CHAPTER 4

INTRODUCTION

The logarithm of *Y* to the base *a* is the exponent to which a must be raised to yield *Y*. That is,

$$\boxed{\log_a Y = x \quad \text{if and only if} \quad a^x = Y}$$

$y = a^x$
$x = \log_a y$

Thus $\log_2 4 = 2$ since $2^2 = 4$ and $\log_2 8 = 3$ since $2^3 = 8$

Example 1 a) $3^2 = 9$; then $2 = \log_3 9$
2 is the logarithm of 9 to base 3.

$3^2 = 9$
$\log_3 3^2 = \log_3 9$
$2 \cancel{\log_3 3} = \log_3 9$
$2 = \log_3 9$

b) $10^2 = 100$; so $2 = \log_{10} 100$
2 is the logarithm of 100 to base 10.

Example 2 If $\log_{10} N = 3$, find the value of N.

$\log_a y = x \Longleftrightarrow$
$y = a^x$

$N = 10^3 = 1000$

Example 3 Given that $\log_x 81 = 4$, find the value of x.

$81 = x^4$
$x^4 = 3^4$

$81 = 9^2 = (3^2)^2 = 3^4$
$x = 3$

Example 4 Evaluate, $\log_8 2$ without using calculator.

$\log_a b = \dfrac{\log_n b}{\log_n a}$

$\log_8 2 = \log_8 8^{1/3}$?
$= \frac{1}{3} \cancel{\log_8 8} = \frac{1}{3}$

$2 \to (8)^?$
$2 \to 8^{\frac{1}{3}}$

$\log_8 2 = \dfrac{\cancel{\log_2 2}}{\log_2 8} = \dfrac{1}{\log_2 2^3} = \dfrac{1}{3\cancel{\log_2 2}} = \dfrac{1}{3}$

— napier

4.1 LAWS OF LOGARITHMS

A. *Product* ×	$\log_a xy = \log_a x + \log_a y$ +
B. *Quotient* ÷	$\log_a \frac{x}{y} = \log_a x - \log_a y$ −
C. *Power*	$\log_a x^n = n\log_a x$

$\log_a x^{-1} = -\log_a x$

4.2 SPECIAL LOGARITHMS

1. $\log_a a = 1$ e.g. $\log_2 2 = 1$, $\log_5 5 = 1$
2. $\log_a 1 = 0$ e.g. $\log_2 1 = 0$, $\log_9 1 = 0$
3. $\log_a \frac{1}{x} = -\log_a x$
4. Natural logarithm $\ln N = \log_e N$ (where e = 2.71828183)

Leonard Euler

$y = a^x$

$x = \log_a y$

$\log_a b = \frac{\log_n b}{\log_n a}$

n → anything

Example 5 *Simplify* $\log_3 2 + \log_3 5 + \log_3 20 - \log_3 25$

Product $\log_a x + \log_a y = \log_a xy$

Quotient $\log_a x - \log_a y = \log_a \frac{x}{y}$

$\log_3 2 + \log_3 5$

$= \log_3 10$

$\log_3 10 + \log_3 20 - \log_3 25$

$\log_3 10 + \log_3 20$

$= \log_3 200$

$\log_3 200 - \log_3 25$

$= \log_3 \frac{200}{25}$

$= \log_3 8$

Example 6 Simplify $3\log 3 + \log 10 - \log 3$ (assume same base)

$3\log 3 = \log 3^3 = \log 27$

$\underbrace{\log 27 + \log 10}_{\log 270} - \log 3 = \log 270 - \log 3$

$= \log \frac{270}{3}$

$= \log 90$

Example 7 *Simplify* $\log_2 16 - \log_2 8 + \log_2 4$

special log → $\log_a a = 1$

Quotient law

$\log_2 \frac{16}{8} = \log_2 2 = 1$

$1 + \underbrace{\log_2 4}_{\log_2 2^2 \to 2\cancel{\log_2 2} = 2} \Rightarrow 1 + 2 = 3$

Example 8 Write the following expression as a single logarithm.

