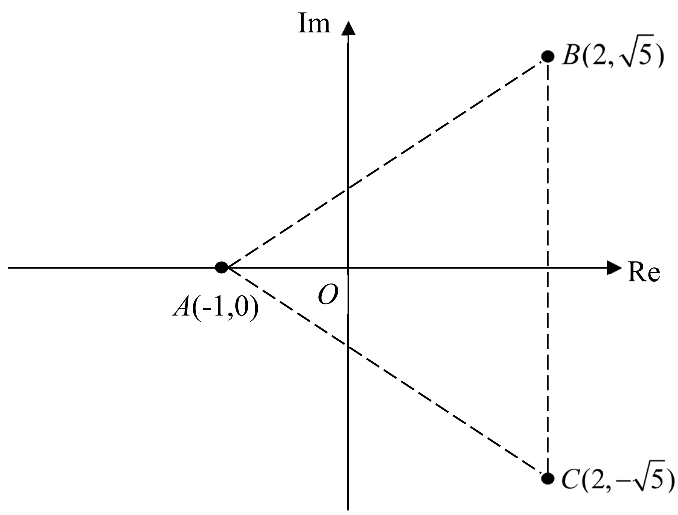
Therefore the required complex number is $-1+3\mathrm{i}$.

9 SAJC/2017/FM/Promo/4

(i) Solve the equation $z^3-3z^2+5z=-9$, giving your answers in exact form. [4]

(ii) Show the 3 roots of the equation in **(i)** on an Argand diagram and find the area of the triangle formed by joining the 3 points. [3]

(iii) Write down two roots of the equation $z^{300}-3z^{200}+5z^{100}=-9$ in polar form. [1]

(i)	$z^3-3z^2+5z=-9$ By observation, $z=-1$ is a root $\Rightarrow$ $z+1$ is a factor. $z^3-3z^2+5z+9=0$ By long division or comparing coefficients, $z^3-3z^2+5z+9=(z+1)(z^2-4z+9)=0$ Solving $z^2-4z+9=0$, $z=2\pm\sqrt{5}\mathrm{i}$ The roots are $-1,\ 2\pm\sqrt{5}\mathrm{i}$.
(ii)	 Area of triangle = $\frac{1}{2}$ base × height $=\frac{1}{2}\times 2\sqrt{5}\times 3$ $=3\sqrt{5}$ units^2
(iii)	$z^{100}=-1=1\mathrm{e}^{\mathrm{i}(\pi+2k\pi)}$ Two possible roots are $\cos\frac{\pi}{100}+\mathrm{i}\sin\frac{\pi}{100}$ and $\cos\frac{-\pi}{100}+\mathrm{i}\sin\frac{-\pi}{100}$.

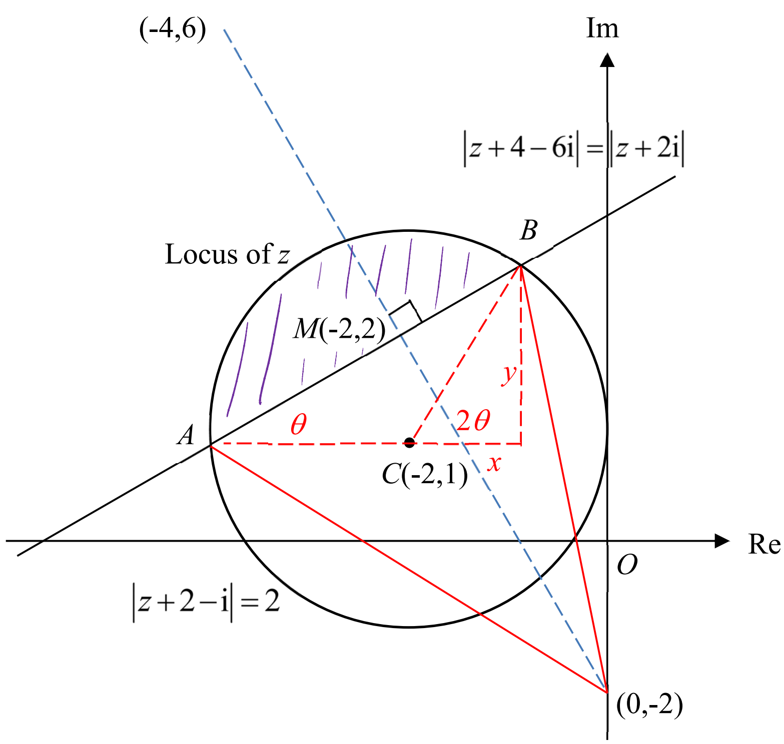
10 SAJC/2017/FM/Promo/9

(ai) Sketch, on a single Argand diagram, the locus of z which satisfies both $|z+2-\text{i}|\le 2$ and $|z+4-6\text{i}|\le|z+2\text{i}|$. [3]

