Chapter 7 Inequalities

The study of inequalities is important in mathematics. For starters, we often need to compare the magnitude of two mathematical objects. This is essentially with the aid of inequalities. You will find that in university, the use of inequalities in analysis is particularly important. For example, in describing the concept of a "limit" and "convergence".

SYLLABUS INCLUDES

- Equations and inequalities (such as Triangle inequality, AM-GM inequality, Cauchy-Schwarz Inequality)

CONTENT

1 Introduction

1.1 Estimation
1.2 Absolute values, estimating size

2 Triangle Inequality

3 Arithmetic-Mean and Geometric Mean (AM-GM) inequality

4 Cauchy-Schwarz Inequality

5 Inequalities involving other functions

5.1 Inequalities involving monotonic functions
5.2 Inequalities involving convex functions

1 Introduction

In general, inequalities are used to make comparisons, and to make estimations. It is one of two simple tools that form the language of estimations, the other being absolute values for measuring size and distance.

1.1 Estimations

One of the major uses of inequalities, as we mentioned, is to make estimations. These have a language of their own that you need to get used to.

Definition 1.1.1 If c is a number we are estimating, and $m < c < M$, we say that m is a **lower estimate (lower bound)** for c and M is an **upper estimate (upper bound)** for c.

If two sets of upper and lower bounds satisfy the inequalities

$$m < m' < c < M' < M$$

we say that m', M' are **stronger** or **sharper** estimates for c, while m, M are **weaker** estimates.

Estimates are obtained in many ways; the inequality laws that you have been introduced to in H2 Mathematics usually play an important role. Calculus can also be used. Here are some examples to illustrate.

Example 1 Give upper and lower bounds for $\dfrac{1}{a^4+3a^2+1}$.

Solution We are not told what a is, so our bounds will have to be valid no matter what a is, i.e. valid for all a. Let us start with the denominator. Since any square is nonnegative, the polynomial has its smallest value 1 when $a = 0$, and it has no upper bound since a can be arbitrarily large. Hence we have

$$1 \le a^4+3a^2+1$$

which implies

$$0 < \frac{1}{a^4+3a^2+1} \le 1$$

These are the sharpest possible estimates valid for all a: In fact, the upper bound 1 is attained when $a = 0$, and the lower bound is 0 since the fraction can be made arbitrarily close to 0 by taking a sufficiently large.

Exercise 2 Give upper and lower estimates for $\dfrac{1+\sin^2 n}{1+\cos^2 n}$, for $n \ge 0$.

Exercise 3 By interpreting the integral as the area under the graph of $y = \frac{1}{x}$ over the interval [1, 2], estimate $\ln 2 = \int_1^2 \frac{\mathrm{d}x}{x}$.

1.2 Absolute values. Estimating size.

We now look to the other tool of estimation, the absolute value. As with inequalities, we have introduced the basic rules in manipulating them in H2 Mathematics.

Here are two good ways to think about absolute value.

Absolute value measures *magnitude*: $|a|$ is the size of a.

Negative numbers can be big too: a million dollar loss is a big sum for a small business.

Absolute value measures distance: $|a - b|$ is the distance between a and b.

The latter is often used to describe intervals on the real axis. For instance, the interval (2, 4) can be described as $\{x: |x - 3| < 1\}$, i.e. the set of points whose distance from the point 3 is at most 1.

Two important and common use of the absolute value are as follow:

(1) $|a| \geq 0$ for all real values a.

(2) $|a| \leq M \Leftrightarrow -M \leq a \leq M$. The absolute value is also an efficient way to give **symmetric bounds**. In fact, the inequality on the left is often more convenient to use. In the bounds are not symmetric to start with, they can be made so by doing the following:

$$K \leq a \leq L \Rightarrow |a| \leq M, \text{ where } M = \max\left(|K|, |L|\right).$$

In working with absolute values, we will make frequent use of the simple property $|ab| = |a||b|$ as well as the triangle inequality, a law that connects the absolute value with sums.

