RAFFLES INSTITUTION
H2 Mathematics (9758)
2016 Year 5

Tutorial 4B : Complex Numbers II: Polar Form

Section A (Basic Questions)

Do these questions without a GC first, then verify your answers using a GC wherever possible.

1 For each of the following complex number, express it in polar form:

(a) $\sqrt{3}+\mathrm{i}$ **(b)** $-1+\mathrm{i}\sqrt{3}$ **(c)** $\mathrm{i}^2(1+\mathrm{i})$

(d) $-\mathrm{i}(1+\mathrm{i})$ **(e)** $\sin\theta+\mathrm{i}\cos\theta$, where $0<\theta<\dfrac{\pi}{2}$

[**a)** $2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ **b)** $2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ **c)** $\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}$ **d)** $\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}$ **e)** $\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\theta\right)}$]

2 Express the following complex numbers in the form $x+\mathrm{i}y$, where $x,\ y\in\mathbb{R}$:

(a) $2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$ **(b)** $5\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$

[**a)** $-\sqrt{3}+\mathrm{i}$ **b)** $\dfrac{5\sqrt{2}}{2}(1-\mathrm{i})$]

3 The complex numbers z_1 and z_2 are such that

$$|z_1|=2,\ \arg(z_1)=\frac{\pi}{3},\ |z_2|=\sqrt{2},\ \arg(z_2)=-\frac{3\pi}{4}.$$

Find the modulus and argument of the following:

(a) $\dfrac{1}{z_1}$ **(b)** z_1^* **(c)** $-z_1$

(d) z_1z_2 **(e)** $z_2^{\,2}$ **(f)** $\dfrac{z_1}{z_2}$

[**a)** $\dfrac{1}{2},\ -\dfrac{\pi}{3}$ **b)** $2,\ -\dfrac{\pi}{3}$ **c)** $2,\ -\dfrac{2\pi}{3}$ **d)** $2\sqrt{2},\ -\dfrac{5\pi}{12}$ **e)** $2,\ \dfrac{\pi}{2}$ **f)** $\sqrt{2},\ -\dfrac{11\pi}{12}$]

4 The complex numbers z and w are such that

$$|z|=2,\ \arg(z)=-\frac{2\pi}{3},\ |w|=5,\ \arg(w)=\frac{3\pi}{4}.$$

Find the exact values of **(a)** $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, **(b)** $\left|\dfrac{w^2}{z}\right|$, $\arg(\dfrac{w^2}{z})$ and $\mathrm{Im}(\dfrac{w^2}{z})$.

[**a)** $-1,\ -\sqrt{3}$ **b)** $\dfrac{25}{2},\ \dfrac{\pi}{6},\ \dfrac{25}{4}$]

Section B (Standard Questions)

5 The complex number w has modulus $\sqrt{2}$ and argument $-\frac{3\pi}{4}$, and the complex number z has modulus 2 and argument $-\frac{\pi}{3}$.

Find the exact values of the modulus and argument of wz.

By expressing w and z in the form $x+\mathrm{i}y$, $x, y \in \mathbb{R}$, find the exact real and imaginary parts of wz. Hence show that $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

$[2\sqrt{2},\ \frac{11\pi}{12};\ -1-\sqrt{3},\ \sqrt{3}-1]$

6 Express the complex numbers $z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}}$ and $z_2 = \frac{\sqrt{2}}{1-\mathrm{i}}$ in polar form.

Draw an Argand diagram showing the points representing z_1, z_2 and z_1+z_2.

Deduce from your diagram that $\tan\frac{3\pi}{8} = 1+\sqrt{2}$.

$[z_1 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}},\ z_2 = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}]$

7 **[9740/2012/01/Q6]**

Do not use a calculator in answering this question

The complex number z is given by $z = 1+\mathrm{i}c$ where c is a non-zero real number.

(i) Find z^3 in the form $x+\mathrm{i}y$.

(ii) Given that z^3 is real, find the possible values of z.

(iii) For the value of z found in part **(ii)** for which $c<0$, find the smallest positive integer n such that $|z^n| > 1000$. State the modulus and argument of z^n when n takes this value.

[(i) $z^3 = 1-3c^2+\mathrm{i}(3c-c^3)$ **(ii)** $z = 1\pm\sqrt{3}\mathrm{i}$ **(iii)** smallest $n=10$, $|z^{10}| = 1024$, $\arg z^{10} = \frac{2\pi}{3}$ **]**

8 **[9740/2014/02/Q4(b)]**

It is given that $w = \sqrt{3}-\mathrm{i}$.