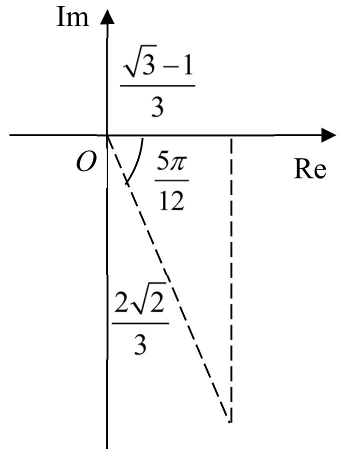
(ii) It is given that $-\pi<\arg(z+2\text{i})\le\pi$. Find the complex numbers v and w that give the greatest and least values of $\arg(z+2\text{i})$ respectively. [4]

(bi) The complex number w has modulus 6 and argument $-\frac{5\pi}{6}$, and the complex number z has modulus $4\sqrt{2}$ and argument $\frac{3\pi}{4}$. Find the modulus and argument of $\frac{z}{w}$, giving each answer exactly. [3]

(ii) Given that the Cartesian forms of w and z are $-3\sqrt{3}-3\text{i}$ and $-4+4\text{i}$ respectively, find the exact real part of $\frac{z}{w}$ and deduce that $\cos\frac{5\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}$. [4]

(a)(i)	
(a)(ii)	The points that give the greatest and least arguments are A and B respectively. <u>Either</u> By observation, as AB has gradient $\frac{1}{2}$, A is the point $(-4,1)$.

	$\frac{y}{2} = \sin 2\theta$ $= 2\sin\theta\cos\theta$ $= 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{4}{5}$ $y = \frac{8}{5}$ By Pythagoras' Theorem, $x^2 + \left(\frac{8}{5}\right)^2 = 2^2 \Rightarrow x = \frac{6}{5}$ $B = (-2 + \frac{6}{5}, 1 + \frac{8}{5}) = (-\frac{4}{5}, \frac{13}{5})$ $\therefore v = -4 + \mathrm{i},\ w = -\frac{4}{5} + \frac{13}{5}\mathrm{i}$ <u>Or</u> $(x+2)^2 + (y-1)^2 = 4 \quad ---(1)$ $y - 2 = \frac{1}{2}(x+2)$ $y = \frac{1}{2}x + 3 \quad ---(2)$ Sub (2) into (1): $(x+2)^2 + (\frac{1}{2}x + 2)^2 = 4$ $\frac{5}{4}x^2 + 6x + 4 = 0$ $5x^2 + 24x + 16 = 0$ $x = -4 \quad \text{or} \quad x = -\frac{4}{5}$ Sub into (2): $y = 1 \quad \text{or} \quad y = \frac{13}{5}$ $\therefore v = -4 + \mathrm{i},\ w = -\frac{4}{5} + \frac{13}{5}\mathrm{i}$
(b)(i)	$\left\|\frac{z}{w}\right\| = \frac{\|z\|}{\|w\|} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$ $\arg\left(\frac{z}{w}\right) = \arg z - \arg w = \left(\frac{3}{4}\pi\right) - \left(-\frac{5\pi}{6}\right) = \frac{19\pi}{12} \equiv -\frac{5\pi}{12}$

(b)(ii)	$\frac{z}{w}=\frac{-4(1-\mathrm{i})}{-3(\sqrt{3}+\mathrm{i})}$ $=\frac{4(1-\mathrm{i})}{3(\sqrt{3}+\mathrm{i})}\times\frac{(\sqrt{3}-\mathrm{i})}{(\sqrt{3}-\mathrm{i})}$ $=\frac{4(\sqrt{3}-1-(\sqrt{3}+1)\mathrm{i})}{3(4)}$ $\mathrm{Re}\left(\frac{z}{w}\right)=\frac{\sqrt{3}-1}{3}$ From the diagram, $\cos\left(\frac{5\pi}{12}\right)=\frac{\sqrt{3}-1}{3}\div\frac{2\sqrt{2}}{3}$ $=\frac{\sqrt{3}-1}{2\sqrt{2}}$ 