2 Triangle Inequality

Theorem 2.1 (Triangle Inequality) For real numbers a and b,

$$|a+b| \le |a| + |b| \text{ with equality if and only if } ab \ge 0.$$

Proof

Other forms of Triangle Inequality

Exercise 4 Show that

(a) $\big||a|-|b|\big| \le |a-b|$ (hint: $|a| \le |a-b| + |b|$)

(b) $\big||a|-|b|\big| \le |a+b|$

Exercise 5 If $|a| \ge 3$ and $|b| \le 1$ what is a lower "estimate" for $|a-b|$ and $|a+b|$?

Exercise 6 In Fourier Analysis, one uses trigonometric sums of the form

$$S_n = c_1 \cos t + c_2 \cos 2t + ... + c_n \cos nt$$

If $c_i = \dfrac{1}{2^i}$, give an upper bound for S_n.

The triangle inequality, as the name suggests, has a geometric interpretation. It states that the sum of the lengths of any two sides of a triangle is greater than the length of the remaining side. We have seen the proof of the triangle inequality in the case of real numbers above. If we extend it to a higher dimension, and replace the variables with vectors and the absolute value with the norm (distance function), we have $|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}|+|\mathbf{b}|$, where the length of the third side is represented by $|\mathbf{a}+\mathbf{b}|$. The proof, however, is best done in the following manner:

$|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}|+|\mathbf{b}|$
$\Leftrightarrow |\mathbf{a}+\mathbf{b}|^2 \le \left(|\mathbf{a}|+|\mathbf{b}|\right)^2$
$\Leftrightarrow (\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}+\mathbf{b}) \le |\mathbf{a}|^2+2|\mathbf{a}||\mathbf{b}|+|\mathbf{b}|^2$
$\Leftrightarrow |\mathbf{a}|^2+2\mathbf{a}\bullet\mathbf{b}+|\mathbf{b}|^2 \le |\mathbf{a}|^2+2|\mathbf{a}||\mathbf{b}|+|\mathbf{b}|^2$
$\Leftrightarrow \mathbf{a}\bullet\mathbf{b} \le |\mathbf{a}||\mathbf{b}|$
$\Leftrightarrow |\mathbf{a}||\mathbf{b}|\cos\theta \le |\mathbf{a}||\mathbf{b}|$
which is true since $\cos\theta \le 1$.

There is of course a geometrical proof, which you can read up easily if you are interested. The triangle inequality can be extended by mathematical induction to arbitrary polygonal paths, showing that the total length of such a path is no less than the length of the straight line between its endpoints. Consequently, the length of any polygon side is always less than the sum of the other polygon side lengths, i.e. $|\mathbf{a}_1+\mathbf{a}_2+\ldots+\mathbf{a}_n| \le |\mathbf{a}_1|+|\mathbf{a}_2|+\ldots+|\mathbf{a}_n|$.

Exercise 7

Determine all triangles whose side lengths are in (i) arithmetic progression, (ii) geometric progression.

For the remaining sections of this chapter, we will look at two classical inequalities as well as a couple of other methods that can be used to provide estimates. Let us first begin with the AM-GM inequality.

3 Arithmetic Mean-Geometric Mean (AM-GM) Inequality

The **arithmetic mean** (AM) of *n* real numbers is defined as the average of the *n* numbers. In the context of this section, we will consider the case where the numbers are nonnegative. Thus for $x_1, x_2, ..., x_n \geq 0,$ then $\text{AM} = \dfrac{x_1 + x_2 + ... + x_n}{n}$.

The **geometric mean** (GM) of the *n* real numbers is like the average of the product, thus for $x_1, x_2, ..., x_n \geq 0,$ then $\text{GM} = \left(x_1 x_2 ... x_n\right)^{\frac{1}{n}}$.

In fact, the arithmetic and geometric means are special cases of the *r*-th power means, P_r, which are defined as $P_r = \left(\dfrac{x_1^r + x_2^r + ... + x_n^r}{n}\right)^{\frac{1}{r}}$, for $r \neq 0$. When $r = 0$[1], it is defined as the geometric mean. The arithmetic mean in particular corresponds to the case $r = 1$. We will come back to the discussion of the power means later in the chapter, and will contend ourselves for now with just the case $n = 2$ and $r = 0, 1$, the AM-GM inequality for 2 variables.