(i) Without using a calculator, find an exact expression for w^6. Give your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $0 \le \theta < 2\pi$.

(ii) Without using a calculator, find the three smallest positive whole number values of n for which $\frac{w^n}{w^*}$ is a real number.

[(i) $64\mathrm{e}^{\mathrm{i}\pi}$ **(ii)** $n = 5, 11, 17$ **]**

9 **[JJC Prelim 9740/2014/02/Q4(i)]**

The complex numbers p and q are given by

$$p = 1 - \mathrm{i} \text{ and } q = 1 + \sqrt{3}\,\mathrm{i}.$$

Two other complex numbers r and s are given by

$$r = k + \mathrm{i} \text{ and } s = \frac{p^2 r}{q^3}, \text{ where } k \text{ is a real number.}$$

Given that $|s| = \frac{1}{2}$ and $\arg(s) = -\frac{2}{3}\pi$, find the value of k.

$\ln\left(\frac{p^2 r}{q^3}\right)$. $2\ln p + \ln r - 3\ln q$

[$k = -\sqrt{3}$]

10 **[NJC Prelim 9740/2014/02/Q3]**

(a) Given that $\left|\frac{2\mathrm{i} - z^*}{z} - 1\right|^2 - z = \mathrm{i}$, find z in the form $x + \mathrm{i}y$.

(b) Let $p = -\sqrt{3} + \mathrm{i}$ and $q = -4\mathrm{i}$.

(i) Write down p and q exactly in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$.

(ii) Find the exact value of $\frac{p^{10}}{q^5} + \frac{q^5}{p^{10}}$.

[(a) $4 - \mathrm{i}$, **(b)(i)** $2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$, $4\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$ **(ii)** $-\sqrt{3}$]

11 **(a)** In an Argand diagram, the points A, B, C, D represent the complex numbers a, b, c, d respectively. Given that $ABCD$ is a square described in an anticlockwise sense, with $a = 1 + \mathrm{i}$ and $c = 7 + 3\mathrm{i}$, find b and d.

(b) In an Argand diagram, O is the origin, and the point P represents the complex number $3 + \mathrm{i}$. The point Q represents the complex number $a + \mathrm{i}b$, where $a, b > 0$, and triangle OPQ is equilateral. Find the exact values of a and b.

[(a) $b = 5 - \mathrm{i}$, $d = 3 + 5\mathrm{i}$ **(b)** $a = \frac{1}{2}\left(3 - \sqrt{3}\right)$, $b = \frac{1}{2}\left(1 + 3\sqrt{3}\right)$**]**

Section C (Extensions / Challenging Questions)

12 Let z_1 and z_2 be distinct complex numbers.

Classify the following statements as true or false, justifying your answers.

(i) If $|z_1| = |z_2|$, then $\mathrm{Re}\left(\frac{z_1 + z_2}{z_1 - z_2}\right) = 0$. **(ii)** If $\mathrm{Re}\left(\frac{z_1 + z_2}{z_1 - z_2}\right) = 0$, then $|z_1| = |z_2|$.

13 9205/1999/02/Q12b

The complex numbers z and w each have modulus R, and have arguments α and β respectively, where $0 < \alpha < \beta < \frac{\pi}{2}$.

In either order,

(i) show that $z + w = 2R\left\{\cos\left(\frac{1}{2}(\beta - \alpha)\right)\right\}e^{\frac{1}{2}(\alpha+\beta)i}$,

(ii) express $|z + w|$ and $\arg(z + w)$ in terms of R, α and β, as appropriate.

Show also that $|z - w| = 2R\sin\left(\frac{1}{2}(\beta - \alpha)\right)$.

The complex numbers z and w are represented by the points Z and W respectively on an Argand diagram with origin O. Triangle OZW has area Δ.
Show that $\left|z^2 - w^2\right| = 4\Delta$.

Section D (Self-Practice Questions)

1(i) Find the exact values of $x,\ y \in \mathbb{R}$ such that $\dfrac{3}{x+iy} + \dfrac{5}{2+6i} = 4$. **[3]**

(ii) The complex number z is given by $-\sqrt{3} + i$.
Find the modulus and argument of z.

Find the smallest positive integer value k such that $\left(\dfrac{1}{z^*}\right)^k$ is real. **[5]**

[1i) $x = \dfrac{10}{13}$, $y = -\dfrac{2}{13}$ **ii)** 2, $\dfrac{5\pi}{6}$; smallest $k = 6$]

2 Show that the complex number $z = 1 + e^{-i\frac{\pi}{3}}$ can be expressed as $\sqrt{3}e^{-i\frac{\pi}{6}}$.

The complex number zw has modulus 5 and argument $\dfrac{3\pi}{4}$. Find the complex number w in polar form. **[5]**

[2) $w = \dfrac{5\sqrt{3}}{3}e^{i\frac{11\pi}{12}}$]

3 CJC Prelim 9740/2009/01/Q8a

One of the roots of the equation $z^4 - z^3 + 4z^2 + 3z + p = 0$ is $1 - 2i$, where p is a constant.