11 TJC/2017/FM/Promo/6

(i) Show that if $z = \mathrm{e}^{\mathrm{i}\theta}$, then

$$z^k - \frac{1}{z^k} = 2\mathrm{i}\sin k\theta,$$

where k is a positive integer. [1]

(ii) Show that $\sin^5\theta$ can be expressed in the form

$$A\sin\theta + B\sin 3\theta + C\sin 5\theta,$$

where the values of A, B and C are to be determined. [4]

(iii) Find the particular solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\mathrm{e}^x \operatorname{cosec} y\right)^5$, given that $y = 0$ when $x = 0$. [3]

(i) $LHS = z^k - \frac{1}{z^k} = \left(\cos\theta + \mathrm{i}\sin\theta\right)^k - \left(\cos\theta + \mathrm{i}\sin\theta\right)^{-k}$

$= \left(\cos k\theta + \mathrm{i}\sin k\theta\right) - \left(\cos k\theta - \mathrm{i}\sin k\theta\right)$

$= 2\mathrm{i}\sin k\theta$

(ii) When $k = 1$, $z - \frac{1}{z} = 2\mathrm{i}\sin\theta$

$$\sin^5\theta = \left(\frac{1}{2i}\left(z - \frac{1}{z}\right)\right)^5 = \left(\frac{1}{32i}\left(z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \left(\frac{1}{z^5}\right)\right)\right)$$

$$= \frac{1}{32i}\left(z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)\right)$$

$$= \frac{1}{32\mathrm{i}}\left(2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin 5\theta\right)$$

$$= \frac{5}{8}\sin\theta - \frac{5}{16}\sin 3\theta + \frac{1}{16}\sin 5\theta$$

(iii) $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\mathrm{e}^x \operatorname{cosec} y\right)^5$

$$\int \sin^5 y \,\mathrm{d}y = \int \mathrm{e}^{5x}\,\mathrm{d}x$$

$$\int \frac{5}{8}\sin y - \frac{5}{16}\sin 3y + \frac{1}{16}\sin 5y \,\mathrm{d}y = \int \mathrm{e}^{5x}\,\mathrm{d}x$$

$$-\frac{5}{8}\cos y + \frac{5}{48}\cos 3y - \frac{1}{80}\cos 5y = \frac{\mathrm{e}^{5x}}{5} + c$$

When $x = 0$, $y = 0$,

$$-\frac{5}{8} + \frac{5}{48} - \frac{1}{80} = \frac{1}{5} + c \quad \Rightarrow c = -\frac{11}{15}$$

Therefore the particular solution is $-\frac{5}{8}\cos y + \frac{5}{48}\cos 3y - \frac{1}{80}\cos 5y = \frac{\mathrm{e}^{5x}}{5} - \frac{11}{15}$

12	**ACJC/2018/FM/Prelim/I/Q6** Let $$C = 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta + \ldots + \cos 20\theta,$$ $$S = \sin 2\theta + \sin 4\theta + \sin 6\theta + \ldots + \sin 20\theta.$$ **(i)** Show that $C + \mathrm{i}S = \dfrac{\sin 11\theta}{\sin\theta}\mathrm{e}^{\mathrm{i}10\theta}$ for all $\theta \neq n\pi,\ n \in \mathbb{Z}$. [3] **(ii)** Hence, show that $\cos 2\theta + \cos 4\theta + \cos 6\theta + \ldots + \cos 20\theta = \dfrac{\sin 21\theta}{2\sin\theta} - \dfrac{1}{2}$. [3] **(iii)** Deduce that $\displaystyle\sum_{r=1}^{10} r\sin 2r\theta = \frac{\sin 21\theta\cos\theta}{4\sin^2\theta} - \frac{21\cos 21\theta}{4\sin\theta}$. [3]

(i)

$$C + \mathrm{i}S = 1 + \mathrm{e}^{2\mathrm{i}\theta} + \mathrm{e}^{4\mathrm{i}\theta} + \mathrm{e}^{6\mathrm{i}\theta} + \ldots + \mathrm{e}^{20\mathrm{i}\theta}$$

$$= \frac{1\left(1 - \left(\mathrm{e}^{2\mathrm{i}\theta}\right)^{11}\right)}{1 - \mathrm{e}^{2\mathrm{i}\theta}} = \frac{1 - \mathrm{e}^{22\mathrm{i}\theta}}{1 - \mathrm{e}^{2\mathrm{i}\theta}} = \frac{\mathrm{e}^{11\mathrm{i}\theta}\left(\mathrm{e}^{-11\mathrm{i}\theta} - \mathrm{e}^{11\mathrm{i}\theta}\right)}{\mathrm{e}^{\mathrm{i}\theta}\left(\mathrm{e}^{-\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta}\right)}$$

