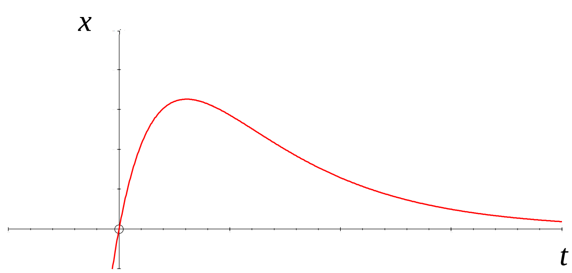
2024 J2 H2 FM Prelims P1 (Solutions)

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>1</td><td>
Let P_n be the statement that $\sum_{r=1}^{n}\frac{1}{r^2}<2-\frac{1}{n}$ for all integers $n\geq 2$.
To show P_2 is true:
LHS = $\sum_{r=1}^{2}\frac{1}{r^2}=\frac{5}{4}$.
RHS = $2-\frac{1}{2}=\frac{3}{2}>\frac{5}{4}$. Hence P_2 is true.
Suppose P_k is true for some integer $k\geq 2$, i.e. $\sum_{r=1}^{k}\frac{1}{r^2}<2-\frac{1}{k}$.
We want to show P_{k+1} is also true, i.e. $\sum_{r=1}^{k+1}\frac{1}{r^2}<2-\frac{1}{k+1}$.
$$\begin{aligned}\sum_{r=1}^{k+1}\frac{1}{r^2}&=\sum_{r=1}^{k}\frac{1}{r^2}+\frac{1}{(k+1)^2}\\&<2-\frac{1}{k}+\frac{1}{(k+1)^2}\\&=2-\frac{k^2+k+1}{k(k+1)^2}\\&<2-\frac{k^2+k}{k(k+1)^2}\\&=2-\frac{1}{k+1}\end{aligned}$$
Hence P_{k+1} is true.
Since P_2 is true and P_k implies P_{k+1}, by the Principle of Mathematical Induction, P_n is true for all integers $n\geq 2$.
</td></tr>
<tr><td>2(i)</td><td>
Using GC, $\text{rref}(\mathbf{A})=\begin{pmatrix}1&0&2&2\\0&1&0&2\\0&0&0&0\end{pmatrix}$ or $\text{ref}(\mathbf{A})=\begin{pmatrix}1&\frac{7}{3}&2&\frac{20}{3}\\0&1&0&2\\0&0&0&0\end{pmatrix}$
$\text{Dim}(R_T)=\text{Rank}(A)=2$
By the Rank-Nullity Theorem,
$\text{Dim}(K_T)=4-2=2$
</td></tr>
</table>

2(ii)	Basis for $R_{\mathrm{T}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} \right\}$ Let $\begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then $\begin{matrix} w+2y+2z=0 \\ x+2z=0 \end{matrix} \Rightarrow \begin{matrix} w=-2y-2z \\ x=-2z \end{matrix}$ $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y-2z \\ -2z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$ Basis for $K_{\mathrm{T}} = \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$ As $\begin{pmatrix} 3 \\ 1 \\ 11 \end{pmatrix} \in R_{\mathrm{T}}$, $\mathbf{Ax} = \begin{pmatrix} 3 \\ 1 \\ 11 \end{pmatrix}$ for some $\mathbf{x} \in \mathbb{R}^4$. Observe $\begin{pmatrix} 3 \\ 1 \\ 11 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}$, so a possible solution $\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ $\therefore$ The general solution of $\mathrm{T}(\mathbf{x}) = \begin{pmatrix} 3 \\ 1 \\ 11 \end{pmatrix}$ is $\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}$