Theorem 3.1 Arithmetic-Geometric Mean Inequality (2 variables)

If $a,\ b \geq 0,$ then $\dfrac{a+b}{2} \geq \sqrt{ab}$ with equality when $a = b$.

Proof

It is important to check the condition for when the minimum is attained.

Example 2 Show that for positive *x*, $x + \dfrac{1}{x} \geq 2$.

Solution By the AM-GM inequality, $\dfrac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \dfrac{1}{x}} = 1 \Rightarrow x + \dfrac{1}{x} \geq 2$. Equality holds when $x = \dfrac{1}{x} \Rightarrow x = 1 \text{ (since } x > 0)$. The same result can be obtained via differentiation.

[1] This is a result of $\lim\limits_{r \to 0^+} P_r = \lim\limits_{r \to 0^+} \left(\dfrac{x_1^r + x_2^r + ... + x_n^r}{n}\right)^{\frac{1}{r}} = \left(x_1 x_2 ... x_n\right)^{\frac{1}{n}}$ which can be proven using L'Hopital's Rule.

Exercise 8 Explain why $\sqrt{x^2+2}+\dfrac{1}{\sqrt{x^2+2}}\geq 2$ is not the best lower bound.

Provide the correct minimum value and when it is attained.

As mentioned earlier, we can generalize the AM-GM inequality to more than 2 terms. Let us show that for 3 and 4 variables.

Example 3

(a) If $a,\ b, c\geq 0,$ then $\dfrac{a+b+c}{3}\geq \sqrt[3]{abc}$ with equality when $a=b=c$.

(b) If $a,\ b, c, d\ \geq 0,$ then $\dfrac{a+b+c+d}{4}\geq \sqrt[4]{abcd}$ with equality when $a=b=c=d$.

Solution Interestingly we prove (b) first !

$\dfrac{a+b+c+d}{4}\geq \dfrac{\sqrt{ab}+\sqrt{cd}}{2}\geq \sqrt{\sqrt{ab}\sqrt{cd}}=\sqrt[4]{abcd}$. (show equality yourself)

Now we use (b) to show (a). Let $d=\sqrt[3]{abc}$. Then

$\dfrac{a+b+c+\sqrt[3]{abc}}{4}\geq \sqrt[4]{abc\sqrt[3]{abc}}=\sqrt[4]{(abc)^{\frac{4}{3}}}=\sqrt[3]{abc}$

$\Rightarrow \dfrac{a+b+c}{3}\geq \sqrt[3]{abc}$ (show the condition for equality yourself)

The complete proof via induction for the AM-GM inequality with *n* variables will be done in the tutorial.

Example 4 Let $a_0>a_1>...>a_n$ be real numbers. By rewriting a_0-a_n as $(a_0-a_1)+(a_1-a_2)+...+(a_{n-1}-a_n)$, prove that

$$a_0+\frac{1}{a_0-a_1}+\frac{1}{a_1-a_2}+...+\frac{1}{a_{n-1}-a_n}\geq a_n+2n.$$

Solution Moving the term a_n to the LHS, so that the constant $2n$ is isolated on the right, and using the hint suggested, it leads us to think of using **the** AM-GM **inequality with a suitable pairing of terms** .

$$a_0-a_n+\frac{1}{a_0-a_1}+\frac{1}{a_1-a_2}+...+\frac{1}{a_{n-1}-a_n}$$

$$=\left(a_0-a_1+\frac{1}{a_0-a_1}\right)+\left(a_1-a_2+\frac{1}{a_1-a_2}\right)+...+\left(a_{n-1}-a_n+\frac{1}{a_{n-1}-a_n}\right)$$

$$\geq 2n\sqrt[2n]{1}=2n$$

Exercise 9 Let $a_1, a_2, ..., a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Show that

