RAFFLES INSTITUTION
H2 Mathematics (9758)
2024 Year 6

2024 Year 6 Timed Practice Revision Practice Paper 2 Solutions

Source: 2020 Year 6 Term 3 Timed Practice

Section A: Pure Mathematics (60 marks)

1	The complex number z is given by $x+\mathrm{i}y$, where x and y are real numbers.		
	(i)	Express y in terms of x if $\arg\left(z^2\right)=-\frac{\pi}{2}$.	[2]
	(ii)	State the values of x and y if $\mathrm{Re}(z)>0$ and $\lvert z\rvert=2$.	[1]
		Using these values of x and y, find the smallest positive integer n for which $\frac{z^*}{z^n}$ is a negative real number.	[2]

(i) **[2]**	$z=r\mathrm{e}^{\mathrm{i}\theta}$ $z^2=r^2\mathrm{e}^{\mathrm{i}(2\theta)}$ $2\theta=-\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}$ $\theta=-\frac{\pi}{4}+k\pi,\ k\in\mathbb{Z}$ $\tan(\theta)=-1=\frac{y}{x}\Rightarrow y=-x\ ,\ x\neq 0$ **Alternative** Given $z=x+\mathrm{i}y$ $\arg\left(z^2\right)=-\frac{\pi}{2}$ $2\arg z=-\frac{\pi}{2}$ or $\frac{3\pi}{2}$ $\arg z=-\frac{\pi}{4}$ or $\frac{3\pi}{4}\Rightarrow z$ is in the 2nd or 4th quadrant $\therefore y=-x,\ x\neq 0$	Represent z^2 on an argand diagram!
(ii) **[1]**	$x=\sqrt{2},\ y=-\sqrt{2}$	

(iii) **[2]**	$z=\sqrt{2}-\mathrm{i}\sqrt{2}$ $\arg(z)=-\frac{\pi}{4}$ $\frac{z^*}{z^n}=\frac{r\mathrm{e}^{\mathrm{i}(-\theta)}}{r^n\mathrm{e}^{\mathrm{i}(n\theta)}}=r^{1-n}\ \mathrm{e}^{-\mathrm{i}(n+1)\theta}$ $-(n+1)\theta=\pi+k(2\pi),\ k\in\mathbb{Z}$ $-(n+1)\left(-\frac{\pi}{4}\right)=(2k+1)\pi,\ k\in\mathbb{Z}$ $\frac{n+1}{4}=2k+1,\ k\in\mathbb{Z}$ $n=8k+3,\ k\in\mathbb{Z}$ The smallest positive integer n is 3. **<u>Alternative</u>** $n>0,\quad (n+1)\left(\frac{\pi}{4}\right)=\pi,\ 3\pi,\ 5\pi...$ The smallest positive integer n is 3.	For $\frac{z^*}{z^n}$ to be a negative real number means that $\arg\left(\frac{z^*}{z^n}\right)=\pi$

2	**(a)**	By writing $\frac{2}{r\left(r^2-1\right)}$ in partial fractions, find an expression for $\sum_{r=2}^{n}\frac{2}{r\left(r^2-1\right)}$. [3]
	(b)	A geometric series has first term a and common ratio r, where a and r are non-zero and $r\neq 1$. The 3rd and 9th terms of the series are 448 and 7 respectively. Given also that the sum of the first n terms is 1197, find the values of a, r and n. [4]

(a) **[3]**	Let $\frac{2}{r\left(r^2-1\right)}=\frac{A}{r}+\frac{B}{r+1}+\frac{C}{r-1}$ $2=A(r+1)(r-1)+Br(r-1)+Cr(r+1)$ Let $r=0,\ A=-2$ Let $r=1,\ C=1$ Let $r=-1,\ B=1$ $\therefore\frac{2}{r\left(r^2-1\right)}=\frac{-2}{r}+\frac{1}{r+1}+\frac{1}{r-1}$ $\sum_{r=2}^{n}\frac{2}{r\left(r^2-1\right)}=\sum_{r=2}^{n}\left[\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}\right]$	To apply MOD write in "order" $\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}$.

$$
\begin{aligned}
&= \frac{1}{1} - \frac{2}{2} + \frac{1}{3} \\
&+ \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \\
&+ \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \\
&+ \frac{1}{4} - \frac{2}{5} + \frac{1}{6} \\
&\vdots \\
&+ \frac{1}{n-3} - \frac{2}{n-2} + \frac{1}{n-1} \\
&+ \frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n} \\
&+ \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}
\end{aligned}
$$

$$\sum_{r=2}^{n} \frac{2}{r\left(r^2-1\right)} = \frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}$$

(b)
[4]

$ar^2 = 448, \quad ar^8 = 7$

$\Rightarrow r^6 = \frac{1}{64} \Rightarrow r = \pm\frac{1}{2}, \quad a = 1792$

Sum of first n terms $= \dfrac{1792\left(1-r^n\right)}{1-r} = 1197$

if $r = \frac{1}{2}$, $\dfrac{1792\left(1-\left(\frac{1}{2}\right)^n\right)}{\frac{1}{2}} = 1197$