A student says that "One of the other roots must be $1 + 2i$."
Explain why this statement is not entirely correct.

(i) Determine the value of p.

(ii) Find the exact values of all the other roots of the equation.

[3(i) $p = 5$ **ii)** $-\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$]

4 TPJC Prelim 9740/2008/01/Q6 (modified)

Given that $z_1 = 1-2i$ is a root of the quadratic equation $z^2+az+b=0$ where $a,\ b \in \mathbb{R}$, find the values of a and b. Mark on an Argand diagram the points P_1 and P_2, which represent the roots z_1 and z_2 of the quadratic equation $z^2+az+b=0$. A third point P_3 is on the real axis such that P_1, P_2 and P_3 form a triangle which encloses the origin with an area of 8 square units. Find the complex number represented by the point P_3. **[5]**

P_3 -3

[(4) $a=-2,\ b=5$; -3]

5 NJC Prelim 9740/2007/01/Q8 (modified)

Given that $z = \dfrac{i-1}{(\sqrt{3}+i)^2}$, find the exact modulus of z, and show that the argument of z is $\dfrac{5\pi}{12}$. **[3]**

Find the exact values of a and b which satisfy the equation $z^3 = e^{a+ib}$, where $a \in \mathbb{R}$ and $-\pi < b \le \pi$. **[3]**

[(5) $|z| = \dfrac{1}{2\sqrt{2}}$; $a = -\dfrac{9}{2}\ln 2,\ b = -\dfrac{3\pi}{4}$]

doing, got $\ln \frac{1}{16\sqrt{2}}$

6 JJC Prelim 9233/2006/02/Q3

Given that $z = \dfrac{\left(-1+i\sqrt{3}\right)^{10}}{(i)^{14}}$, express z in the form $a+ib$, $a,\ b \in \mathbb{R}$, giving the exact values of a and b. **[4]**

[$a=512,\ b=-512\sqrt{3}$]

7 DHS Prelim 9740/2013/02/Q4

The equation $z^4-4z^3+az^2-20z+25=0$, $a \in \mathbb{R}$, has a root ki, where k is a real number.

(i) Explain clearly why there is more than one possible value for k. **[1]**

(ii) Find the possible exact values of k and show that $a=10$.

Hence find the roots of the above equation. **[7]**

(iii) Deduce the roots of the following equations in the form $x+iy$, where x and y are real,

(a) $w^4+4iw^3-10w^2-20iw+25=0$,

(b) $25v^4-20v^3+10v^2-4v+1=0$. **[4]**

[$k=\pm\sqrt{5}$, $z=\pm\sqrt{5}i,\ 2\pm i$, $w=\pm\sqrt{5},\ \pm1-2i$, $v=\pm\dfrac{i}{\sqrt{5}},\ \dfrac{1}{5}(2\pm i)$]

8 **PJC Common Test 2 9740/2007/Q2**

(i) Find the fourth roots of the complex number $8(1+\mathrm{i}\sqrt{3})$, giving your answers exactly in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. **[4]**

(ii) Sketch the points P_1, P_2, P_3 and P_4 on an Argand diagram, where P_1, P_2, P_3 and P_4 represent the roots of $8(1+\mathrm{i}\sqrt{3})$. Prove that $P_1P_2P_3P_4$ form a square. **[3]**

[(i) $2\mathrm{e}^{\mathrm{i}\left(\frac{6k+1}{12}\right)\pi}$, $k=-2,-1,0,1$ **]**

not in syllabus

~~9~~ **HCI Prelim 9740/2007/02/Q2**

The equation E is given by $z^4+az^2+b=0$, where a and b are constants.

(a) Solve E when $a=0$ and $b=1-\mathrm{i}\sqrt{3}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. **[3]**

The points A, B, C and D represent the roots of the equation E on an Argand diagram. By considering the modulus and argument of each of the roots found, identify the shape of the quadrilateral $ABCD$. Justify your answer. **[5]**

(b) Find the range of values of b such that all the roots to E are real when $a=-2$. **[3]**

[(a) $2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{3k+1}{6}\right)\pi}$, $k=-2,-1,0,1$; square **(b)** $0\le b\le 1$ **]**