$$2\log_{10} x + 3\log_{10}(x+2) - \log_{10}(x^2+5)$$

product
quotient
power

$$\log_{10} x^2 + \log_{10}(x+2)^3 - \log_{10}(x^2+5)$$

$$= \log_{10} \frac{x^2(x+2)^3}{(x^2+5)}$$

Example 9 Solve the equation $2^x = 5$

apply log on both sides

$$\log 2^x = \log 5$$

$$x \log 2 = \log 5$$

$$x = \frac{\log 5}{\log 2}$$

apply

$$\left\{ \begin{array}{l} y = a^x \\ x = \log_a y \end{array} \right\}$$

$$5 = 2^x$$

$$x = \log_2 5$$

$$= \frac{\log 5}{\log 2}$$

$$\log_b a = \frac{\log a}{\log b}$$

Example 10 *Given that* $\log_3 2 = 0.631, \log_3 5 = 1.465$; *find the value of* $\log_3 1.2$

$$\left\{ \begin{array}{l} \log_3 2 = 0.631 \\ \log_3 5 = 1.465 \end{array} \right\}$$

$$\log_3 1.2 = \log_3 \frac{12}{10}$$

quotient law

$$= \log_3 12 - \log_3 10$$

$$1.2 = \frac{12}{10}$$

$$= \log_3 (3\times 4) - \log_3 (2\times 5)$$

$12 \to 3\times 4$

$10 \to 2\times 5$

$$= \cancel{\log_3 3}^{\,1} + \log_3 4 - [\log_3 2 + \log_3 5]$$

$$\log_3 2^2 = 2\log_3 2$$

$$1 + 2\log_3 2 - \log_3 2 - \log_3 5$$

$$= 1 + \log_3 2 - \log_3 5$$

$$= 1 + 0.631 - 1.465$$

$$=$$

4.3 CHANGE OF BASE

a) Logarithms to *base 10* are called "common logarithms", and are denoted by lg. When the base is 10, this number is generally omitted.

i.e. *log N* denotes the logarithm of N to the base 10.

b) Logarithms to *base e* are called "natural logarithms", and are denoted by ln. e has approximately the value 2.718.

i.e. $\log_e N$ is written as *ln N*

$\log \to \log_{10}$

$\ln \to \log_e$

c) Logarithms can be to any base; however common logarithms are exclusively used for calculations at this stage.

d) Where logarithms to other bases are encountered, they have to be changed to base 10 for a numerical answer.

To change from base a to base b :

$$\boxed{\log_a N = \frac{\log_b N}{\log_b a}}$$

$\log_3 7 = \dfrac{\log 7}{\log 3}$

Example 11 Find the value of (a) $\log_3 4$ (b) $\log_2 10$

(a) $\log_3 4 = \dfrac{\log 4}{\log 3}$

$= \_\_\_\_\_\_$

(b) $\log_2 10 = \dfrac{\cancel{\log 10}\ 1}{\log 2}$

$= \dfrac{1}{\log 2}$

$= \_\_\_\_\_\_$

Example 12 Find the value of x : $\log_2 x + \log_4 x = \dfrac{3}{2}$

$4 = 2^2$

$\log_2 x + \frac{1}{2}\log_2 x = \frac{3}{2}$

$\frac{3}{2}\log_2 x = \frac{3}{2}$

$\log_2 x = 1 \Rightarrow x = 2^1 = 2$

Q.E.D.

$\log_4 x = \log_2 ?$

$= \dfrac{\log_2 x}{\log_2 4}$

$= \dfrac{\log_2 x}{2\log_2 2}$

$= \frac{1}{2}\log_2 x$

Example 13 Find the positive value of x : $\log_2 x = \log_4 (x+6)$

$\log_2 x = \dfrac{\log_2 (x+6)}{\log_2 4}$

$\log_2 x = \dfrac{\log_2 (x+6)}{2}$

$2\log_2 x = \log_2 (x+6)$

$\log_2 x^2 = \log_2 (x+6)$

$x^2 = x+6$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3$ ✓, $\cancel{x = -2}$

$x \quad -3$

$x \quad 2$

$\log_2 4 = \log_2 2^2$

$= 2\,\cancel{\log_2 2}\ 1$

$= 2$

Example 14 Find the value of x: $\log_3 x - 4\log_x 3 + 3 = 0$

$4\log_x 3 = \log_x 3^4 = \frac{\log_3 81}{\log_3 x} = \frac{4}{\log_3 x}$

common base 3

$\log_3 x - \frac{4}{\log_3 x} + 3 = 0$

Let $p = \log_3 x$. $p - \frac{4}{p} + 3 = 0$

multiply both sides by p

$p^2 - 4 + 3p = 0$

$(p+4)(p-1) = 0$

$p = 1$ or $p = -4$

$p = 1$, $\log_3 x = 1$, $x = 3^1$, $x = 3$

$p = -4$, $\log_3 x = -4$, $x = 3^{-4} = \frac{1}{81}$

TUTORIAL 4

1. Write each of the following in logarithmic form. For example, $3^4 = 81$ can be written as $\log_3 81 = 4$.

   (a) $2^4 = 16$ (b) $125 = 5^3$ (c) $64 = 16^{\frac{3}{2}}$ (d) $81 = (\frac{1}{3})^{-4}$