$1-\left(\frac{1}{2}\right)^n = \frac{1}{2}\left(\frac{1197}{1792}\right)$

$\left(\frac{1}{2}\right)^n = \frac{341}{512}$

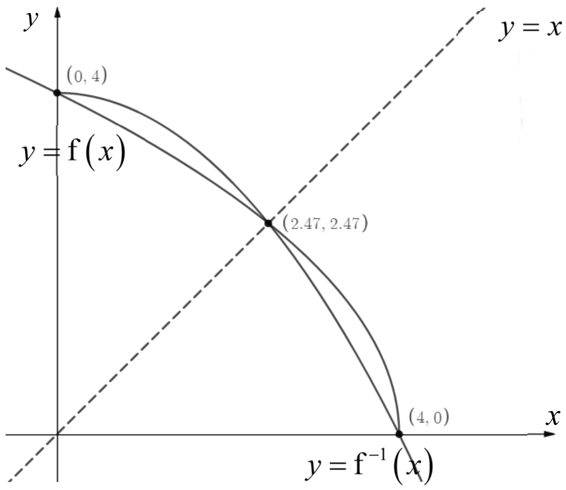
$n = 0.586$ (rej as n is an integer)

if $r = -\frac{1}{2}$, $\dfrac{1792\left(1-\left(-\frac{1}{2}\right)^n\right)}{1-\left(-\frac{1}{2}\right)} = 1197$

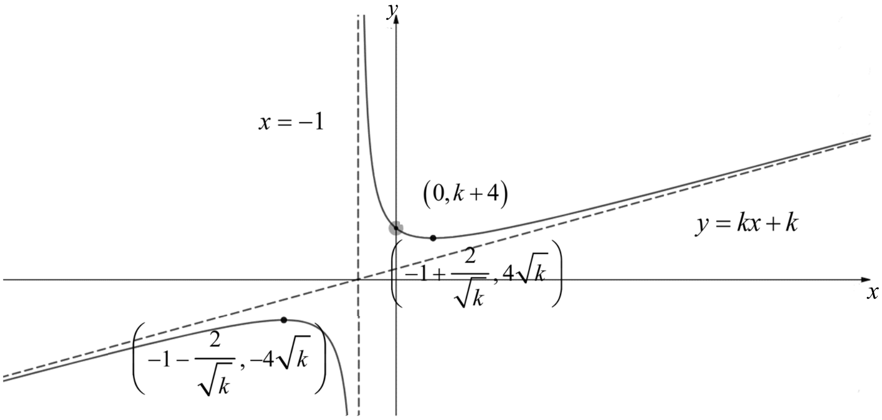
$1-\left(-\frac{1}{2}\right)^n = \frac{3}{2}\left(\frac{1197}{1792}\right)$

	$(-\frac{1}{2})^n = -\frac{1}{512}$ Note that n must be an odd integer, $(\frac{1}{2})^n = \frac{1}{512}$ $n\ln\frac{1}{2} = \ln\frac{1}{512}$ $\therefore n = 9,\ a = 1792,\ r = -\frac{1}{2}$ **<u>Alternative</u>** Observe that since every term is positive, and the first term $a = 1792 >$ sum of n terms $= 1197$, $r = -\frac{1}{2}$ otherwise the sum can only get larger from 1792. Sum of n terms $= \dfrac{1792\left(1-\left(-\frac{1}{2}\right)^n\right)}{1-\left(-\frac{1}{2}\right)} = 1197$ $1-\left(-\frac{1}{2}\right)^n = \frac{3}{2}(\frac{1197}{1792})$ $(-\frac{1}{2})^n = -\frac{1}{512}$ Note that n must be an odd integer, $(\frac{1}{2})^n = \frac{1}{512}$ $n\ln\frac{1}{2} = \ln\frac{1}{512}$ $n = 9$	

3	The functions f and g are defined by $\mathrm{f}: x \mapsto \sqrt{16-4x},\ x \in \mathbb{R},\ x \le 4,$ $\mathrm{g}: x \mapsto x^2,\ x \in \mathbb{R}.$	
	(i)	Sketch on the same diagram, the graphs of f and f^{-1}, giving the coordinates of all points of intersection. [4]
	(ii)	Explain why the composite function fg does not exist. [1]
	(iii)	Find gf in similar form and state its range. [2]

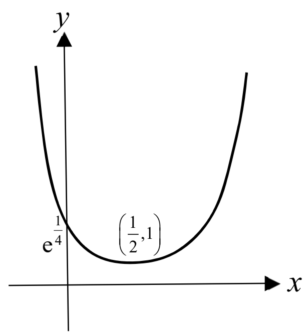
(i)	 Clearly (0, 4), (4, 0) are solutions to $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ Another solution to $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ lies on $y=x$. $\sqrt{16-4x}=x$. By GC, x = 2.47 (3sf) The coordinates of points of intersections are (0, 4), (4, 0) and (2.47, 2.47).	The graphs of f and f^{-1} should appear as "reflections about the line $y=x$". There are 3 points of intersection. Note that $\sqrt{16-4x}=x$ $x=-2+2\sqrt{5}$
(ii)	$\mathrm{D_f}=(-\infty,4]$ $\quad\quad$ $\mathrm{R_g}=[0,\infty)$ Since $\mathrm{R_g}\not\subset \mathrm{D_f}$, hence fg does not exist.	
(iii)	$\mathrm{gf}: x \mapsto 16-4x,\ x\in\mathbb{R},\ x\le 4$ $\mathrm{R_{gf}}=[0,\infty)$	"similar form" means arrow notation which includes "domain"