$$= \frac{\mathrm{e}^{10\mathrm{i}\theta}\left[\cos(-11\theta) + \mathrm{i}\sin(-11\theta) - \cos 11\theta - \mathrm{i}\sin 11\theta\right]}{\cos(-\theta) + \mathrm{i}\sin(-\theta) - \cos\theta - \mathrm{i}\sin\theta}$$

$$= \frac{\mathrm{e}^{10\mathrm{i}\theta}\left[-2\mathrm{i}\sin 11\theta\right]}{-2\mathrm{i}\sin\theta}$$

$$= \frac{\sin 11\theta}{\sin\theta}\mathrm{e}^{10\mathrm{i}\theta}$$

(ii)

$$\cos 2\theta + \cos 4\theta + \cos 6\theta + \ldots + \cos 20\theta$$

$$= \mathrm{Re}\left(\frac{\sin 11\theta}{\sin\theta}\mathrm{e}^{10\mathrm{i}\theta}\right) - 1 = \mathrm{Re}\left(\frac{(\cos 10\theta + \mathrm{i}\sin 10\theta)[\sin 11\theta]}{\sin\theta}\right) - 1$$

$$= \frac{\sin 11\theta\cos 10\theta}{\sin\theta} - 1$$

$$= \frac{\sin 21\theta + \sin\theta}{2\sin\theta} - 1 = \frac{\sin 21\theta}{2\sin\theta} - \frac{1}{2}$$

(iii)

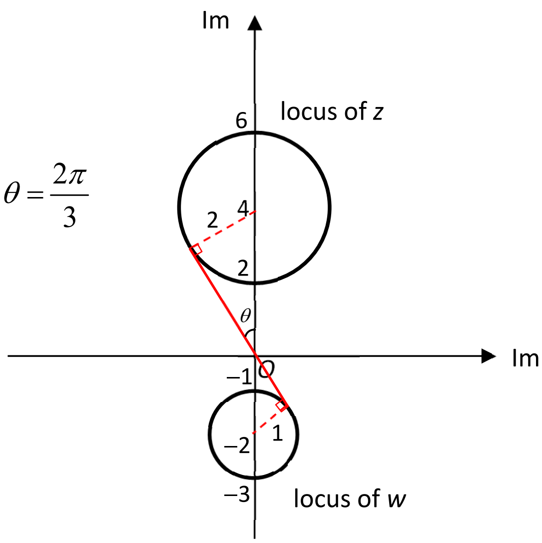
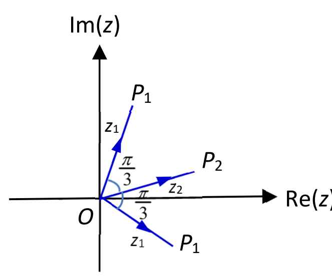
Differentiate (ii) w.r.t θ,

$$-2\sin 2\theta - 4\sin 4\theta - 6\sin 6\theta - \ldots - 20\sin 20\theta$$

$$= \frac{21\sin\theta\cos 21\theta - \cos\theta\sin 21\theta}{2\sin^2\theta}$$

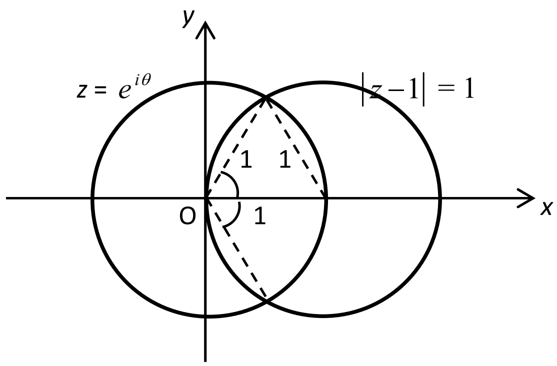
$$-2\sum_{r=1}^{10} r\sin 2r\theta = \frac{21\cos 21\theta}{2\sin\theta} - \frac{\cos\theta\sin 21\theta}{2\sin^2\theta}$$

$$\sum_{r=1}^{10} r\sin 2r\theta = \frac{\sin 21\theta\cos\theta}{4\sin^2\theta} - \frac{21\cos 21\theta}{4\sin\theta}$$