3(i)	Since the daily removal of concentration level is 7%, i.e. the concentration remaining end of each day is 93%. The recurrence relation is $u_n = (0.93)^7 u_{n-1} + 2.5, \quad u_0 = 0$ i.e. $u_n = 0.6017u_{n-1} + 2.5, \quad u_0 = 0$ Hence, $u_n = c(0.93^7)^n + \dfrac{2.5}{1-0.93^7}$ $= c(0.93^7)^n + 6.2767, \quad u_0 = 0$ When $n = 0$, $0 = c + 6.2767 \Rightarrow c = -6.2767$ Therefore, $u_n = 6.28\left(1-(0.602)^n\right)$ (to 3 s.f.), $\quad n \geq 0,\ n \in \mathbb{Z}$
3(i)	**Alternatively:** $u_n = 0.93^7 u_{n-1} + 2.5$ $= 0.93^7\left(0.93^7 u_{n-2} + 2.5\right) + 2.5$ $= \left(0.93^7\right)^2 u_{n-2} + (0.93^7)2.5 + 2.5$ $= \left(0.93^7\right)^2\left(0.93^7 u_{n-3} + 2.5\right) + (0.93^7)2.5 + 2.5$ $= \left(0.93^7\right)^3 u_{n-3} + \left(0.93^7\right)^2 2.5 + (0.93^7)2.5 + 2.5$ Therefore, $u_n = 0.93^{7n}u_0 + \left(0.93^7\right)^{n-1}2.5 + \left(0.93^7\right)^{n-2}2.5 + ... + (0.93^7)2.5 + 2.5$ $= 2.5\left(\dfrac{1-\left(0.93^7\right)^n}{1-0.93^7}\right)$ since $u_0 = 0$ $= 6.28\left(1-(0.602)^n\right)$ (to 3 s.f.), $\quad n \geq 0,\ n \in \mathbb{Z}$
3(ii)	Since $u_n = 6.2767\left(1-\left(0.93^7\right)^n\right) \leq 6.2767$ for all $n \geq 0$, the concentration level will always be less than 7 mg/l. So the Local Authority should allow the factory to go ahead with the discharge
4	Since 2 is an eigenvalue of **A**, $\det(\mathbf{A} - 2\mathbf{I}) = 0$ $\Rightarrow \begin{vmatrix} a-2 & -3 & -3 \\ 2 & a-7 & -3 \\ -2 & 8 & a+4 \end{vmatrix} = 0$ $\Rightarrow (a-2)\left[(a-7)(a+4) - (-3)(8)\right] - (2)[(-3)(a+4) - (-3)(8)]$ $-2[(-3)(-3) - (-3)(a-7)] = 0$ $\Rightarrow (a-2)(a^2 - 3a - 28 + 24) + 6a - 24 - 6a + 24 = 0$ $\Rightarrow (a-2)(a^2 - 3a - 4) = 0$ $\Rightarrow (a-2)(a+1)(a-4) = 0$ $\Rightarrow a = -1, 2$ or 4

4(i)	$\det(\mathbf{A}-\lambda\mathbf{I})=0$ $\Rightarrow \begin{vmatrix} -1-\lambda & -3 & -3 \\ 2 & -6-\lambda & -3 \\ -2 & 8 & 5-\lambda \end{vmatrix} = 0$ $\Rightarrow (-1-\lambda)\left[(-6-\lambda)(5-\lambda)-(-3)(8)\right]-(2)[(-3)(5-\lambda)-(-3)(8)]$ $-2[(-3)(-3)-(-3)(-6-\lambda)]=0$ $\Rightarrow (\lambda+1)(\lambda^2+\lambda-6)=0$ $\Rightarrow \lambda=-3,-1$ or 2 Using a GC, For $\lambda=-3$, $\begin{pmatrix} 2 & -3 & -3 \\ 2 & -3 & -3 \\ -2 & 8 & 8 \end{pmatrix}\mathbf{x}=\mathbf{0}$. An eigenvector is $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. For $\lambda=-1$, $\begin{pmatrix} 0 & -3 & -3 \\ 2 & -5 & -3 \\ -2 & 8 & 6 \end{pmatrix}\mathbf{x}=\mathbf{0}$. An eigenvector is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ For $\lambda=2$, $\begin{pmatrix} -3 & -3 & -3 \\ 2 & -8 & -3 \\ -2 & 8 & 3 \end{pmatrix}\mathbf{x}=\mathbf{0}$. An eigenvector is $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.
4(ii)	If λ is an eigenvalue of **A** with corresponding eigenvector **x**, then $(2\mathbf{A}-3\mathbf{I})\mathbf{x}=2\mathbf{A}\mathbf{x}-3\mathbf{x}=2\lambda\mathbf{x}-3\mathbf{x}=(2\lambda-3)\mathbf{x}$, i.e. $2\lambda-3$ is an eigenvalue of **A** with corresponding eigenvector **x**. Therefore the eigenvalues of **A** are –9, –5, 1, with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ respectively.
5(i)	$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+k\frac{\mathrm{d}x}{\mathrm{d}t}+m\omega^2x=0$ Characteristic equation: $ms^2+ks+m\omega^2=0$ $s=\frac{-k\pm\sqrt{k^2-4(m)(m\omega^2)}}{2m}=\frac{-k}{2m}+\mathrm{i}\omega \quad \because k^2$ can be ignored General solution of x is $x=\mathrm{e}^{-\frac{k}{2m}t}\left(A\cos(\omega t)+B\sin(\omega t)\right)$. At t = 0, x = 0 $\Rightarrow$ A = 0. Since $x=B\mathrm{e}^{-\frac{k}{2m}t}\sin(\omega t)$, $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{Bk}{2m}\mathrm{e}^{-\frac{k}{2m}t}\sin(\omega t)+B\omega\mathrm{e}^{-\frac{k}{2m}t}\cos(\omega t)$