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n$$

Exercise 10 Let $n \geq 3$ be a natural number and let $a_2, ..., a_n$ be positive real numbers such that $a_2 \cdots a_n = 1$. By suitably writing $1 = \underbrace{\frac{1}{k-1} + \frac{1}{k-1} + ... + \frac{1}{k-1}}_{k-1 \text{ copies}}$, show that

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

4 Cauchy-Schwarz Inequality

In H2 Mathematics, you have been introduced to the "dot product" in the topic of vectors.

$$\left|\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \bullet \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right| = \left|\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right| \left|\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right| \cos\theta \leq \left|\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right| \left|\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right| \Leftrightarrow |x_1 x_2 + y_1 y_2| \leq \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}.$$

Equality is attained when $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$. (What does this mean?)

We will see that this is a general property of numbers x_1, x_2, y_1, y_2. Do also note the difference between the restriction on nonnegative reals for the AM-GM inequality and the absence of the restriction for the Cauchy-Schwarz inequality.

Exercise 11 (Proof of Cauchy-Schwarz Inequality)

Consider $\sum_{i=1}^{n}\left(a_i + tb_i\right)^2$ as a quadratic in *t*.

Since $\sum_{i=1}^{n}\left(a_i + tb_i\right)^2 \geq 0$ we can obtain a condition for the discriminant of the quadratic equation in *t*. Find this condition.

Hence, show that $\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^{\,2}}\sqrt{\sum_{i=1}^{n} b_i^{\,2}}$

When does equality hold ?

Hence show that $ab + bc + ca \leq a^2 + b^2 + c^2$ for any real numbers *a*, *b* and *c*.

Example 5 Minimize $\frac{1}{x^2} + \frac{9}{y^2}$ for points on the circle $x^2 + y^2 = 4$.

Solution The idea to use Cauchy-Schwarz is that we can clear the denominators using the constraint given.

$$4 = \left|\begin{pmatrix} \frac{1}{x} \\ \frac{3}{y} \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}\right| \leq \left|\begin{pmatrix} \frac{1}{x} \\ \frac{3}{y} \end{pmatrix}\right| \left|\begin{pmatrix} x \\ y \end{pmatrix}\right| = \sqrt{\frac{1}{x^2} + \frac{9}{y^2}}\sqrt{x^2 + y^2} = 2\sqrt{\frac{1}{x^2} + \frac{9}{y^2}}$$

Hence we have $\frac{1}{x^2} + \frac{9}{y^2} \geq 4$. To show that the minimum is 4, we need to show that the minimum is attained. The equality condition of Cauchy-Schwarz tells us that is only when $\begin{pmatrix} \frac{1}{x} \\ \frac{3}{y} \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$. This means that we have

$1 = \lambda x^2, 3 = \lambda y^2$. Substituting into $x^2 + y^2 = 4$

$\Rightarrow \frac{1}{\lambda} + \frac{3}{\lambda} = 4 \Rightarrow \lambda = 1 \Rightarrow x = \pm 1, y = \pm\sqrt{3}$

Hence there are 4 points on the circle minimizing the quantity.

Exercise 12 Let a, b and c be positive real numbers. Prove that

(a) $2a^2+3b^2+6c^2 \geq (a+b+c)^2 \geq 3(ab+bc+ca)$,

(b) $\dfrac{a^2}{c}+\dfrac{b^2}{a}+\dfrac{c^2}{b} \geq a+b+c$.

Exercise 13 Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$a^2+b^2+c^2 \geq a+b+c\ .$$

5 Inequalities involving other functions

In this section we will look at further techniques in solving inequalities involving trigonometric, exponential and logarithmic functions. A useful technique is by differentiation.

5.1 Inequalities involving monotonic functions

In our previous Chapter on functions, we defined strictly increasing and strictly decreasing functions.
A function that is non-increasing or non-decreasing is said to be monotonic.
It can be shown that if $\mathrm{f}\,'(x) \geq 0$ then f is non-decreasing.
Similarly, we have if $\mathrm{f}\,'(x) \leq 0$ then f is non-increasing.