2. Write each of the following in exponential form. For example, $\log_5 125 = 3$ can be written as $5^3 = 125$.

   (a) $\log_2 32 = 5$ (b) $2 = \log_5 25$ (c) $7 = \log_2 128$ (d) $-2 = \log_3(1/9)$

   (e) $\log_e 1 = 0$ (f) $2 = \log_a X$ (g) $\ln 20.09 = 3$

3. Determine the value of each of the following logarithms.

   (a) $\log_2 64$ (b) $\log_{10} 10^7$ (c) $\log_{27} 3$ (d) $\log_5 125$ (e) $\log_{10} 10^{-6}$

   6 7 $\frac{1}{3}$ 3 -6

   $\log_{27} 3 = \frac{\log_3 3}{\log_3 27} = \frac{1}{3}$

4. Write each of the following as a single logarithm.

   log ( )

   (a) $3\log_a 2 + 2\log_a 3 - 2\log_a 6$ (b) $3\log_2 5 - 2\log_2 7$

   (c) $\frac{1}{2}\log_5 64 + \frac{1}{3}\log_5 27 - \log_5(x^2+4) = \log_5\left(\frac{24}{x^2+4}\right)$ ✓

   (d) $3\log_2(x+2) + \log_2 8x - 2\log_2(x+8)$

   (e) $2\log_5 x - 3\log_5(2x+1) + \log_5(x-4) = \log_5 x^2 - \log_5(2x+1)^3 + \log_5(x-4)$

   $2\log_5 x = \log_5 x^2$

   $3\log_5(2x+1) = \log_5(2x+1)^3$

   $= \log_5 \frac{x^2(x-4)}{(2x+1)^3}$

   QED

5. Evaluate, without using calculator:

(a) $3\log_{10}2 + 2\log_{10}5 - \log_{10}20$ $= \log_{10} 2^3 + \log_{10} 5^2 - \log_{10} 20$

$= \log_{10} 8 + \log_{10} 25 - \log_{10} 20$

$= \log_{10}\left(\frac{8 \times 25}{20}\right)$

$= \log_{10} 10$

$= 1$

(b) $\log_{10}\frac{41}{35} + \log_{10}70 - \log_{10}\frac{41}{2} + 2\log_{10}5$

(c) $\log_{10}\frac{14}{15} + \log_{10}\frac{21}{20} - \log_{10}\frac{49}{50}$

6. Solve the equations: $3^x = 2$ $\log_3 2 = x$

(a) $3^x = 2$ (b) $3^{4x} = 4$ (c) $2^x\, 2^{(x+1)} = 10$

(d) $(\frac{1}{2})^x = 6$ (e) $(\frac{2}{3})^x = \frac{1}{16}$

7. Given that $\log_2 3 = 1.585$, and $\log_2 5 = 2.322$, calculate the values of $\log_2 60$ and $\log_2 0.3$.

8. If $\log_7 2 = 0.356$ and $\log_7 3 = 0.565$, find the value of $\log_7\frac{8}{9} + 2\log_7\frac{9}{2}$

9. Solve each of the following equations.

(a) $\log_2 x + \log_2(x+2) = 3$ (b) $\log_3 x - \log_3(2x+3) = -2$

10. Find the values of x in

(a) $\log_2 x + \log_x 2 = 2$ (b) $\log_3 x - 2\log_x 3 = 1$

Challenging Questions

1 If $u = \log_4 x,$ find in term of u

(a) x

(b) $\log_4 2x$

(c) $\log_x 64$

2 (a) If $\log_8 x = p$, express $\log_2 x$ in terms of p.

Given that $\log_q(xy) = 3$ and $\log_q(x^2 y^3) = 4$.

Calculate the values of $\log_q x$ and $\log_q y$

3 (a) Calculate the value of $\log_3 8$. Giving your answer correct to 3 significant figures.

(b) Evaluate x if $\log_2(1+x) + \log_2(5-x) - \log_2(x-2) = 3$