4	A curve C has equation $y=k(x+1)+\dfrac{4}{x+1},\ x\in\mathbb{R},\ x\neq -1$, where k is a constant, $0<k<2$.	
	(i)	Sketch C, labelling clearly the axial intercept(s), the coordinates of the turning points and equations of the asymptotes. [4]
	The graph of C is transformed by a reflection in the x-axis, followed by a translation of 1 unit in the positive x-direction, followed by a stretch with scale factor $\frac{1}{2}$ parallel to the y-axis.	
	(ii)	Find the equation of the resulting curve in the form $y=\mathrm{f}(x)$. [3]

(i)	$y=k(x+1)+\dfrac{4}{x+1}$ When $x=0$, $y=k+4 \Rightarrow$ coordinates are $(0,k+4)$ When $y=0$, $k(x+1)+\dfrac{4}{x+1}=0$ $k(x+1)=-\dfrac{4}{x+1}$ $(x+1)^2=-\dfrac{4}{k}$ Not applicable since $k>0$. The graph does not cut the x-axis. $\dfrac{dy}{dx}=k-\dfrac{4}{(x+1)^2}=0 \Leftrightarrow x=-1\pm\dfrac{2}{\sqrt{k}}$ When $x=-1-\dfrac{2}{\sqrt{k}}$, $y=k\left(-1-\dfrac{2}{\sqrt{k}}+1\right)+\dfrac{4}{-1-\dfrac{2}{\sqrt{k}}+1}=-4\sqrt{k}$ When $x=-1+\dfrac{2}{\sqrt{k}}$, $y=k\left(-1+\dfrac{2}{\sqrt{k}}+1\right)+\dfrac{4}{-1+\dfrac{2}{\sqrt{k}}+1}=4\sqrt{k}$ Note that $0<k<2$ so $-1+\dfrac{2}{\sqrt{k}}>0$	Turning points are in quadrant 1 and 3.
		
(ii)	Applying transformations to C: $y=k(x+1)+\dfrac{4}{x+1}$ Reflection in the x-axis (replace y by $-y$) $-y=k(x+1)+\dfrac{4}{x+1}$ $y=-k(x+1)-\dfrac{4}{x+1}$ Translate 1 unit in the positive x-direction (replace x by $x-1$)	

	$y=-k\left(\left(x-1\right)+1\right)-\dfrac{4}{\left(x-1\right)+1}$ $y=-kx-\dfrac{4}{x}$ Scale with a factor $\dfrac{1}{2}$ parallel to the y-axis (replace y by $2y$) $2y=-kx-\dfrac{4}{x}$ $y=-\dfrac{kx}{2}-\dfrac{2}{x}$	

5	The curve C has equation $y=\mathrm{e}^{\left(x-\frac{1}{2}\right)^2}$.	
	(i)	Sketch C, labelling clearly the coordinates of the axial intercept(s) and turning point(s), if any. [2]
	(ii)	Show that the equation of the tangent to C at the point where $x=p$ can be expressed as $$(2p-1)x-\mathrm{e}^{-\left(p-\frac{1}{2}\right)^2}y=2p^2-p-1.$$ Hence find the equations of the tangents to C which passes through the origin. [4]
	(iii)	The straight line $y = mx$ intersects C at two distinct points. State the range of values of m. [2]

(i)	 No x-intercepts, y-intercept at $\left(0,\mathrm{e}^{\frac{1}{4}}\right)$ and min turning point at $\left(\dfrac{1}{2},1\right)$.	
(ii)	$y=\mathrm{e}^{\left(x-\frac{1}{2}\right)^2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\left(x-\frac{1}{2}\right)\mathrm{e}^{\left(x-\frac{1}{2}\right)^2}$	

Gradient of tangent at $x = p$ is $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=p} = 2\left(p - \frac{1}{2}\right)\mathrm{e}^{\left(p-\frac{1}{2}\right)^2}$

Equation of tangent is

$$y - \mathrm{e}^{\left(p-\frac{1}{2}\right)^2} = (2p-1)\mathrm{e}^{\left(p-\frac{1}{2}\right)^2}\left(x - p\right)$$

$$\mathrm{e}^{-\left(p-\frac{1}{2}\right)^2} y - 1 = (2p-1)x - 2p^2 + p$$

$$(2p-1)x - \mathrm{e}^{-\left(p-\frac{1}{2}\right)^2} y = 2p^2 - p - 1 \quad \text{(shown)}$$

When tangent passes through the origin, $x = 0,\ y = 0$

$$2p^2 - p - 1 = 0$$

$$\left(2p+1\right)\left(p-1\right) = 0$$

$$\therefore p = -\frac{1}{2} \text{ or } p = 1$$

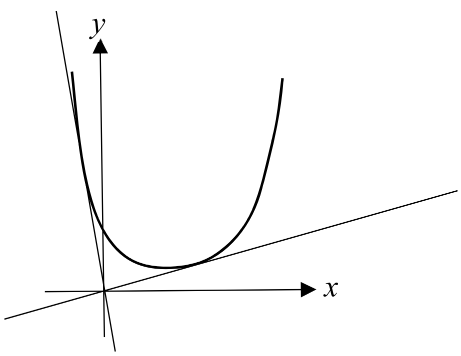
At $p = -\dfrac{1}{2}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2\mathrm{e}$