Example 6 Show that for $x \geq 0$, $\ln(1+x) \leq x$.
Deduce that $\mathrm{e}^x \geq 1+x$

Solution

Let $\mathrm{f}(x) = \ln(1 + x) - x$.

Then $\mathrm{f}(0) = 0$ and $\mathrm{f}\,'(x) = \dfrac{1}{1+x} - 1 \le 0$ for $x \ge 0$.

Hence f is decreasing and since f(0) = 0, we must have $f(x) \leq f(0) = 0$ for x ³ 0.

That is, $\ln(1 + x) - x \leq 0$.

$\Rightarrow \ln(1 + x) \leq x$

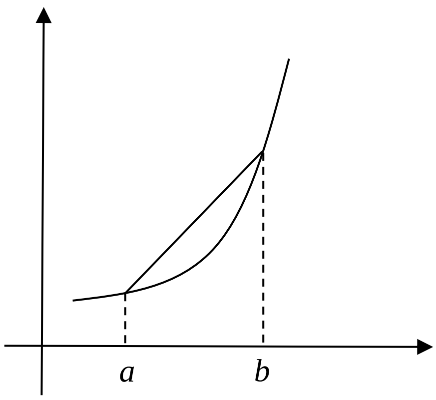
Since the exponential function is strictly increasing, $e^{\ln(1+x)} \leq e^{x} \Rightarrow 1 + x \leq e^{x}$.

Exercise 14 Show that for $0 \leq x \leq \dfrac{\pi}{2}$, $\sin x \leq x \leq \tan x$.

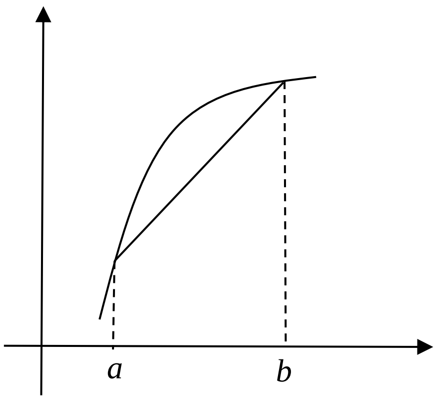
5.2 Inequalities involving convex functions

The definition of a convex function comes from a simple geometric description.

We say a function f is **convex** (concave up) on an interval I if $\forall a, b \in I$, the line segment joining $(a, f(a))$ and $(b, f(b))$ lies above the graph of f *and* **concave** (concave down) if the line segment lies below the graph of f.



f **convex**



f **concave**

In other words, a function f is convex (concave up) on an interval I, if for *a and b* in *I*, $x \in (a, b)$

$$f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \Leftrightarrow f(x) \leq f(b) + \frac{f(b) - f(a)}{b - a}(x - b)$$

We can also compare slopes of line segments. We have by rewriting, for all $x \in (a, b)$

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x} .$$

We could also parametrize the line segment, if we let $x = at + (1-t)b$, $t \in (0,1)$.
We have f is convex if $\mathrm{f}\left(at + (1-t)b\right) \le \mathrm{f}(a)t + \mathrm{f}(b)(1-t)$ and
concave if $\mathrm{f}\left(at + (1-t)b\right) \ge \mathrm{f}(a)t + \mathrm{f}(b)(1-t)$

In particular, $\mathrm{f}\left(\frac{a+b}{2}\right) \le \frac{\mathrm{f}(a) + \mathrm{f}(b)}{2}$ if f is convex.

It can be shown that if f is twice differentiable, then f is convex on an interval if and only if $\mathrm{f}'' \ge 0$ over the interval, and concave if and only if $\mathrm{f}'' \le 0$.

Example 7 Show that for $0 \le x \le \frac{\pi}{2}$, $\sin x \ge \frac{2}{\pi} x$

Solution $\frac{\mathrm{d}^2}{\mathrm{d}x^2} \sin x = -\sin x < 0$ on $0 \le x \le \frac{\pi}{2}$.

Hence the line segment joining $(0,0)$ and $\left(\frac{\pi}{2}, 1\right)$ is below the graph of $y = \sin x$.