	At $t = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = v \Rightarrow B = \frac{v}{\omega}$. $\therefore x = \frac{v}{\omega} \mathrm{e}^{-\frac{k}{2m}t} \sin(\omega t)$
5(ii)	Period of vibrations, $T = \frac{2\pi}{\omega}$. Let $t_0 = \frac{\pi}{2\omega}$, Amplitude at time t_0 after the nth period, $A_n = \frac{v}{\omega} \mathrm{e}^{-\frac{k}{2m}(t_0+nT)}$ Amplitude at time t_0 after the (n+1)th $A_{n+1} = \frac{v}{\omega} \mathrm{e}^{-\frac{k}{2m}(t_0+(n+1)T)}$ $\frac{A_{n+1}}{A_n} = \frac{\frac{v}{\omega} \mathrm{e}^{-\frac{k}{2m}(t_0+(n+1)T)}}{\frac{v}{\omega} \mathrm{e}^{-\frac{k}{2m}(t_0+nT)}} = \mathrm{e}^{-\frac{k}{2m}(T)}$ (constant) Thus, amplitude of successive vibrations follows a geometric progression.
5(iii)	Roots to the characteristic eqn will be real, distinct and negative. Thus the general solution will be $x = A\left(\mathrm{e}^{-m_1 t} - \mathrm{e}^{-m_2 t}\right)$ where m_1 and m_2 are positive constants. x will initially increase, then gradually decrease and tend to zero (overdamped system). A possible graph is 

6(i)	$x_{n+2} = x_{n+1} + kx_n$
6(ii)	(ii) $x_{n+2} = x_{n+1} + 0.11x_n$ $x_{n+2} - x_{n+1} - 0.11x_n = 0$ The auxiliary equation is $m^2 - m - 0.11 = 0$ $m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-0.11)}}{2}$ $= \frac{1 \pm \sqrt{1.44}}{2}$ $m = 1.1$ or -0.1 $\therefore\ x_n = c(1.1^n) + d(-0.1)^n$ Sub. $u_0 = 20$, $20 = c + d$ Sub. $u_1 = 25$, $25 = 1.1c - 0.1d$ Solving, $c = 22.5$ $d = -2.5$ $\therefore\ x_n = \underline{\underline{22.5(1.1^n) - 2.5(-0.1)^n}}$
6(iii)	$y_{n+2} = y_{n+1} + ky_n - ry_{n+1}$ $= (1-r)\,y_{n+1} + ky_n$
7(i)	$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} - 2y + (1+x)\,y^2 = 0$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y - (1+x)\,y^2}{1+x} = \frac{2y}{1+x} - y^2$ Using Euler's method, $y_{n+1} = y_n + h\,\mathrm{f}(x_n, y_n) \Rightarrow y_{n+1} = y_n + (0.5)\left(\frac{2y_n}{1+x_n} - y_n^{\,2}\right)$, where $x_0 = 0, y_0 = 1$. $y_1 = y_0 + (0.5)\left(\frac{2y_0}{1+x_0} - y_0^{\,2}\right)$ $= 1 + (0.5)\left(\frac{2}{1} - 1\right)$ $= 1.5$ $y_2 = y_1 + (0.5)\left(\frac{2y_1}{1+x_1} - y_1^{\,2}\right)$ $= 1.5 + (0.5)\left(\frac{2(1.5)}{1+0.5} - 1.5^2\right)$ $= 1.375$