Equation of tangent: $y = -2\mathrm{e}x$

At $p = 1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{\frac{1}{4}}$

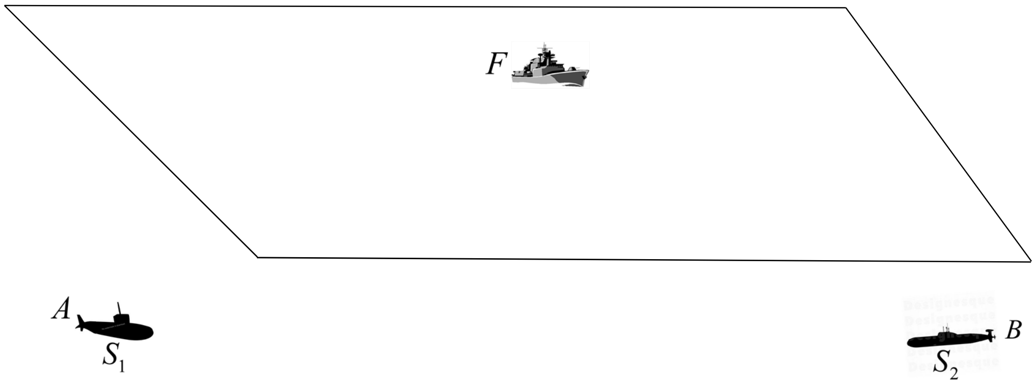
Equation of tangent: $y = \mathrm{e}^{\frac{1}{4}}x$

(iii)



For a straight line passing through the origin to intersect the curve at two points, the straight line must cut the curve in the region bounded by the 2 tangent lines pasing through the origin, $y = \mathrm{e}^{\frac{1}{4}}x$ and $y = -2\mathrm{e}x$.

$\therefore m > \mathrm{e}^{\frac{1}{4}}$ or $m < -2\mathrm{e}$

6	A frigate is stationed at position $F(1,2,0)$. Two submarines S_1 and S_2 are under the sea surface. Submarine S_1 is at position $A(-2,-1,-1)$ and travelling in a path parallel to vector $-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. An enemy submarine S_2 is detected at position $B(3,2,-2)$ travelling in a path parallel to vector $-2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. 	
	(i)	Determine if the paths of the submarines will intersect each other. [3]
	(ii)	The enemy submarine S_1 will launch a torpedo at the frigate when it is at a point P in its path that is closest to F. Find the co-ordinates of P. [4]
	(iii)	Find a cartesian equation of the plane π that contains F and the path of S_2. Calculate the acute angle between π and the x-y plane. [4]
	(iv)	A depth charge is a countermeasure used against submarines. The frigate releases a depth charge which descends to the position $Q(1, 2, -k)$ to target S_2. The depth charge is detonated when the distance from Q to P is at a minimum. Find the value of k. [1] When detonated the depth charge will cause damage within a 0.1 unit radius. Explain whether S_2 will be damaged by the depth charge. [2]

(i)	For S_1, $l_1 : \mathbf{r} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, \ \lambda \in \mathbb{R}$ For S_2, $l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}, \ \mu \in \mathbb{R}$ For point of intersection, $\begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$	Write neatly and be careful when "flipping page over".

	$3\lambda - 2\mu = -5$ ---(1) $-2\lambda - 3\mu = -3$ ---(2) $\lambda + \mu = 1$ ---(3) Using (1) and (2), $\lambda = -\frac{9}{13}$, $\mu = \frac{19}{13}$ which does not satisfy (3) Since there are **no solutions** that satisfy (1), (2) and (3), the 2 paths of the submarines do not cross each other.	
(ii)	$\overrightarrow{OP} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$, for some $\mu \in \mathbb{R}$ $\overrightarrow{PF} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ $\overrightarrow{PF} \perp l_2$ so $\left[\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right] \bullet \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = 0$ $4 + 2 - \mu(4+9+1) = 0$ $\mu = \frac{3}{7}$ $\overrightarrow{OP} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \frac{3}{7} \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 15 \\ 5 \\ -11 \end{pmatrix}$ Co-ordinates are $P\left(\frac{15}{7}, \frac{5}{7}, -\frac{11}{7} \right)$	Question required "co-ordinates"
(iii)	$\overrightarrow{BF} = \overrightarrow{OF} - \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\underset{\sim}{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$ $\pi: \underset{\sim}{r} \bullet \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$ Cartesian Equation of π, $\pi: 3x - y + 3z = 1$ Let angle between the 2 planes be θ.	Question required "Cartesian Equation"

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$$\cos\theta = \frac{\left|\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\3\end{pmatrix}\right|}{\sqrt{3^2+(-1)^2+3^2}} = \frac{3}{\sqrt{19}}$$

$\theta = 46.508°\ (5sf)$

$= 46.5°\ \ (1\text{ dp})$

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x-y plane is z = 0 has

normal $\mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

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$$\overrightarrow{OQ} = \begin{pmatrix}1\\2\\-k\end{pmatrix},\quad \overrightarrow{OP} = \frac{1}{7}\begin{pmatrix}15\\5\\-11\end{pmatrix} \quad\Rightarrow\quad \overrightarrow{PQ} = \frac{1}{7}\begin{pmatrix}-8\\9\\-7k+11\end{pmatrix}$$

$\left|\overrightarrow{PQ}\right|$ is minimized when $-7k+11=0$, so $k=\frac{11}{7}$

Explanation 1

Assume the submarine is at point P when depth charge detonated. $\left|\overrightarrow{PQ}\right| = \frac{1}{7}\sqrt{(-8)^2+9^2+0} = \frac{\sqrt{145}}{7} > 0.1$

Therefore the submarine is not damaged.