The equation of the line is given by $y = \frac{2}{\pi} x$. Hence for $0 \le x \le \frac{\pi}{2}$, $\sin x \ge \frac{2}{\pi} x$.

Example 8 Show that $\mathrm{f}(x) = x^n$, $n \ge 2$ is convex.

Hence, conclude that $\left(\frac{a+b}{2}\right)^n \le \frac{a^n + b^n}{2}$ for $a,b > 0$.

Solution $\mathrm{f}''(x) = n(n-1)x^{n-2} > 0$ for $n \ge 2, x > 0$.

Hence $\mathrm{f}\left(\frac{a+b}{2}\right) \le \frac{\mathrm{f}(a) + \mathrm{f}(b)}{2} \Rightarrow \left(\frac{a+b}{2}\right)^n \le \frac{a^n + b^n}{2}$.

Exercise 15 Show that $\mathrm{f}(x) = \ln x$ is concave.

Hence, conclude that $\frac{a+b}{2} \ge \sqrt{ab}$ for $a,b > 0$. This provides another proof of the AM-GM inequality.

Tutorial

1. J80/1

(i) Given that $a > 0$, $b > 0$, prove that $\dfrac{a}{b}+\dfrac{b}{a} \geq 2$.

(ii) Given that $x > 0$, $y > 0$, $z > 0$ and that $x+y+z=3$, prove that $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z} \geq 3$.

2. J83/1

(a) Let a, b, c be three positive numbers such that $a+b+c=3p$, where p is fixed. By writing $a=p+\alpha$, $b=p+\beta$, $c=p+\gamma$, or otherwise, find the least possible value of $a^2+b^2+c^2$ as a, b and c vary.

(b) Given that x is positive, find the least value of $x+\dfrac{1}{x}$ as x varies.

Hence, or otherwise, given that p, q and r are positive, show that

$$\left(p^2+1\right)\left(q^2+1\right)\left(r^2+1\right) \geq 8pqr.$$

3. Show that

(a) for $0 < x < \dfrac{\pi}{2}$, $3x \leq 2\sin x + \tan x$.

(b) for $0 \leq x \leq \pi$, $\sin^2 x \leq \dfrac{4}{\pi^2} x\left(\pi - x\right)$.

4. **[RI/2014/9824/Lecture Test 1/5]**

Let a, b and c be real numbers and x, y and z positive real numbers.

By considering $\left(ay-bx\right)^2 \geq 0$, show that $\dfrac{(a+b)^2}{x+y} \leq \dfrac{a^2}{x}+\dfrac{b^2}{y}$. [2]

Deduce that $\dfrac{(a+b+c)^2}{x+y+z} \leq \dfrac{a^2}{x}+\dfrac{b^2}{y}+\dfrac{c^2}{z}$. [1]

Hence or otherwise,

(a) show that $\dfrac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \leq \dfrac{x+y+z}{3}$; [2]

(b) determine the minimum value of $\dfrac{1}{p^2}+\dfrac{4}{q^2}+\dfrac{9}{r^2}$ given that $p^2+q^2+r^2=1$ and p, q and r are real numbers. [3]

5. **[2015/9824/2]**

Let s and G be the sum and the geometric mean, respectively, of the positive numbers $a_1, a_2, ..., a_n$.

(i) If a and b are two of the positive numbers $a_1, a_2, ..., a_n$ such that $a \geq G \geq b$, then prove that replacing a and b by G and $\dfrac{ab}{G}$, respectively, does not alter the geometric mean and does not increase the sum. [4]

(ii) Use part (i) and mathematical induction to prove that $s \geq nG$. [6]

(iii) By applying the AM-GM inequality to the numbers $1+a_1, 1+a_2, ..., 1+a_n$, prove that $(1+a_1)(1+a_2)...(1+a_n) \leq 1+s+\dfrac{s^2}{2!}+...+\dfrac{s^n}{n!}$. [5]

6. **[2016/9824/3]**

(i) For some positive integer n, let $x_1 \leq x_2 \leq ... \leq x_n$ and $y_1 \leq y_2 \leq ... \leq y_n$ be real numbers. By considering the sum of all n^2 terms of the form