7(ii)	$y=\frac{1}{z}$ ... (1) $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{z^2}\frac{\mathrm{d}z}{\mathrm{d}x}$ ... (2) Subst. (1) and (2) into $(1+x)\frac{\mathrm{d}y}{\mathrm{d}x}-2y+(1+x)y^2=0$, $(1+x)\left(-\frac{1}{z^2}\right)\frac{\mathrm{d}z}{\mathrm{d}x}-2\left(\frac{1}{z}\right)+(1+x)\left(\frac{1}{z}\right)^2=0$ $(1+x)\frac{\mathrm{d}z}{\mathrm{d}x}+2z-(1+x)=0$ $\frac{\mathrm{d}z}{\mathrm{d}x}+\frac{2}{1+x}z=1$ (shown)
	$\text{I.F.}=\mathrm{e}^{\int\frac{2}{1+x}\mathrm{d}x}$ $=\mathrm{e}^{2\ln(1+x)}=(1+x)^2$ $(1+x)^2\frac{\mathrm{d}z}{\mathrm{d}x}+(1+x)^2\left(\frac{2}{1+x}\right)z=(1+x)^2$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[(1+x)^2 z\right]=(1+x)^2$ $(1+x)^2 z=\int(1+x)^2\,\mathrm{d}x$ $=\frac{(1+x)^3}{3}+c$ $z=\frac{1+x}{3}+\frac{c}{(1+x)^2}$ $\frac{1}{y}=\frac{(1+x)^3+3c}{3(1+x)^2}$ $y=\frac{3(1+x)^2}{(1+x)^3+A}$, where $A=3c$ When x = 0, y = 1, $\frac{3}{1+A}=1\Rightarrow A=2$. $\therefore y=\frac{3(1+x)^2}{(1+x)^3+2}$
7(iii)	When x = 1, $y=\frac{3(1+1)^2}{(1+1)^3+2}=\frac{6}{5}$. % error $=\frac{\lvert 1.2-1.375\rvert}{1.2}\times 100$ $=14.6$ To improve, use a smaller step size h

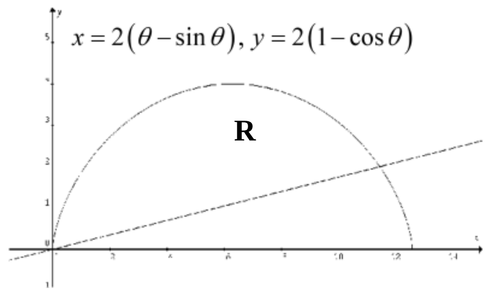
8(i)	$x = r(\theta - \sin\theta),\ y = r(1-\cos\theta) \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = r(1-\cos\theta), \dfrac{\mathrm{d}y}{\mathrm{d}\theta} = r\sin\theta$ Surface area $= \int_0^{2\pi} 2\pi y\sqrt{\left(\dfrac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$ $= 2\pi r\int_0^{2\pi} (1-\cos\theta)\sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\ \mathrm{d}\theta$ $= 2\pi r^2\int_0^{2\pi} (1-\cos\theta)\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\ \mathrm{d}\theta$ $= 2\pi r^2\int_0^{2\pi} (1-\cos\theta)\sqrt{2-2\cos\theta}\ \mathrm{d}\theta$ $= 2\pi r^2\int_0^{2\pi} (1-\cos\theta)\sqrt{2\left(1-1+2\sin^2\dfrac{\theta}{2}\right)}\ \mathrm{d}$ $= 8\pi r^2\int_0^{2\pi} \left(1-\cos^2\dfrac{\theta}{2}\right)\sin\dfrac{\theta}{2}\ \mathrm{d}\theta$ $= 8\pi r^2\int_0^{2\pi} \left(\sin\dfrac{\theta}{2} - \cos^2\dfrac{\theta}{2}\sin\dfrac{\theta}{2}\right)\ \mathrm{d}\theta$ $= 8\pi r^2\left[-2\cos\dfrac{\theta}{2} + \dfrac{2}{3}\cos^3\dfrac{\theta}{2}\right]_0^{2\pi}$ $= 8\pi r^2\left(2-\dfrac{2}{3}+2-\dfrac{2}{3}\right) = \dfrac{64\pi r^2}{3}$ Given $\dfrac{64\pi r^2}{3} = 432\pi.$ $r^2 = \dfrac{81}{4} \Rightarrow r = \dfrac{9}{2}$