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Explanation 2

Assume the submarine is somewhere along the path l_2 where $l_2 : \mathbf{r} = \begin{pmatrix}3\\2\\-2\end{pmatrix} + \mu\begin{pmatrix}-2\\-3\\1\end{pmatrix}$, $\mu\in\mathbb{R}$, when depth charge detonated.

Points along the submarine's path are given by $(3-2\mu, 2-3\mu, -2+\mu)$.

Distance between $(3-2\mu, 2-3\mu, -2+\mu)$ and $Q\left(1, 2, -\frac{11}{7}\right)$ can be expressed as $D(\mu) = \sqrt{(2-2\mu)^2+(-3\mu)^2+\left(-\frac{3}{7}+\mu\right)^2}$

which has minimum (from GC) of 1.67 > 0.1

Therefore the submarine is not damaged.

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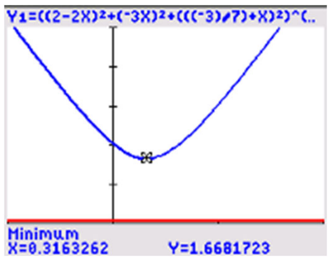

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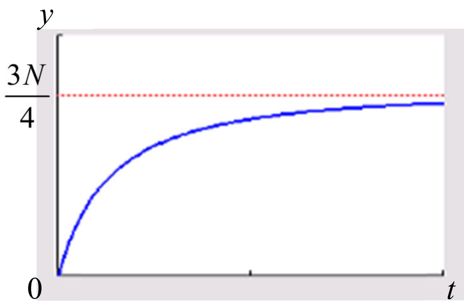
7	A chemical substance Uru starts to form when N units of Vibranium react with M units of Kryptonite, where $M < N$. Let y denote the number of units of Uru formed t hours after the reaction takes place. The differential equation for the chemical reaction is given by $$\frac{\mathrm{d}y}{\mathrm{d}t} = k(N-y)(M-y),$$ where $0 \le y \le M$ and k is a positive constant.

	(i)	By first expressing $\dfrac{1}{(N-y)(M-y)}$ in the form $\dfrac{A}{N-y}+\dfrac{B}{M-y}$, where A and B are constants to be found in terms of N and M, find t in terms of y. [5]
	It is given that $\dfrac{M}{N}=\dfrac{3}{4}$.	
	(ii)	Find the time taken to produce $\dfrac{N}{4}$ units of Uru, giving your answer in terms of N and k. [2]
	(iii)	Express the solution of the differential equation in the form $y=\mathrm{f}\left(t\right)$. [2] Sketch the part of the curve with this equation which is relevant in this context and state what happens to y for large values of t. [3]

(i) [5]	$\dfrac{1}{(N-y)(M-y)}=\dfrac{1}{N-M}\left(\dfrac{1}{M-y}-\dfrac{1}{N-y}\right)$ Therefore $A=-\dfrac{1}{N-M}$, $B=\dfrac{1}{N-M}$ $\dfrac{\mathrm{d}y}{\mathrm{d}t}=k(N-y)(M-y)$ $\dfrac{1}{(N-y)(M-y)}\dfrac{\mathrm{d}y}{\mathrm{d}t}=k$ $\dfrac{1}{N-M}\left(\dfrac{1}{M-y}-\dfrac{1}{N-y}\right)\dfrac{\mathrm{d}y}{\mathrm{d}t}=k$ $\int\dfrac{1}{M-y}-\dfrac{1}{N-y}\ \mathrm{d}y=\int k(N-M)\ \mathrm{d}t$ $\ln\left\|\dfrac{N-y}{M-y}\right\|=k(N-M)t+C$, where $C\in\mathbb{R}$ is an arbitrary constant. $0\le y\le M$ and $M<N$, $\ln\left(\dfrac{N-y}{M-y}\right)=k(N-M)t+C$, $t=0,\ y=0\Rightarrow C=\ln\left(\dfrac{N}{M}\right)$ $\ln\left(\dfrac{N-y}{M-y}\right)=k(N-M)t+\ln\left(\dfrac{N}{M}\right)$ $k(N-M)t=\ln\left(\dfrac{N-y}{M-y}\right)-\ln\left(\dfrac{N}{M}\right)$	**See Tut8C Q11**