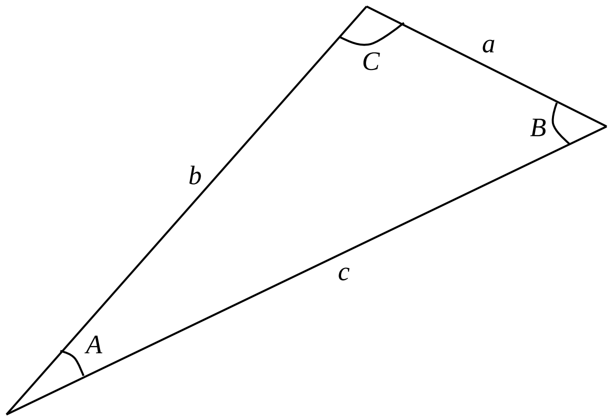
$$\left(x_i - x_j\right)\left(y_i - y_j\right)$$

prove that

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right).$$

[4]

(ii) Let a triangle have angles A, B and C and let the lengths of the opposite sides be a, b and c, respectively.



By applying the result of part **(i)**, prove that $aA+bB+cC \geq \frac{1}{3}\pi(a+b+c)$. [3]

(iii) Let a, b and c be three positive numbers such that $a^2+b^2+c^2=1$. By applying the result of **(i)** with $\{x_i\} = \left\{\dfrac{a+b}{c}, \dfrac{c+a}{b}, \dfrac{b+c}{a}\right\}$, find the minimum possible value of

$$\frac{(a+b)\left(a^2+b^2\right)}{c}+\frac{(c+a)\left(c^2+a^2\right)}{b}+\frac{(b+c)\left(b^2+c^2\right)}{a}.$$

[6]

7. (a) Using Mathematical Induction, prove the AM-GM inequality:

$$\text{If } x_1, x_2, ..., x_n \geq 0, \text{ then } \frac{x_1 + x_2 + ... + x_n}{n} \geq \left(x_1 x_2 \cdots x_n\right)^{\frac{1}{n}}$$

with equality when $x_1 = x_2 = ... = x_n$. (Hint: You should prove it for powers of 2 first. The idea is similar to Example 3 in the notes)

(b) Show by Mathematical Induction that if f is convex on an interval I, then

$$a_1, a_2, ..., a_n \in I, \ \mathrm{f}\left(\frac{a_1 + a_2 + ... + a_n}{n}\right) \leq \frac{\mathrm{f}(a_1) + \mathrm{f}(a_2) + ... + \mathrm{f}(a_n)}{n}.$$

(i) By considering the function $\mathrm{f}(x) = \mathrm{e}^x$, prove the AM-GM inequality in (a).

(ii) Show that if $a, b, c \geq 0$, then $\dfrac{a}{b+c} + \dfrac{b}{a+c} + \dfrac{c}{a+b} \geq \dfrac{3}{2}$.

Hint: Consider $\mathrm{f}(x) = \dfrac{x}{k-x}$.

8. **[9824/2018/3]**

A triangle has sides of length a, b and c units. In each of the following cases, prove that there is a triangle having sides of the given lengths.

(i) $\dfrac{a}{1+a}, \dfrac{b}{1+b}$ and $\dfrac{c}{1+c}$ units. [4]

(ii) $\sqrt{a}, \sqrt{b}$ and $\sqrt{c}$ units. [3]

(iii) $\sqrt{a(b+c-a)}, \sqrt{b(c+a-b)}$ and $\sqrt{c(a+b-c)}$ units. [6]

9. (i) Let $\mathrm{f}(x) = \left(x + 2a\right)^3 - 27a^2 x$, where $a \geq 0$. By sketching $\mathrm{f}(x)$, show that $\mathrm{f}(x) \geq 0$ for $x \geq 0$.

(ii) Use (i) to find the greatest value of $x^2 y$ in the region of the xy-plane given by $x \geq 0$, $y \geq 0$ and $x + 2y \leq 3$. For what values of x and y is this greatest value achieved?