13	**TJC/2018/FM/Prelim/I/7** **(a)** The complex numbers z and w are such that $$\lvert z-4\mathrm{i}\rvert = 2 \text{ and } \lvert w+2\mathrm{i}\rvert = 1.$$ By considering an Argand diagram, find **(i)** the least value of $\lvert z-w\rvert$, **(ii)** the greatest value of $\arg(z-w)$. [4] **(b)** The points P_1 and P_2 represent the complex numbers z_1 and z_2 respectively in an Argand diagram with origin O. Given that $$z_1^2 - z_1z_2 + z_2^2 = 0,$$ show that $$z_1 = z_2\left(\cos\frac{\pi}{3} \pm \mathrm{i}\sin\frac{\pi}{3}\right).$$ Hence or otherwise, prove that the triangle OP_1P_2 is equilateral. [7]
	(a) Least value of $\lvert z-w\rvert = 3$  $$\sin\theta = \frac{2}{4} \Rightarrow \theta = \frac{\pi}{6}$$ Greatest value of $\arg(z-w) = \frac{\pi}{2} + \theta = \frac{2\pi}{3}$ **(b)** $z_1^2 - z_1z_2 + z_2^2 = 0$ $$\left(z_1 - \frac{1}{2}z_2\right)^2 - \left(\frac{1}{2}z_2\right)^2 + z_2^2 = 0$$ $$\left(z_1 - \frac{1}{2}z_2\right)^2 = -\frac{3}{4}z_2^2$$ $$z_1 - \frac{1}{2}z_2 = \pm\,\mathrm{i}\frac{\sqrt{3}}{2}z_2$$ $$z_1 = z_2\left(\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}\right)$$ $$= z_2\left(\cos\frac{\pi}{3} \pm \mathrm{i}\sin\frac{\pi}{3}\right) \text{ (shown)}$$  Since $\left\lvert\cos\frac{\pi}{3} \pm \mathrm{i}\sin\frac{\pi}{3}\right\rvert = 1$, $\lvert z_1\rvert = \lvert z_2\rvert\left\lvert\cos\frac{\pi}{3} \pm \mathrm{i}\sin\frac{\pi}{3}\right\rvert = \lvert z_2\rvert$ Since $\arg z_1 = \arg z_2 \pm \frac{\pi}{3}$, $\angle P_1OP_2 = \frac{\pi}{3}$ Thus ΔOP_1P_2 is equilateral.

14	**AJC/2018/FM/Prelim/I/5** The complex number z can be expressed as $\mathrm{e}^{\mathrm{i}\theta}$ where $-\pi < \theta \le \pi$. **(a)** Given that z satisfies the equation $\lvert z-1 \rvert = 1$, find the possible values of θ by means of a geometrical argument or otherwise. [3] **(b)** It is given, instead, that z satisfies the equation $\arg\left(1+z+z^2+...+z^{n-1}\right)=0$ for some positive integer $n \ge 2$ and $z \ne 1$. Determine the set of possible values of θ, giving your answer in terms of n. [7]

(a)



From the Argand diagram, by considering the equilateral triangles, the argument of z satisfying both given conditions is either $\dfrac{\pi}{3}$ or $-\dfrac{\pi}{3}$.

Alternative method

$$\lvert e^{i\theta}-1 \rvert = 1 \quad\Rightarrow\quad \lvert(\cos\theta-1)+i\sin\theta\rvert = 1$$
$$\Rightarrow\quad \left\lvert -2\sin^2\left(\tfrac{\theta}{2}\right)+2i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right\rvert = 1$$
$$\Rightarrow\quad \left\lvert 2i\sin\tfrac{\theta}{2}\right\rvert\left\lvert\cos\tfrac{\theta}{2}+i\sin\tfrac{\theta}{2}\right\rvert = 1$$
$$\Rightarrow\quad \left\lvert\left(2\sin\tfrac{\theta}{2}\right)e^{\frac{\theta}{2}i}\right\rvert = 1 \quad\Rightarrow\quad \sin\tfrac{\theta}{2} = \pm\tfrac{1}{2}$$
$$\Rightarrow\quad \tfrac{\theta}{2} = \pm\tfrac{\pi}{6} \quad \text{since } -\tfrac{\pi}{2} < \tfrac{\theta}{2} \le \tfrac{\pi}{2}$$
$$\Rightarrow\quad \theta = \pm\tfrac{\pi}{3}$$

(b) $$\arg\left(1+z+z^2+...+z^{n-1}\right)=0 \quad\Rightarrow\quad \arg\left(\frac{z^n-1}{z-1}\right)=0$$
$$\Rightarrow\quad \arg\left(z^n-1\right)-\arg\left(z-1\right)=0$$
$$\Rightarrow\quad \arg\left(z^n-1\right)=\arg\left(z-1\right)$$

Since the locus of P representing z is a circle of unit radius centred at the origin, each point, other than the point (1,0), corresponds to a unique value of $\arg(z-1)$. Moreover, the point representing z^n also lies on the same circle.