	$t=\dfrac{1}{k\left(N-M\right)}\ln\left(\dfrac{M}{N}\left(\dfrac{N-y}{M-y}\right)\right)$	
(ii) **[2]**	Given that $\dfrac{M}{N}=\dfrac{3}{4}\Leftrightarrow M=\dfrac{3}{4}N$. When $y=\dfrac{1}{4}N$, $t=\dfrac{1}{k\left(N-\frac{3}{4}N\right)}\ln\left(\dfrac{3}{4}\left(\dfrac{N-\frac{1}{4}N}{\frac{3}{4}N-\frac{1}{4}N}\right)\right)=\dfrac{4}{kN}\ln\left(\dfrac{3}{4}\left(\dfrac{3}{2}\right)\right)=\dfrac{4}{kN}\ln\left(\dfrac{9}{8}\right)$	
(iii) **[5]**	$t=\dfrac{4}{kN}\ln\left(\dfrac{3}{4}\left(\dfrac{N-y}{\frac{3}{4}N-y}\right)\right)\Leftrightarrow\dfrac{3N-3y}{3N-4y}=\mathrm{e}^{\frac{kNt}{4}}$ $3N-3y=3N\mathrm{e}^{\frac{kNt}{4}}-4y\mathrm{e}^{\frac{kNt}{4}}$ $\left(4\mathrm{e}^{\frac{kNt}{4}}-3\right)y=3N\left(\mathrm{e}^{\frac{kNt}{4}}-1\right)$ $y=3N\left(\dfrac{\mathrm{e}^{\frac{kNt}{4}}-1}{4\mathrm{e}^{\frac{kNt}{4}}-3}\right)$ *or* $y=3N\left(\dfrac{1-\mathrm{e}^{-\frac{kNt}{4}}}{4-3\mathrm{e}^{-\frac{kNt}{4}}}\right)$ *or* $y=N\left(1-\dfrac{1}{4-3\mathrm{e}^{-\frac{kNt}{4}}}\right)$  As $kN>0$, so as $t\to\infty$, $\mathrm{e}^{-\left(\frac{kN}{4}\right)t}\to 0\Rightarrow y=3N\left(\dfrac{1-\mathrm{e}^{-\frac{kNt}{4}}}{4-3\mathrm{e}^{-\frac{kNt}{4}}}\right)\to\dfrac{3N}{4}$ OR As $kN>0$, so as t gets large, $\mathrm{e}^{\frac{kNt}{4}}$ also gets large so $\dfrac{\mathrm{e}^{\frac{kNt}{4}}-1}{4\mathrm{e}^{\frac{kNt}{4}}-3}\approx\dfrac{\mathrm{e}^{\frac{kNt}{4}}}{4\mathrm{e}^{\frac{kNt}{4}}}=\dfrac{1}{4}$ and hence y ***increases*** and tends towards $\dfrac{3N}{4}$.	

Section B: Probability and Statistics (40 marks)

8	Find the number of ways in which the letters of the word INTEGRITY can be arranged if	
	(i)	there are no restrictions, [1]
	(ii)	any 2 vowels must be separated by exactly 2 consonants, [2]
	(iii)	no adjacent letters are the same. [3]

(b) (i)	II, N, TT, E, G, R, Y Number of ways = $\frac{9!}{2!2!} = 90720$	
(ii)	[VCCVCCVCC or CCVCCVCCV or CVCCVCCVC] C taken from N, TT, G, R, Y V taken from II, E Number of ways = $\frac{6!}{2!} \times \frac{3!}{2!} \times 3 = 3240$	"Act out" the possibilities.
(iii)	Number of ways with TT as a unit = $\frac{8!}{2!}$, and similar with II as unit. Both these include possibility of both TT and II as units which occur in 7! ways Number of ways = $\frac{9!}{2!2!} - \frac{8!}{2!} - \frac{8!}{2!} + 7! = 55440$ **<u>Alternative 1</u>** Number of ways = total way – TT as an unit but II separated – II as an unit but TT separated – TT and II both as units. So number of ways = $\frac{9!}{2!2!} - 6! \times {}^{7}C_2 \times 2 - 7! = 55440$ **<u>Alternative 2</u>** Number of ways = arrange except TT and II then slot in TT then slot in II + the case with TIT as a group So number of ways = $5! \times {}^{6}C_2 \times {}^{8}C_2 + 7! = 55440$	Please see TutS1A Q7(iv)

9	During training, the time in seconds for a soldier to dismantle a certain type of equipment is a normally distributed continuous random variable *T*. The standard deviation of *T* is 3.5 and the expected value of *T* is 35.1. After a new set of instructional material is introduced, *n* soldiers are selected at random and the mean time taken for this sample of soldiers to dismantle the equipment is found to be $\bar{t}$ seconds. A test is carried out, at 5% significance level, to determine whether the mean time taken to dismantle the equipment has been reduced.	
	(i)	State the appropriate hypothesis for the test. [1]
	(ii)	Given that $n = 20$, find the set of values of $\bar{t}$ for which the result of the test would be to reject the null hypothesis. [3]
	(iii)	Given instead that $\bar{t} = 33.2$ and the result of the test is that the null hypothesis is not rejected. Find the largest possible value of *n*. [3]

(i) **[1]**	Let μ denote the population mean time taken to dismantle equipment, after the new set of instructional material is introduced. H0: $\mu = 35.1$ vs H1: $\mu < 35.1$	**See Tut S5 Q9**
(ii) **[3]**	Given *T* is the time taken for a soldiers to dismantle the equipment Under H0, $\bar{T} \sim \mathrm{N}\left(35.1, \frac{3.5^2}{20}\right)$ Level of significance: 5% Reject H0 if p-value < 0.05 using a one-tail z-test. $\mathrm{P}\left(\bar{T} \le \bar{t}\right) \le 0.05$ From G.C., $\mathrm{P}\left(\bar{T} \le 33.81269842\right) = 0.05 \Rightarrow \bar{t} \le 33.8$ Therefore, the set of values is $\left\{\bar{t} \in \mathbb{R} : 0 < \bar{t} \le 33.8\right\}$ *or* $\left(0, 33.8\right]$.	We **do not** use the Central Limit Theorem as the population is already given to be normally distributed. Note differences between $\bar{T}$ which is a random variable and $\bar{t}$ which is a value.
(iii) **[3]**	Under H0, $\bar{T} \sim \mathrm{N}\left(35.1, \frac{3.5^2}{n}\right)$ Level of significance: 5% Reject H0 if p-value ≤ 0.05 i.e. For H0 not to be rejected, p-value > 0.05 $\mathrm{P}\left(\bar{T} \le 33.2\right) > 0.05$ **<u>Method 1 by standardisation:</u>** $\mathrm{P}\left(Z \le \frac{33.2 - 35.1}{3.5/\sqrt{n}}\right) > 0.05$ By G.C., $\mathrm{P}\left(Z \le -1.6449\right) > 0.05$ $\Rightarrow \frac{33.2 - 35.1}{3.5/\sqrt{n}} > -1.6449$	