8(ii)

At $\left(3\pi+2,2\right)$,

$2=2\left(1-\cos\theta\right)\Rightarrow\cos\theta=0\Rightarrow\theta=\frac{3\pi}{2}$

Equation of L: $y=\frac{2}{3\pi+2}x$



Volume of solid generated

$$=2\pi\int_0^{1.5\pi}2\left(\theta-\sin\theta\right)\cdot2\left(1-\cos\theta\right)\cdot2\left(1-\cos\theta\right)\,\mathrm{d}\theta \quad - \quad \left(\pi-\frac{\pi}{3}\right)\left(3\pi+2\right)^2\left(2\right)$$

$$=16\pi\int_0^{1.5\pi}\left(\theta-\sin\theta\right)\cdot\left(1-\cos\theta\right)^2\,\mathrm{d}\theta \quad - \quad \frac{4\pi}{3}\left(3\pi+2\right)^2$$

$=835$ (3 s.f.)

9(i)

$\frac{\mathrm{d}P}{\mathrm{d}t}=2P-\frac{1}{150}P^2=2P\left(1-\frac{P}{300}\right)$. The carrying capacity is 300.

9(ii)

$$\frac{\mathrm{d}P}{\mathrm{d}t}=2P-\frac{1}{150}P^2=\frac{1}{150}P\left(300-P\right)$$

$$\int\frac{1}{P\left(300-P\right)}\,\mathrm{d}P=\int\frac{1}{150}\,\mathrm{d}t$$

$$\int\frac{1}{150^2-\left(P-150\right)^2}\,\mathrm{d}P=\int\frac{1}{150}\,\mathrm{d}t$$

$$\frac{1}{300}\ln\left(\frac{P}{300-P}\right)=\frac{1}{150}t+C$$

$$\ln\left(\frac{P}{300-P}\right)=2t+300C$$

$$\frac{P}{300-P}=A\mathrm{e}^{2t}\ \text{ where }\ A=\mathrm{e}^{300C}$$

$$P=\frac{300A\mathrm{e}^{2t}}{1+A\mathrm{e}^{2t}}\ \text{ or }\ P=\frac{300}{B\mathrm{e}^{-2t}+1}.$$

9(iii)

$$\frac{\mathrm{d}P}{\mathrm{d}t}=2P-\frac{1}{150}P^2-h$$

$$=-\frac{1}{150}\left(P^2-300P\right)-h$$

$$=-\frac{1}{150}\left[\left(P-150\right)^2-150^2\right]-h$$

$$=-\frac{1}{150}\left(P-150\right)^2+150-h$$

$\frac{\mathrm{d}P}{\mathrm{d}t}<0$ for all P if $150-h<0\ \Rightarrow\ h>150$.

Thus $h\le150$ and $h_{\max}=150$, and $\frac{\mathrm{d}P}{\mathrm{d}t}=0$ when $P=150=\frac{300}{2}$ (shown).

9(iii)

Alternatively:

$$\frac{dP}{dt} = 2P - \frac{1}{150}P^2 - h \quad \_\_\_(*)$$

$$\frac{d}{dP}\frac{dP}{dt} = 2 - \frac{1}{75}P$$

$$\frac{d}{dP}\frac{dP}{dt} = 0 \Rightarrow P = 150.$$

Substitute $P = 150$ into (*):

$$\frac{dP}{dt} = 2(150) - \frac{1}{150}(150)^2 - h \quad = 150 - h$$

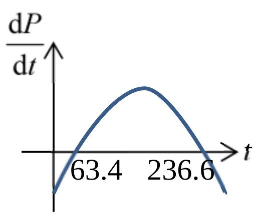
$$\frac{dP}{dt} < 0 \text{ if } h > 150$$

So $h \le 150$ and $h_{max} = 150$, and $\frac{dP}{dt} = 0$ when $P = 150 = \frac{300}{2}$ (shown).

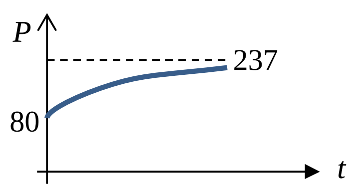
9(iv)

$$\frac{dP}{dt} = 2P - \frac{1}{150}P^2 - 100$$

$\frac{dP}{dt} = 0 \Rightarrow P = 63$ (unstable) or $P = 237$ (stable) (nearest integer)



$P_0 = 80$: Since $63 < P_0 < 237$, $\frac{dP}{dt} > 0$, the population will increase to 237.