(iii) Use (i) to show that $\left(p + q + r\right)^3 \geq 27pqr$, for any non-negative numbers p, q and r. If $\left(p + q + r\right)^3 = 27pqr$, what relationship must p, q and r satisfy?

10. [RI/2013/Lecture Test 1/Q8]

Let f be a real-valued, continuous and strictly increasing function on [0, c] with $c > 0$ and f(0) = 0. Let f^{-1} denote the inverse function of f.

(i) Explain using a graphical method that for all $a \in [0, c]$ and $b \in [0, \mathrm{f}(c)]$,

$$ab \le \int_0^a \mathrm{f}(x)\ \mathrm{d}x + \int_0^b \mathrm{f}^{-1}(x)\ \mathrm{d}x.$$

When does equality hold? [3]

(ii) Let $p, q \ge 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. By considering the function $\mathrm{f}(x) = x^{p-1}$ or otherwise, show that for all non-negative a and b, $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

When is equality attained? [4]

Assignment

1. [9824 Specimen Paper]

Let a, b, c and d be positive numbers.

(i) Sketch the graph of $y = \ln x$ and hence explain why

$$\frac{\ln a + 2\ln b}{3} \le \ln\left(\frac{a+2b}{3}\right).$$ [4]

(ii) Hence show that $\sqrt[3]{ab^2} \le \frac{a+2b}{3}$. [2]

(iii) By writing $\sqrt{\frac{d}{c}}$ as $2\sqrt{\frac{d}{4c}}$, find the minimum value of $\frac{c}{d} + \sqrt{\frac{d}{c}}$ and find the exact value of $\frac{c}{d}$ for which the minimum is attained. [4]

2. [2014/9824/1]

(i) It is given that a, b and c are real numbers and that a is positive. Prove that

$ax^2 + 2bx + c \ge 0$ for all real values of x if and only if $b^2 \le ac$. [4]

(ii) For some positive integer n, let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be real numbers. By considering $\sum_{i=1}^{n} (a_i x + b_i)^2$, prove that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$ [4]

(iii) Let u, v and w be positive real numbers such that $uv + vw + wu \ge 12$. By using the result of part (ii), or otherwise, prove that $u + v + w \ge 6$. [6]

3. A function $\mathrm{f}(x)$ is said to be concave on some interval if $\mathrm{f}''(x) < 0$ in that interval.

Let $\mathrm{f}(x)$ be concave on a given interval and let $x_1, x_2, ..., x_n$ lie in the interval. Jensen's inequality states that

$$\frac{1}{n}\sum_{k=1}^{n}\mathrm{f}(x_k) \le \mathrm{f}\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)$$

and that equality holds if and only if $x_1 = x_2 = ... = x_n$. You may use this result without proving it in this question.

(i) Use a sketch to illustrate why Jensen's inequality holds when $n = 2$.

(ii) Given that A, B and C are angles of a triangle, show that

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$$

and

$$\sin A \times \sin B \times \sin C \le \frac{3\sqrt{3}}{8}.$$

(iii) By choosing a suitable function f, prove that

$$\sqrt[n]{t_1 t_2 ... t_n} \le \frac{t_1 + t_2 + ... + t_n}{n}$$

for any positive integer n and for any positive numbers $t_1, t_2, ..., t_n$.
Hence

(a) show that $x^4 + y^4 + z^4 + 16 \ge 8xyz$, where x, y and z are any positive numbers;

(b) find the minimum value of $x^5 + y^5 + z^5 - 5xyz$, where x, y and z are any positive numbers.

Additional Practice Questions

Refer to the compilation of 2010 to 2019 STEP I and II problems. A graphing calculator should not be used for all these questions.

1. **2010/STEP III/1**

2. **2011/STEP II/3**

3. **2011/STEP III/4**

4. **2013/STEP II/8**

5. **2017/STEP I/2**

6. **2017/STEP II/4**

7. **2018/STEP II/3**

8. **2018/STEP III/5**

9. **2019/STEP II/3**