	$\Rightarrow \frac{-1.9}{3.5}\sqrt{n} > -1.6449$ $\Rightarrow n < \left[1.6449\left(\frac{3.5}{1.9}\right)\right]^2 \approx 9.181378$ **<u>Method 2 by GC table:</u>** Let $y = \text{normalcdf}\left(-1E99,\ 33.2,\ 35.1,\ 3.5/\sqrt{x}\right)$ <table><tr><td>x</td><td>y</td></tr><tr><td>8</td><td>0.0623</td></tr><tr><td>9</td><td>0.0517</td></tr><tr><td>10</td><td>0.043</td></tr></table>Therefore, largest possible number of n is 9.	

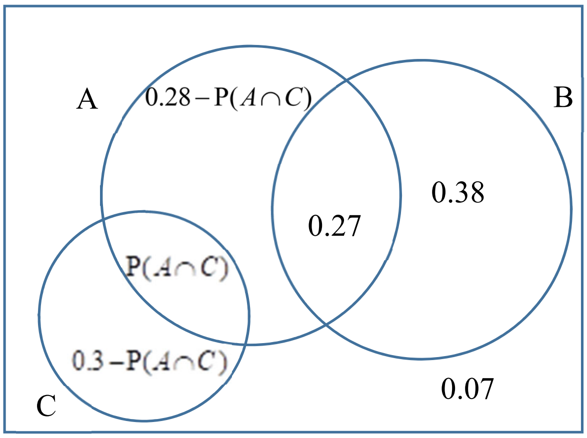
10	A bag contains 3 red balls and n yellow balls, where $n \geq 2$. In a game, Joe removes 2 balls at random from the bag one at a time without replacement. The number of red balls Joe removes from the bag is denoted by T.		
	(i)	Find $\mathrm{P}(T = t)$ for all possible values of t.	[2]
	(ii)	Find $\mathrm{E}(T)$ and $\mathrm{Var}(T)$.	[5]

(i)	<table><tr><td>t</td><td>$\mathrm{P}(T=t)$</td></tr><tr><td>0</td><td>$\frac{\binom{3}{0}\binom{n}{2}}{\binom{n+3}{2}} = \frac{\frac{n(n-1)}{2}}{\frac{(n+3)(n+2)}{2}} = \frac{n(n-1)}{(n+3)(n+2)}$</td></tr><tr><td>1</td><td>$\frac{\binom{3}{1}\binom{n}{1}}{\binom{n+3}{2}} = \frac{3n}{\frac{(n+3)(n+2)}{2}} = \frac{6n}{(n+3)(n+2)}$</td></tr><tr><td>2</td><td>$\frac{\binom{3}{2}\binom{n}{0}}{\binom{n+3}{2}} = \frac{3}{\frac{(n+3)(n+2)}{2}} = \frac{6}{(n+3)(n+2)}$</td></tr></table>	Check that $\frac{n(n-1)+6n+6}{(n+3)(n+2)} = 1$
(ii)	$E(T) = \frac{6n}{(n+3)(n+2)} + 2\times\frac{6}{(n+3)(n+2)} = \frac{6n+12}{(n+3)(n+2)} = \frac{6}{n+3}$	Simplify your answers.

	$E\left(T^2\right) = \frac{6n}{(n+3)(n+2)} + 4\times\frac{6}{(n+3)(n+2)} = \frac{6(n+4)}{(n+3)(n+2)}$ $Var(T) = \mathrm{E}\left(T^2\right) - \left[\mathrm{E}\left(T\right)\right]^2 = \frac{6(n+4)}{(n+3)(n+2)} - \left(\frac{6}{n+3}\right)^2$ $= \frac{6\left[(n+3)(n+4) - 6(n+2)\right]}{(n+3)^2(n+2)}$ $= \frac{6\left(n^2+n\right)}{(n+2)(n+3)^2}$ $= \frac{6n(n+1)}{(n+2)(n+3)^2}$	

11	For events A and B it is given that $\mathrm{P}(A) = 0.55$, $\mathrm{P}(B) = 0.65$ and $\mathrm{P}(A \cup B) = 0.93$.	
	Find	
	(i)	$\mathrm{P}(A \cap B)$ [2]
	(ii)	$\mathrm{P}(A \cup B')$ [2]
	(iii)	Determine if A and B are independent. [1]
	It is further given that $\mathrm{P}(C) = 0.3$ and that events B and C are mutually exclusive.	
	(iv)	Find the greatest and least possible values of $\mathrm{P}(A \cap C)$. [3]