9(v)

Without harvesting,

$$P = \frac{300}{Be^{-2t} + 1}$$

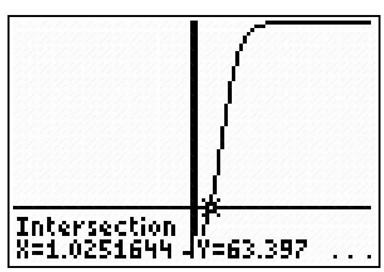
When $t = 0$, $P = 10$,

$$10 = \frac{300}{B + 1} \Rightarrow B = 29$$

$$P = \frac{300}{29e^{-2t} + 1}$$

For population not to die out, $P > 63.397$.

$$\frac{300}{29e^{-2t} + 1} > 63.397$$



Using GC, $t > 1.025$

Hence, least $t = 1.03$ (2 d.p.)

10(a) (i)	Since $\mathbf{A}^2 = \mathbf{A}$, $\det \mathbf{A}^2 = \det \mathbf{A}$ $\Rightarrow (\det \mathbf{A})^2 = \det \mathbf{A}$ $\Rightarrow (\det \mathbf{A})^2 - \det \mathbf{A} = 0$ $\Rightarrow (\det \mathbf{A} - 1)(\det \mathbf{A}) = 0$ $\Rightarrow \det \mathbf{A} = 1$ or 0
10(a) (ii)	When det **A** = 1, then **A** is invertible (non-singular). There is an inverse $\mathbf{A}^{-1}$, such that $\mathbf{AA}^{-1} = \mathbf{I}$. From $\mathbf{A}^2 = \mathbf{A}$, multiply $\mathbf{A}^{-1}$ to both sides of the equation, $(\mathbf{A}^{-1}\mathbf{A})\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ $\Rightarrow \mathbf{IA} = \mathbf{I}$ $\Rightarrow \mathbf{A} = \mathbf{I}$
10(a) (iii)	Suppose det **A** = 0, then $ad = bc$ ------- (1) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\Rightarrow \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Comparing the (1,1) entry, $\Rightarrow a^2 + bc = a$ ----- (2) Substituting (1) into (2), $a^2 + ad = a$ $\Rightarrow a + d = 1$ (since $a \neq 0$) (shown)
10(b) (i)	Let $\mathbf{u}, \mathbf{v} \in A$. $(\mathbf{u}+\mathbf{v}) \bullet \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \mathbf{u} \bullet \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \mathbf{v} \bullet \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0+0=0$ Therefore $\mathbf{u}+\mathbf{v} \in A$ and A is closed under vector addition. $(a\mathbf{u}) \bullet \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = a\left(\mathbf{u} \bullet \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right) = 0$ where a is a scalar. Therefore $a\mathbf{u} \in A$ and A is closed under scalar multiplication. In particular, the zero vector is in A. Therefore A is a subspace of $\mathbb{R}^3$. The zero vector is not in B since $2(0)+3(0)-5(0)=0 \neq 1$ Therefore B is a not a subspace of $\mathbb{R}^3$.

10(b) (ii)	Both $\begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \in B' \subseteq A \cup B'$ However, $\begin{pmatrix}0\\2\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\2\\1\end{pmatrix}$ and $\begin{pmatrix}0\\2\\1\end{pmatrix} \notin A$, $\begin{pmatrix}0\\2\\1\end{pmatrix} \in B$. Therefore $\begin{pmatrix}0\\2\\1\end{pmatrix} \notin A \cup B'$ and since $A \cup B'$ is not closed under vector addition, it is not a subspace of $\mathbb{R}^3$. Also, $\begin{pmatrix}-2\\2\\0\end{pmatrix} \in A \cap B'$ However, $\frac{1}{2}\begin{pmatrix}-2\\2\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix} \in B$ Therefore $\begin{pmatrix}-1\\1\\0\end{pmatrix} \notin A \cap B'$ and since $A \cap B'$ is not closed under scalar multiplication, it is not a subspace of $\mathbb{R}^3$.