So $\arg\left(z^n-1\right)=\arg\left(z-1\right)$ only if $z^n = z \quad\Rightarrow\quad z^{n-1}=1$ (since $z \ne 0$).

$$z^{n-1}=1 \quad\Rightarrow\quad e^{(n-1)\theta i}=e^{2k\pi i}, \quad k\in\mathbb{Z}$$
$$\Rightarrow\quad (n-1)\theta = 2k\pi \quad\Rightarrow\quad \theta=\frac{2k\pi}{n-1}, \quad k\in\mathbb{Z}$$

$\therefore$ the set of values of θ is $\left\{\theta:\ \theta=\dfrac{2k\pi}{n-1} \text{ where } k\in\mathbb{Z}\setminus\{0\}, \dfrac{1-n}{2}<k\le\dfrac{n-1}{2}\right\}$.

Q	**Answers**
1	$a=-2+3\mathrm{i},\ -1$ or $3-\mathrm{i}$
2	(ii) least $n=4$ (iv) $z=1+i,\ 1-i,\ -1+i,\ -1-i$
3	(ii) $\frac{5\sqrt{2}}{2}$, (iii)(a) $b=2$, (b) $\left(3-\sqrt{2}\right)+\left(-3+\sqrt{2}\right)\mathrm{i}$, (c) $y=x-3+\sqrt{2}$
4	(a)(i) $z=2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right)},\quad k=0,\pm1,\pm2$; (ii) $w=\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{5}\right)}$. (ii) $16-8\sqrt{3}$ or 2.14; (iii) $2.36\le\arg\left(z-5+\mathrm{i}\right)<2.94$
5	(ii) $h=\sqrt{3}\sin\left(\frac{1}{6}\pi\right)=\frac{1}{2}\sqrt{3}$ (iii) $-4\sqrt{3}+3\mathrm{i}$ (iv) $0\le\arg\left(z+4\sqrt{3}+\mathrm{i}\right)<\frac{2}{3}\pi$
6	(i) $z=2\mathrm{e}^{i\left(\frac{\pi}{6}+\frac{k\pi}{3}\right)}, k=0,\pm1,\pm2,-3$ (v) $\frac{1}{6}\pi<\arg\left(w\right)<\frac{5}{6}\pi$
7	(i) $\lvert z\rvert=2\cos\frac{\theta}{2}$ (ii) $\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$, $\left(2\cos\frac{\theta}{2}\right)^n\sin\frac{n\theta}{2}$
8	(a) $-1+3\mathrm{i}$
9	(i) $-1,\ 2\pm\sqrt{5}\mathrm{i}$ (iii) $\cos\frac{\pi}{100}+\mathrm{i}\sin\frac{\pi}{100}$ and $\cos\frac{-\pi}{100}+\mathrm{i}\sin\frac{-\pi}{100}$
10	(a)(ii) $v=-4+\mathrm{i},\ w=-\frac{4}{5}+\frac{13}{5}\mathrm{i}$; b (i) $\frac{2}{3}\sqrt{2}$, $-\frac{5}{12}\pi$, (ii) $\frac{1}{3}(\sqrt{3}-1)$
11	(ii) $A=\frac{5}{8}, B=-\frac{5}{16}, C=\frac{1}{16}$ (iii) $-\frac{5}{8}\cos y+\frac{5}{48}\cos 3y-\frac{1}{80}\cos 5y=\frac{\mathrm{e}^{5x}}{5}-\frac{11}{15}$
13	(a) (i) 3 (ii) $\frac{2\pi}{3}$
14	(a) $\frac{\pi}{3}$ or $-\frac{\pi}{3}$ (b) $\left\{\theta:\ \theta=\frac{2k\pi}{n-1}\ \text{where } k\in\mathbb{Z}\setminus\{0\}, \frac{1-n}{2}<k\le\frac{n-1}{2}\right\}$