(i) **[2]**	$\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)$ $= 0.55 + 0.65 - 0.93$ $= 0.27$	See Venn Diagram below in (iv)
(ii) **[2]**	$\mathrm{P}(A \cup B') = \mathrm{P}(A) + \mathrm{P}(A \cup B)'$ $= 0.55 + (1 - 0.93)$ $= 0.62$ OR $\mathrm{P}(A \cup B') = 1 - \mathrm{P}(A' \cap B)$ $= 1 - \left[\mathrm{P}(B) - \mathrm{P}(A \cap B)\right]$ $= 1 - (0.65 - 0.27)$ $= 0.62$	
(iii) **[1]**	Since $\mathrm{P}(A \cap B) = 0.27 \neq \mathrm{P}(A) \times \mathrm{P}(B) = 0.3575$, A and B are not independent	

	OR $\mathrm{P}(A\mid B)=\frac{27}{65}\neq\frac{11}{20}=\mathrm{P}(A)$ OR $\mathrm{P}(B\mid A)=\frac{27}{55}\neq\frac{13}{20}=\mathrm{P}(B)$	
(iv) **[2]**	 From the Venn diagram, it is clear that since $0.28-\mathrm{P}(A\cap C)\geq 0$, $\mathrm{P}(A\cap C)\leq 0.28$ Also, since $0.3-\mathrm{P}(A\cap C)\leq 0.07$, $\mathrm{P}(A\cap C)\geq 0.23$ $\therefore 0.23\leq \mathrm{P}(A\cap C)\leq 0.28$ Least $\mathrm{P}(A\cap C)=0.23$ Greatest $\mathrm{P}(A\cap C)=0.28$	

12	**In this question you should state clearly all the distributions that you use, together with the values of the appropriate parameters.** A local supermarket sells two types of potatoes, Russet potatoes and Holland potatoes. The masses, in grams, of the Russet potatoes have the distribution $\mathrm{N}\left(200,30^2\right)$ and the masses, in grams, of the Holland potatoes have the distribution $\mathrm{N}\left(150,24^2\right)$.	
	(i)	Find the probability that the total mass of 3 randomly chosen Holland potatoes is less than twice the mass of a randomly chosen Russet potato. [3]
	(ii)	The supermarket decides to pack the potatoes into a variety pack. Each variety pack consists of 5 randomly chosen Russet and 4 randomly chosen Holland potatoes. The probability that a randomly chosen variety pack is within k grams of 1600 grams is found to be 0.775. Find k. [3]
	(iii)	40 variety packs are randomly selected and are to be donated to needy families. Using the value of k found in **(ii)**, find the probability that at most 5 variety packs are not within k grams of 1600 grams, [2]
	(iv)	State an assumption needed for your calculations in parts **(i), (ii)** and **(iii)**. [1]

	The supermarket recently introduced a new type of potatoes, called New potatoes. The mean mass of New potatoes is 55 grams and standard deviation is 13 grams.	
	(v)	Find an approximate value for the probability that the average mass from a random sample of 100 New potatoes is not more than 58 grams. [3]

(i)	Let R be the random variable denoting the mass, in grams, of a Russet potato. Let H be the random variable denoting the mass, in grams, of a Holland potato. $\mathrm{E}\left(\left(H_1+H_2+H_3\right)-2R\right)=3\times150-2\times200=50$ $\mathrm{Var}\left(H_1+H_2+H_3-2R\right)=3\times24^2+2^2\times30^2=5328$ $H_1+H_2+H_3-2R\sim \mathrm{N}\left(50,5328\right)$ $\mathrm{P}\left(H_1+H_2+H_3<2R\right)$ $=\mathrm{P}\left(H_1+H_2+H_3-2R<0\right)$ $=0.247$ (to 3s.f.)	Define random variable clearly. Do NOT use "Z" which is for N(0,1) Read Question clearly. Indicate the distribution clearly in ALL parts.
(ii)	Let V be the random variable denoting the mass, in grams, of a randomly chosen variety pack. $\mathrm{E}\left(V\right)=5\times200+4\times150=1600$ $\mathrm{Var}\left(V\right)=5\times30^2+4\times24^2=6804$ $V\sim \mathrm{N}\left(1600,\ 6804\right)$ $\mathrm{P}\left(1600-k<V<1600+k\right)=0.775$ $k=100$ (to 3s.f.)	
(iii)	Let W be the random variable denoting the number of variety packs not within "k" grams of the mean out of 40 packs. $W\sim \mathrm{B}\left(40,\ 1-0.775=0.225\right)$ $\mathrm{P}\left(W\le5\right)=0.0869$ (3sf)	Define the binomial random variable clearly
(iv)	The mass of (randomly chosen) Russet potatoes and Holland potatoes are independent of each other. **Please note:** "Distributions" or "Probabilities" CANNOT be described as being independent of each other. "Events" or "Random Variables" can be described as independent.	
(v)	Let C be the random variable denoting the mass of a randomly chosen New potato. $\mathrm{E}\left(C\right)=55$ and $\mathrm{Var}\left(C\right)=13^2$ Since $n=100$ is large, by Central Limit Theorem, $\overline{C}=\dfrac{C_1+\ldots C_{100}}{100}\sim \mathrm{N}\left(55,\ \dfrac{13^2}{100}\right)$ approximately	You cannot claim that C is normally distributed as that **is not** specified. Notice also that the question asks to "approximate" the probability.

	$\mathrm{P}\left(\overline{C} \le 58\right) = 0.989 \quad (\text{to 3s.f.})$	